and

$$\begin{aligned} d_3(x) &= P\varphi(x) + P(d_2)^+(x) - \varphi(x) \\ &= E^x\left[\max\left\{\varphi(X_1), \mathbf{E}^{X_1}\left[\max\left\{\varphi(X_2), \mathbf{E}^{X_2}[\varphi(X_3)]\right\}\right]\right\}\right] - \varphi(x), \end{aligned}$$

and so forth.

By this lemma, if $x \in B^k$, then it is included by the following joint sets:

$$P\varphi(x) \le \varphi(x), \qquad P^2\varphi(x) \le \varphi(x), \dots, P^k\varphi(x) \le \varphi(x). \tag{3.6}$$

This shows that, when one comes to stop under the k-SLA rule, one already has been considering the previous degree of stopping rules.

LEMMA 3.2.

(1) $$\mathbb{N}(P\varphi - \varphi)^+(x) < \infty, \qquad \text{for } x \in S. \tag{3.7}$$

(2) $$\mathbb{N}[(d_k)^+ - P(d_{k-1})^+](x) < \infty, \qquad \text{for } x \in S. \tag{3.8}$$

PROOF.

(1) By Lemma 3.1, we have that $B^1 \supset B^k$, and hence, $\nu(B^1) \le \nu(B^k)$ a.e. P^x, $x \in S$. Assumptions 3.1(2) and 2.1(1) imply that $\mathbb{N}_{C^1}[P_{B^1}\varphi](x) < \infty$ and $\lim_{n\to\infty}(P_{C^1})^n\varphi(x) = 0$ for $x \in S$.

(2) From the definition of (3.1), we have

$$(d_i)^+(x) - P(d_{i-1})^+(x) \le (P\varphi - \varphi)^+(x), \qquad x \in S, \quad i = 1, 2, \dots, k.$$

The conclusion is immediately obtained by Lemma 3.2(1).

THEOREM 3.3. *Under Assumptions 2.1 and 3.1, $\nu(B^k)$ is the optimal stopping time and the optimal value of (1.1) is given by*

$$v(x) = \varphi(x) + \mathbb{N}\left[(d_k)^+ - P(d_{k-1})^+\right](x), \qquad x \in S, \tag{3.9}$$

$$= \begin{cases} \varphi(x), & x \in B^k, \\ \mathbb{N}_{C^k}[P_{B^k}\varphi](x), & x \in C^k. \end{cases} \tag{3.10}$$

PROOF. Let $w(x) = E^x[\varphi(X_{\nu(B^k)})]$, $x \in S$. Immediately,

$$w(x) = \begin{cases} \varphi(x), & x \in B^k, \\ Pw(x), & x \in C^k \end{cases}$$

by the definition of the strategy. If $x \in B^k$, then (3.2) and Lemma 3.1 yield that $Pw(x) = P\varphi(x) \le \varphi(x)$. Therefore, $w(x)$, $x \in S$ satisfies the optimality equation (1.2). Let $\tau^* = \inf\{n \ge 0; w(X_n) \le \varphi(X_n)\} = \inf\{n \ge 0; X_n \in B^k\} = \nu(B^k)$. Following the Martingale system theory [1], we see that $w(x)$ is equal to the optimal value $v(x)$ and τ^* is the optimal stopping time. We will calculate the optimal value. When $x \in C^k$,

$$v(x) = Pv(x) = P_{B^k}\varphi(x) + P_{C^k}v(x),$$

dividing S of the integral P into B^k and C^k. Hence,

$$v(x) = \mathbb{N}_{C^k}[P_{B^k}\varphi](x), \qquad \text{for } x \in C_k.$$