4. GAME VALUE FOR THE k-SLA AND INFINITY RULE

The result of the previous section is applied to the problem of Dynkin's stopping game, in which Player I is to maximize and Player II to minimize as already defined in Section 2. The optimal strategy of each player is defined by (2.4) and (2.5). The result of this section is an extension of the discussion in Section 2, which considers a policy from the OLA rule to the k-SLA and the infinity rule.

For the standard maximization problem of (2.19), the next sequence will be defined analogously to $d_i(x)$, $i = 1, 2, \ldots, k$ in (3.1) for the minimization of (2.20). We set $k \geq 1$ a fixed integer as before. Define $e_i(x)$, $x \in S$, $i = 1, 2, \ldots, k$ by

$$e_i(x) = (P\psi - \psi)(x) - P(e_{i-1})^-(x), \tag{4.1}$$

where we put $e_0(x) = 0$. Denote the stopping region for Player I and II by

$$\begin{aligned} B_1^k &= \{x \in S; d_k(x) \leq 0\}, \\ B_2^k &= \{x \in S; e_k(x) \geq 0\}, \end{aligned} \tag{4.2}$$

respectively, and C_k be the complement of $B_1^k \cup B_2^k$. We shall refer *k-SLA rule* of the game variant to the stopping rule based on the first hitting time of set B_1^k or B_2^k.

ASSUMPTION 4.1.

(1) *Either of B_1^k or B_2^k is assumed to be nonempty and each set B_i^k is closed with respect to P for $i = 1, 2$; that is,*

$$P(x, B_i^k) = 1, \qquad x \in B_i^k, \quad i = 1, 2. \tag{4.3}$$

(2)
$$\nu(B_1^k \cup B_2^k) < \infty \text{ a.e. } P^x, \qquad X_0 = x \in S. \tag{4.4}$$

(3) *We assume that*

$$\begin{aligned} \liminf_n E^x[\psi(X_n)] &\leq \varphi(x), \qquad x \in B_1^k, \\ \limsup_n E^x[\varphi(X_n)] &\geq \psi(x), \qquad x \in B_2^k. \end{aligned} \tag{4.5}$$

The result on the k-SLA rule for the stopping problem by the previous discussion would be as follows.

THEOREM 4.1. *Under Assumptions 2.1 and 4.1, the sets B_1^k and B_2^k are disjoint. Further, the k-SLA rule is optimal and the game value is given by*

$$v(x) = \begin{cases} \varphi(x), & x \in B_1^k, \\ \mathrm{N}_{C^k}\left[P_{B_1^k}\varphi + P_{B_2^k}\psi\right](x), & x \in C^k, \\ \psi(x), & x \in B_2^k. \end{cases} \tag{4.6}$$

To consider the infinity-SLA rule of the game variant, we take the limit of k to infinity. Similar to Lemma 3.5, we see that the sequence $\{e_i(x); i \geq 1\}$ is monotone decreasing and bounded below. So

$$e^*(x) = \lim_{i \to \infty} e_i(x), \qquad x \in S \tag{4.7}$$

exists and satisfies that

$$e^*(x) = (P\psi - \psi)(x) - P(e^*)^-(x), \qquad x \in S. \tag{4.8}$$