COROLLARY 2.4. *The game value of Dynkin's stopping problem is expressed by the sum of two functions*

$$v(x) = v_1(x) + v_2(x), \qquad x \in S. \tag{2.28}$$

We note that, from (2.26) and (2.27), $v_1(x) = Pv_1(x)$, $x \in B_2 \cup C$ and $v_2(x) = Pv_2(x)$, $x \in B_1 \cup C$, respectively.

This would be compared with Bismut's result [8]. Theorem III.1 in [8] is as follows. The simultaneous equation

$$\begin{aligned} u_1(x) &= Pu_1(x) + (\varphi - u_2 - Pu_1)^+(x), \\ u_2(x) &= Pu_2(x) - (\psi - u_1 - Pu_2)^-(x), \qquad x \in S \end{aligned} \tag{2.29}$$

has the unique solution under Assumption 2.1, and imposing a discount factor on the payoff of the formulation, and

$$u(x) = u_1(x) + u_2(x), \qquad x \in S \tag{2.30}$$

satisfies the optimality equation (2.8) and $\varphi(x) \le u(x) \le \psi(x)$, $x \in S$.

3. EXTENSION OF THE OLA RULE TO THE k-SLA RULE

The natural requirement of the extension from the OLA rule to the k-Step ($k \ge 1$), the Look Ahead rule [9] is considered. For the sake of simplicity, we do not treat the game problem, but the standard stopping problem (1.1) in this section. The game variant is discussed in the next section.

Let $k \ge 1$ be a fixed integer. Define iteratively the following sequence of $d_i(x)$, $x \in S$:

$$d_i(x) = (P\varphi - \varphi)(x) + P(d_{i-1})^+(x), \qquad i = 1, 2, \ldots, k, \tag{3.1}$$

where we put $d_0(x) = 0$. We will consider a region defined by

$$\begin{aligned} B^k &= \{x \in S; d_k(x) \le 0\}, \\ C^k &= \text{ the complement of } B^k. \end{aligned} \tag{3.2}$$

ASSUMPTION 3.1.

(1) *The set B^k and C^k are nonempty and the stopping set B^k is closed with respect to P; that is,*

$$P(x, B^k) = 1, \qquad x \in B^k. \tag{3.3}$$

(2) *The first hitting time $\nu(B^k)$ satisfies*

$$\nu(B^k) < \infty \text{ a.e. } P^x, \qquad X_0 = x \in S. \tag{3.4}$$

We shall refer to the *k-SLA rule* if the rule is based on the first hitting time $\nu(B^k)$. The procedure is as follows. First, one starts by considering the OLA (that is, the 1-SLA) rule. If it reaches into the stopping region, one switches to the 2-SLA rule and considers whether to continue or stop, and so on.

LEMMA 3.1. *The sequence B^i, $i = 1, 2, \ldots, k$ is monotone decreasing; that is,*

$$B^1 \supset B^2 \supset \cdots \supset B^k. \tag{3.5}$$

PROOF. It is clear because of the definition of $d_k(x)$, $x \in S$. In fact, for $X_0 = x \in S$,

$$\begin{aligned} d_1(x) &= P\varphi(x) - \varphi(x) = E^x[\varphi(X_1)] - \varphi(x), \\ d_2(x) &= P\varphi(x) + P(P\varphi - \varphi)^+(x) - \varphi(x) = E^x\left[\left\{\varphi + (P\varphi - \varphi)^+\right\}(X_1)\right] - \varphi(x) \\ &= E^x\left[\max\left\{\varphi(X_1), \mathbf{E}^{X_1}[\varphi(X_2)]\right\}\right] - \varphi(x), \end{aligned}$$