On $B_2 = \{x \in S; \psi(x) \geq P\psi(x)\}$, it is apparent that $w(x) - \varphi(x) \leq 0$ is contradictory to $w(x) = \psi(x) > \varphi(x)$, $x \in B_2$ by Assumption 2.1(2). On $C = \{x \in S; \varphi(x) < P\varphi(x) < P\psi(x) < \psi(x)\}$, it holds that $w(x) = Pw(x)$, as we have seen already in Lemma 2.1(2). Generally, $\varphi(x) \leq w(x) \leq \psi(x)$, $x \in S$ by the definition, so $P\varphi(x) \leq Pw(x) \leq P\psi(x)$, $x \in S$. Hence, the claim that $w(x) - \varphi(x) \leq 0$, $x \in C$ contradicts $\varphi(x) < P\varphi(x) \leq Pw(x) = w(x)$. This concludes that if $x \in \{\varphi(x) \geq Pw(x)\}$, it never occurs that $x \in C$ nor $x \in B_2$, but only the rest case $x \in B_1$ occurs. Therefore, the set $\{\varphi(x) \geq Pw(x)\}$ equals B_1. Similar arguments could be applied to the set B_2. For the set C, it is immediate from the definition and the claim for B_1 and B_2.

Because $w(x) = \varphi(x) = \overline{\varphi}(x)$, $x \in B_1$, $w(x) = Pw(x)$, $\overline{\varphi}(x) \leq P\overline{\varphi}(x)$, $x \in C$ and $w(x) = \psi(x) \leq \limsup_n E^x[\varphi(X_n)]$, $x \in B_2$ all hold, the inequality (2.10), $w(x) \leq \overline{\varphi}(x)$, $x \in S$ can be obtained. The proof for another side of the inequality is similar.

To prove the latter part of the theorem, it suffices to show that $\overline{v}(x) \leq w(x)$ for each $x \in S$. Because the alternative discussion implies that $\underline{v}(x) \geq w(x)$ and $w(x) = \overline{v}(x) = \underline{v}(x)$, we will show that, for some σ, $\sup_\tau E^x[R(\tau, \sigma)] \leq w(x)$, $x \in S$.

Define

$$\begin{aligned} \tau^* &= \inf\{n \geq 0; w(X_n) \leq \varphi(X_n)\}, \\ \sigma^* &= \inf\{n \geq 0; w(X_n) \geq \psi(X_n)\}, \end{aligned} \tag{2.24}$$

and ∞ if there exists no such n. Clearly, $\tau^* = \nu(B_1)$, $\sigma^* = \nu(B_2)$ and $\tau^* \wedge \sigma^* < \infty$ a.e. P^x, $x \in S$ by Assumption 2.2.

Since $w(x)$ satisfies (2.18), $\{w(X_{n\wedge\sigma^*}); n \geq 0\}$ is a super-Martingale with respect to $\{\mathcal{F}_n\}$ by following the discussion of [1, Chapter 3]. We have, for any stopping time $\tau < \infty$, $w(x) \geq E^x[w(X_{\tau\wedge\sigma^*})]$ by using Doob's optional sampling theorem. For $x \in S$ such that $\sigma^* < \infty$ a.e. P^x, it holds that $E^x[R(\tau, \sigma^*)] \leq w(x)$ provided $0 \leq \tau < \infty$, and that $E^x[R(\infty, \sigma^*)] = E^x[\psi(\sigma^*)] = \inf_\sigma E^x[\psi(\sigma)] \leq w(x)$. If $\sigma^* = \infty$, then $\tau^* < \infty$ by the assumption. In this case, $\sup_\tau E^x[R(\tau, \infty)] = \sup_\tau E^x[\varphi(X_\tau)] = E^x[\varphi(X_{\tau^*})] = w(x)$. The state of $\sigma^* \wedge \tau^* < \infty$ a.e. P^x covers S by Assumption 2.2(2). Therefore, being combined with these cases, $\sup_\tau E^x[R(\tau, \sigma^*)] \leq w(x)$ is shown for all $x \in S$. Analogously, since $\{w(X_{n\wedge\tau^*}); n \geq 0\}$ become a sub-Martingale, and so $w(x) \leq \inf_{0\leq\sigma<\infty} E^x[R(\tau^*, \sigma)] = \underline{v}(x)$, $x \in S$. Thus, we obtain that $w(x)$, $x \in S$ is the game value. ∎

We can show the explicit game value as follows.

Theorem 2.3. *Under Assumptions 2.1 and 2.2, the game value of Dynkin's stopping problem is given by*

$$v(x) = \begin{cases} \varphi(x), & x \in B_1, \\ \mathbb{N}_C[P_{B_1}\varphi + P_{B_2}\psi](x), & x \in C, \\ \psi(x), & x \in B_2. \end{cases} \tag{2.25}$$

Proof. By Theorem 2.2, $w(x) = E^x[R(\nu_{B_1}, \nu_{B_2})]$ satisfies the optimality equation (2.8) and the inequality (2.10). Hence, the game has value and its value $v(x)$ equals $w(x)$ for all $x \in S$. The explicit form of $w(x)$ is easily obtained from (2.18) in Lemma 2.1. ∎

If we define two functions as

$$v_1(x) = \begin{cases} \varphi(x), & x \in B_1, \\ \mathbb{N}_C[P_{B_1}\varphi](x), & x \in C, \\ 0, & x \in B_2, \end{cases} \tag{2.26}$$

$$v_2(x) = \begin{cases} \psi(x), & x \in B_2, \\ \mathbb{N}_C[P_{B_2}\psi](x), & x \in C, \\ 0, & x \in B_1. \end{cases} \tag{2.27}$$

Then, we can obtain the next corollary.