Since $v(x) = \varphi(x)$ for $x \in B^k$, this proves (3.10). Next, we shall show (3.9). If $x \in B^k$, $v(x) = \varphi(x)$ and so (3.2) implies that $Pv(x) = P\varphi(x)$. And also, $(d_k)^+(x) = P(d_{k-1})^+(x) = 0$, $x \in B^k$ by Lemma 3.1 and (3.2). On the other hand, if $x \in C^k$, then $v(x) - Pv(x) = 0$ and $\varphi(x) - P\varphi(x) + (d_k)^+(x) - P(d_{k-1})^+(x) = 0$ from the definition of d_k. Thus, we have

$$v(x) - Pv(x) = \varphi(x) - P\varphi(x) + (d_k)^+(x) - P(d_{k-1})^+(x), \qquad x \in S.$$

Hence, (3.9) is proved by Lemma 3.2. ∎

COROLLARY 3.4.

If, for some $j \geq 1$, the j-SLA rule is optimal, then the value of its stopping problem is dominated as follows:

$$\varphi(x) \leq v(x) \leq \varphi(x) + \mathbb{N}(P\varphi - \varphi)^+(x), \tag{3.11}$$

for all x in S.

PROOF. Since $(d_j)^+(x) \leq (P\varphi - \varphi)^+(x) + P(d_{j-1})^+(x)$, $x \in S$, the upper bound could be obtained from (3.10). The lower bound is immediate by (1.1).

Note that, if $j = 1$, that is, the OLA rule is optimal, the upper bound holds with equality. This upper bound of the optimal value is consistant with the result of [14, Lemma 3.3]. In the conclusion of this section, we should like to discuss the infinity-SLA rule, which is a limiting case of tending k to infinity.

LEMMA 3.5. $\lim_{i\to\infty} d_i(x) = d^*(x)$, $x \in S$ *exists. It is integrable with respect to P and satisfies*

$$d^*(x) = (P\varphi - \varphi)(x) + P(d^*)^+(x), \qquad x \in S, \tag{3.12}$$

and also $B^* = \{x \in S; d^*(x) \leq 0\}$ is equal to $\bigcap_{k=1}^\infty B^k$.

PROOF. The sequence is shown to be monotone increasing by the induction and is dominated by $\mathbb{N}(P\varphi - \varphi)^+(x)$, which is integrable with respect to P. The assertion (3.12) holds by the Dominated Convergence Theorem. The last assertion is clear from Lemma 3.1.

ASSUMPTION 3.2.

(1) *The set $B^* = \{x \in S; d^*(x) \leq 0\}$ and its complement C^* are nonempty, and B^* is closed with respect to P; that is,*

$$P(x, B^*) = 1, \qquad x \in B^*. \tag{3.13}$$

(2) *The first hitting time $\nu(B^*)$ satisfies*

$$\nu(B^*) < \infty \text{ a.e. } P^x, \qquad X_0 = x \in S. \tag{3.14}$$

Similarly as before, we shall refer to the *infinity-SLA rule* if the rule is based on the first hitting time $\nu(B^*)$.

THEOREM 3.6. *Under Assumptions 2.1 and 3.2, the infinity-SLA rule is optimal, its optimal value equals*

$$v(x) = \varphi(x) + (d^*)^+(x), \qquad x \in S, \tag{3.15}$$

or equivalently

$$v(x) = \begin{cases} \varphi(x), & x \in B^*, \\ \mathbb{N}_{C^*}[P_{B^*}\varphi](x), & x \in C^*, \end{cases} \tag{3.16}$$

and the stopping region is $B^* = \{x \in S; d^*(x) \leq 0\}$.

PROOF. The assertion follows from Lemma 3.5 and Theorem 3.3 because

$$\mathbb{N}(d^*)^+ - P(d^*)^+ = \mathbb{N}(I - P)(d^*)^+ = (d^*)^+.$$ ∎

According to Chow and Schechner [9], they claim that the infinity-SLA is optimal. We have confirmed this result under the assumption.