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Explicit Optimal Value for Dynkin's Stopping Game

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Abstract—Under the One-step Look Ahead rule of Dynamic Programming, an explicit game value of Dynkin's stopping problem for a Markov chain is obtained by using a potential operator. The condition on the *One-step rule* could be extended to the k-step and infinity-step rule. We shall also decompose the game value as the sum of two explicit functions under these rules.

Keywords—Optimal stopping problem, Dynkin's game, One-step Look Ahead rule, Markov potential theory.

1. INTRODUCTION AND SUMMARY

Let $\{(X_n, \mathcal{F}_n); n \geq 0\}$ be a Markov chain with a countable state space S having stationary transition probabilities $P(x, A)$, $x \in S$. Suppose a function $\varphi(x)$, $x \in S$ is given. The standard optimal stopping problem is to find a stopping time which maximizes $E^x[\varphi(X_\tau)] = \mathbf{E}[\varphi(X_\tau) \mid X_0 = x]$ in the class of all finite stopping times τ adapted to $\{\mathcal{F}_n; n \geq 0\}$. The optimal value is denoted by

$$v(x) = \sup_{0 \leq \tau < \infty} E^x[\varphi(X_\tau)], \qquad x \in S. \tag{1.1}$$

The detailed analyses are discussed by many authors such as Chow, Robbins and Siegmund [1], Shiryaev [2], etc. By the Dynamic Programming method, the optimality equation becomes

$$v(x) = \max \left\{ \begin{matrix} \text{stop} & \text{conti.} \\ \varphi(x), & Pv(x) \end{matrix} \right\}, \qquad x \in S, \tag{1.2}$$

where $Pv(x) = \sum_{y \in S} v(y) P(x, y)$. We shall rewrite this equation as

$$v(x) = \begin{cases} \varphi(x) & \text{on } \{x \in S;\ \varphi(x) \geq Pv(x)\}, \\ Pv(x) & \text{on } \{x \in S;\ \varphi(x) < Pv(x)\}, \end{cases} \tag{1.3}$$

or

$$v(x) - Pv(x) = (\varphi - Pv)^+(x), \qquad x \in S \tag{1.4}$$

in comparison with the game variant of the problem in the later section. Hereafter, we shall use the superscript $\pm$ as $a^+ = \max(a, 0)$, $a^- = -\min(a, 0)$.

In the previous paper [3], we obtained the explicit expression of the optimal value as

$$v(x) = \varphi(x) + \mathbb{N}(P\varphi - \varphi)^+(x), \qquad x \in S, \tag{1.5}$$

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or equivalently

$$v(x) = \begin{cases} \varphi(x), & x \in B, \\ \mathbb{N}_C[P_B\varphi](x), & x \in C, \end{cases} \tag{1.6}$$

if the One-step Look Ahead (abbreviated to OLA) rule is optimal, where $\mathbb{N} = \lim_{n\to\infty} \sum_{k=0}^{n} P^k$ is a potential operator and P_B, $\mathbb{N}_C$ mean the restriction of P and $\mathbb{N}$ on the set B or C, respectively.

The sets B and C are defined by the following (1.7), which are the regions where the decision is made either to stop or to continue, respectively. The OLA rule is that stops are made at the first time when the process enters a state in which stopping is at least as good as continuing for exactly one more period and then stopping. The rule is also well known as the monotone case of Markov sequence in [1]. To be precise, let

$$\begin{aligned} B &= \{x \in S; P\varphi(x) - \varphi(x) \le 0\}, \\ C &= \text{the complement of } B. \end{aligned} \tag{1.7}$$

The decision rule based on the stopping time ν_B, the first hitting time of the set B, is referred to frequently as the OLA rule, hereafter. If the rule is optimal for the problem, we shall say that the OLA rule is optimal. If the set B is closed, that is,

$$P(x, C) = 0, \qquad \text{for } x \in B, \tag{1.8}$$

and if it satisfies

$$\nu_B < \infty \text{ a.e. } P^x, \qquad X_0 = x \in S, \tag{1.9}$$

then the stopping time is optimal; that is, the OLA rule is optimal [4,5]. In [3], the explicit optimal value is obtained when the OLA rule is optimal, and it is applied to the best choice problem. Unlike the problem in which assumption (1.9) holds for all S, one must consider, in the game variant of the OLA rule, the case of $\nu_B = \infty$ on some set.

