is the admissible class of the problem. If either of the players does not stop in a finite horizon, let

$$R(\tau,\infty) = \varphi(X_\tau) \quad \text{and} \quad R(\infty,\sigma) = \psi(X_\sigma). \tag{2.2}$$

Player I's objective is to maximize $E^x[R(\tau,\sigma)]$, $X_0 = x$ with respect to τ such that $\tau \wedge \sigma < \infty$ for fixed σ and on the other hand, Player II is to minimize it with respect to σ for fixed τ. The minimax and the maxmin value of this zero-sum game are defined by

$$\overline{v}(x) = \inf_\sigma \sup_\tau E^x[R(\tau,\sigma)], \tag{2.3}$$

$$\underline{v}(x) = \sup_\tau \inf_\sigma E^x[R(\tau,\sigma)], \tag{2.4}$$

respectively. We say that the problem has a game value if $\overline{v}(x) = \underline{v}(x)$, $x \in S$, and we denote the value function by $v(x)$, $x \in S$. A pair of strategies (τ^*,σ^*) is optimal (equilibrium) if

$$\tau^* \wedge \sigma^* < \infty \text{ a.e. } P^x, \qquad x \in S, \tag{2.5}$$

and

$$E^x[R(\tau^*,\sigma^*)] = \underline{v}(x) = \overline{v}(x), \qquad x \in S. \tag{2.6}$$

Generally, the zero-sum stopping game does not have value and so it is natural to consider a class of randomized stopping times [10] as the admissible class. However, we impose a condition for the payoff functions in order to have the optimal strategy in the class of stopping times adapted to $\{\mathcal{F}_n\}$. We assume Assumption 2.1, whose condition implies that there exists an optimal pure strategy, that is, an optimal stopping time in the zero-sum matrix game for all of randomized strategies. Stettner [11], Elbakidze [12], and others discussed the zero-sum stopping problem under this separability condition (2.12).

The discussion starts from the description of the optimality equation for the game variant. By the argument of recursive games of Everett [13], the value function satisfies *the optimality equation* of the game variant

$$v(x) = \mathrm{VAL}\begin{pmatrix} & II:\text{stop} & II:\text{conti.} \\ I:\text{stop} & \chi(x) & \varphi(x) \\ I:\text{conti.} & \psi(x) & Pv(x) \end{pmatrix}, \tag{2.7}$$

where VAL means the value of the 2 by 2 matrix game. Condition (2.12) (there is no need to be strict with the inequality in this argument) implies that VAL ( ) $= \max_I \min_{II}(\,) = \min_{II}\max_I(\,)$ for any value of (2,2)-element of the matrix. Immediately, we see that (2.7) is equivalent to

$$v(x) = \begin{cases} \varphi(x) & \text{on } \{x \in S; Pv(x) \le \varphi(x)\}, \\ Pv(x) & \text{on } \{x \in S; \varphi(x) < Pv(x) < \psi(x)\}, \\ \psi(x) & \text{on } \{x \in S; \psi(x) \le Pv(x)\}. \end{cases} \tag{2.8}$$

Also, it is equivalent to

$$v(x) - Pv(x) = (\varphi - Pv)^+(x) - (\psi - Pv)^-(x). \tag{2.9}$$

We note that the case of the simultaneous stopping for Player I and II with the reward $\chi(x)$, $x \in S$ does not occur, and hence, it is insignificant in the equations of (2.8) and (2.9). Also, considering the trivial case, one could evaluate the value as $v(x) = \inf_\sigma \sup_\tau E^x[R(\tau,\sigma)] \le \sup_{0\le\tau<\infty} E^x[R(\tau,\infty)] = \sup_{0\le\tau<\infty} E^x[\varphi(X_\tau)]$. The similar inequality is also obtained by exchanging sup and inf. Then, combined with these two inequalities,

$$\inf_{0\le\sigma<\infty} E^x[\psi(X_\sigma)] \le v(x) \le \sup_{0\le\tau<\infty} E^x[\varphi(X_\tau)], \qquad x \in S. \tag{2.10}$$