Already, Neveu [7] had discussed the equality (2.8) and the inequality (2.10) under a general stochastic process.

Now Dynkin's stopping game under the OLA rule is considered.

ASSUMPTION 2.1.

(1) *For each* $x \in S$,

$$E^x\left[\sup_{n\geq 0}\varphi^+(X_n)\right] < \infty, \qquad E^x\left[\inf_{n\geq 0}\{-\psi^-(X_n)\}\right] > -\infty. \tag{2.11}$$

(2) *For the given reward functions,*

$$\varphi(x) < \chi(x) < \psi(x), \tag{2.12}$$

for all x *in* S.

Let us denote

$$\begin{aligned} B_1 &= \{x \in S; P\varphi(x) \leq \varphi(x)\}, \\ B_2 &= \{x \in S; \psi(x) \leq P\psi(x)\}, \\ C &= \text{the complement of } B_1 \cup B_2. \end{aligned} \tag{2.13}$$

ASSUMPTION 2.2.

(1) *Either* B_1 *or* B_2 *is assumed to be nonempty and each set* B_i, $i = 1, 2$ *is closed with respect to* P; *that is,*

$$P(x, B_i) = 1, \qquad x \in B_i. \tag{2.14}$$

(2) *The process eventually hits either of these sets; that is,*

$$\nu(B_1 \cup B_2) < \infty \text{ a.e. } P^x, \qquad X_0 = x \in S, \tag{2.15}$$

where $\nu(B) = \nu_B$ *denotes the first hitting time of set* B.

(3) *We assume that*

$$\begin{aligned} \liminf_n E^x[\psi(X_n)] &\leq \varphi(x), \qquad x \in B_1, \\ \limsup_n E^x[\varphi(X_n)] &\geq \psi(x), \qquad x \in B_2. \end{aligned} \tag{2.16}$$

We shall discuss the problem under Assumption 2.1 throughout the paper, but Assumption 2.2 is tentative for considering the OLA rule in this section. The set B_i, $i = 1, 2$ means the stopping region of the OLA rule for each player, and Assumption 2.1(2), 2.2(3) implies the simultaneous stopping decision does not occur for the OLA rule. So the stopping regions for each player are disjoint, and a receivable reward $\chi(x)$ in the formulation does not appear. Intuitively, we note that (2.12) requires that the reward for one player is disengaged from the stopping region of the opposite player.

LEMMA 2.1.

(1) *The sets* B_1 *and* B_2 *are disjoint.*

(2) *Let* $w(x) = E^x[R(\nu_{B_1}, \nu_{B_2})]$, $x \in S$. *Then,*

$$w(x) - Pw(x) = (\varphi - P\varphi)^+(x) - (\psi - P\psi)^-(x), \qquad x \in S; \tag{2.17}$$

that is,

$$w(x) = \begin{cases} \varphi(x), & x \in B_1, \\ Pw(x), & x \in C, \\ \psi(x), & x \in B_2. \end{cases} \tag{2.18}$$