In this paper, our aim is to show an explicit expression for the value of zero-sum game variant, the so-called Dynkin's stopping game [6,7]. Furthermore, it is proved that the game value in this case is the sum of two independent maximal/minimal values with a zero reward at nonstopping for the standard stopping problems.

In Section 2, Dynkin's stopping game is considered when the OLA rule is optimal. The game value of the problem is expressed by using a potential operator and is decomposed as the sum of two independent maximal/minimal stopping problems. This is simpler than that of Bismut [8]. To discuss the standard stopping problem under an extended condition of the OLA rule, Section 3 considers it under the k-SLA rule, an abbreviation for the *k-Step* ($k \ge 1$) *Look Ahead* rule [9]. We shall express the optimal value of the standard problem under this rule. Again, Dynkin's stopping game is considered in Section 4. By taking k tend to infinity, the relation between the value of Dynkin's game and that of the standard stopping problem is obtained under the infinity-SLA rule.

2. DYNKIN'S STOPPING GAME UNDER THE OLA RULE

The formulation of Dynkin's stopping game is as follows. Two players I and II observe a Markov chain $\{(X_n, \mathcal{F}_n); n \ge 0\}$ with the stationary transition probability P on the countable state space S. Each of them chooses a stopping time adapted to $\{\mathcal{F}_n\}$ as one's strategy. If τ and σ are strategies of Player I and II, respectively, the payoff function is of the form

$$R(\tau, \sigma) = \varphi(X_\tau)\mathbf{1}_{\{\tau<\sigma\}} + \psi(X_\sigma)\mathbf{1}_{\{\tau>\sigma\}} + \chi(X_\tau)\mathbf{1}_{\{\tau=\sigma\}}, \tag{2.1}$$

where the function $\varphi(x)$, $\psi(x)$, and $\chi(x)$ on $x \in S$ are supposed to be given. Earlier stopping of players can stop the observation, and so $\tau \wedge \sigma = \min\{\tau, \sigma\}$ is the termination for the process. To avoid the nonterminated case, a pair of strategies

$$\tau \wedge \sigma < \infty$$

is the admissible class of the problem. If either of the players does not stop in a finite horizon, let

$$R(\tau,\infty)=\varphi(X_\tau) \quad\text{and}\quad R(\infty,\sigma)=\psi(X_\sigma). \tag{2.2}$$

Player I's objective is to maximize $E^x[R(\tau,\sigma)]$, $X_0=x$ with respect to τ such that $\tau\wedge\sigma<\infty$ for fixed σ and on the other hand, Player II is to minimize it with respect to σ for fixed τ. The minimax and the maxmin value of this zero-sum game are defined by

$$\overline{v}(x)=\inf_\sigma\sup_\tau E^x[R(\tau,\sigma)], \tag{2.3}$$

$$\underline{v}(x)=\sup_\tau\inf_\sigma E^x[R(\tau,\sigma)], \tag{2.4}$$

respectively. We say that the problem has a game value if $\overline{v}(x)=\underline{v}(x)$, $x\in S$, and we denote the value function by $v(x)$, $x\in S$. A pair of strategies (τ^*,σ^*) is optimal (equilibrium) if

$$\tau^*\wedge\sigma^*<\infty \text{ a.e. } P^x, \qquad x\in S, \tag{2.5}$$

and

$$E^x[R(\tau^*,\sigma^*)]=\underline{v}(x)=\overline{v}(x), \qquad x\in S. \tag{2.6}$$

Generally, the zero-sum stopping game does not have value and so it is natural to consider a class of randomized stopping times [10] as the admissible class. However, we impose a condition for the payoff functions in order to have the optimal strategy in the class of stopping times adapted to $\{\mathcal{F}_n\}$. We assume Assumption 2.1, whose condition implies that there exists an optimal pure strategy, that is, an optimal stopping time in the zero-sum matrix game for all of randomized strategies. Stettner [11], Elbakidze [12], and others discussed the zero-sum stopping problem under this separability condition (2.12).

