PROOF.

(1) If there exist some $x \in B_1 \cap B_2$, the inequality

$$P\varphi(x) \le \varphi(x) < \psi(x) \le P\psi(x)$$

must hold simultaneously. Since B_1 and B_2 are closed by Assumption 2.2(1), $X_0 = x$ implies $X_1 \in B_1 \cap B_2$ a.e. P^x, $x \in S$. So $E^{X_1}[\varphi(X_2)] \le \varphi(X_1) < \psi(X_1) \le E^{X_1}[\psi(X_2)]$. Repeating this, we have, for each $n, E^x[\varphi(X_n)] \le \varphi(x) < \psi(x) \le E^x[\psi(X_n)], x \in B_1 \cap B_2$. So $\limsup_n E^x[\varphi(X_n)] \le \varphi(x) < \psi(x) \le \liminf_n E^x[\psi(X_n)]$, for $x \in B_1 \cap B_2$. But this contradicts (2.16). Hence, the sets B_1 and B_2 must be disjoint.

(2) Since B_1 and B_2 are the stopping region of each player, if $x \in B_1$, then $\nu_{B_1} = 0$ and $0 < \nu_{B_2} = \infty$ a.e. P^x, $x \in B_1$. So we have $w(x) = \varphi(x)$, $x \in B_1$. Similarly $w(x) = \psi(x)$, $x \in B_2$ and $w(x) = Pw(x)$, $x \in C$. To show the relation (2.17), note that $Pw(x) = P\varphi(x)$, $x \in B_1$ and $Pw(x) = P\psi(x)$, $x \in B_2$ by the closedness of Assumption 2.2(1). Therefore, the conclusion (2.17) follows easily.

(3) The result is immediate from the closedness of sets by Assumption 2.2(1).

REMARK. In the case of the one-player problem, the finiteness of the hitting time (1.9) is assumed to obtain the expression (1.6). But, for this game version, we are not in this situation because $\nu(B_1) = \infty$ a.e. P^x in $x \in B_2$ and $\nu(B_2) = \infty$ a.e. P^x in $x \in B_1$.

Let us consider two standard stopping problems:

$$\overline{\varphi}(x) = \sup_{0 \le \tau < \infty} E^x[\varphi(X_\tau)], \tag{2.19}$$

$$\underline{\psi}(x) = \inf_{0 \le \sigma < \infty} E^x[\psi(X_\sigma)], \qquad x \in S; \tag{2.20}$$

then, each value function is obtained by the OLA rule. Define $C_1 = \{x \in C; \nu(B_1) < \infty \text{ a.e. } P^x\}$ and $C_2 = \{x \in C; \nu(B_2) < \infty \text{ a.e. } P^x\}$. Since the OLA rule is the least criterion of considering one-period-after and $0 \le \nu(B_1) < \infty$ a.e. P^x, $x \in B_1 \cup C_1$, the optimal strategy exists in this region and the value is $E^x[\varphi(X_{\nu(B_1)})] = \varphi(x) + \mathbb{N}(P\varphi - \varphi)^+(x)$. If X_n is not in B_1, it would be foolish to stop at such a state and forego. Hence, we have

$$\overline{\varphi}(x) = \begin{cases} \varphi(x) + \mathbb{N}(P\varphi - \varphi)^+(x), & x \in B_1 \cup C_1, \\ \limsup_n E^x[\varphi(X_{n \wedge \nu(B_1)})], & \text{otherwise.} \end{cases} \tag{2.21}$$

Similarly,

$$\underline{\psi}(x) = \begin{cases} \psi(x) - \mathbb{N}(P\psi - \psi)^-(x), & x \in B_2 \cup C_2, \\ \liminf_n E^x[\psi(X_{n \wedge \nu(B_2)})], & \text{otherwise.} \end{cases} \tag{2.22}$$

By Assumption 2.2(1), we note that $\overline{\varphi}(x) = \limsup_n E^x[\varphi(X_n)]$ for $x \in B_2$ and $\underline{\psi}(x) = \liminf_n E^x[\psi(X_n)]$ for $x \in B_1$ hold.

THEOREM 2.2. *Under Assumption 2.1 and 2.2,*

$$w(x) = E^x[R(\nu_{B_1}, \nu_{B_2})], \qquad x \in S \tag{2.23}$$

satisfies the optimality equation (2.7) and the inequality (2.10). The OLA rule is optimal; that is, the stopping times $\nu(B_i), i = 1, 2$ are the optimal strategy for each player.

PROOF. To prove that $w(x), x \in S$ satisfies (2.7), it is enough to show that the set B_1 equals $\{x \in S; \varphi(x) \ge Pw(x)\}$, the set C equals $\{x \in S; \varphi(x) < Pw(x) < \psi(x)\}$ and the set B_2 equals $\{x \in S; \psi(x) \le Pw(x)\}$ by the comparison of (2.9) and (2.17).

First, the inclusive relation $B_1 \subset \{x \in S; \psi(x) \ge Pw(x)\}$ is clear because $Pw(x) = P\varphi(x)$, $x \in B_1$. Inversely, if $x \in S$ such that $\varphi(x) \ge Pw(x)$, then

$$w(x) - \varphi(x) \le w(x) - Pw(x) = \begin{cases} 0, & \text{for } x \in C, \\ \psi(x) - P\psi(x) \le 0, & \text{for } x \in B_2. \end{cases}$$