The discussion starts from the description of the optimality equation for the game variant. By the argument of recursive games of Everett [13], the value function satisfies *the optimality equation* of the game variant

$$v(x)=\mathrm{VAL}\begin{pmatrix} & II:\text{stop} & II:\text{conti.}\\ I:\text{stop} & \chi(x) & \varphi(x)\\ I:\text{conti.} & \psi(x) & Pv(x)\end{pmatrix}, \tag{2.7}$$

where VAL means the value of the 2 by 2 matrix game. Condition (2.12) (there is no need to be strict with the inequality in this argument) implies that VAL ( ) $=\max_I\min_{II}(\,)=\min_{II}\max_I(\,)$ for any value of (2,2)-element of the matrix. Immediately, we see that (2.7) is equivalent to

$$v(x)=\begin{cases}\varphi(x) & \text{on } \{x\in S;Pv(x)\le\varphi(x)\},\\ Pv(x) & \text{on } \{x\in S;\varphi(x)<Pv(x)<\psi(x)\},\\ \psi(x) & \text{on } \{x\in S;\psi(x)\le Pv(x)\}.\end{cases} \tag{2.8}$$

Also, it is equivalent to

$$v(x)-Pv(x)=(\varphi-Pv)^+(x)-(\psi-Pv)^-(x). \tag{2.9}$$

We note that the case of the simultaneous stopping for Player I and II with the reward $\chi(x)$, $x\in S$ does not occur, and hence, it is insignificant in the equations of (2.8) and (2.9). Also, considering the trivial case, one could evaluate the value as $v(x)=\inf_\sigma\sup_\tau E^x[R(\tau,\sigma)]\le\sup_{0\le\tau<\infty}E^x[R(\tau,\infty)]=\sup_{0\le\tau<\infty}E^x[\varphi(X_\tau)]$. The similar inequality is also obtained by exchanging sup and inf. Then, combined with these two inequalities,

$$\inf_{0\le\sigma<\infty}E^x[\psi(X_\sigma)]\le v(x)\le\sup_{0\le\tau<\infty}E^x[\varphi(X_\tau)], \qquad x\in S. \tag{2.10}$$

Already, Neveu [7] had discussed the equality (2.8) and the inequality (2.10) under a general stochastic process.

Now Dynkin's stopping game under the OLA rule is considered.

ASSUMPTION 2.1.

(1) *For each* $x \in S$,

$$E^x\left[\sup_{n\geq 0}\varphi^+(X_n)\right] < \infty, \qquad E^x\left[\inf_{n\geq 0}\{-\psi^-(X_n)\}\right] > -\infty. \tag{2.11}$$

(2) *For the given reward functions,*

$$\varphi(x) < \chi(x) < \psi(x), \tag{2.12}$$

for all x *in* S.

Let us denote

$$\begin{aligned} B_1 &= \{x \in S; P\varphi(x) \leq \varphi(x)\}, \\ B_2 &= \{x \in S; \psi(x) \leq P\psi(x)\}, \\ C &= \text{the complement of } B_1 \cup B_2. \end{aligned} \tag{2.13}$$

ASSUMPTION 2.2.

(1) *Either* B_1 *or* B_2 *is assumed to be nonempty and each set* B_i, $i = 1, 2$ *is closed with respect to* P*; that is,*

$$P(x, B_i) = 1, \qquad x \in B_i. \tag{2.14}$$

(2) *The process eventually hits either of these sets; that is,*

$$\nu(B_1 \cup B_2) < \infty \text{ a.e. } P^x, \qquad X_0 = x \in S, \tag{2.15}$$

where $\nu(B) = \nu_B$ *denotes the first hitting time of set* B.

(3) *We assume that*

$$\begin{aligned} \liminf_n E^x[\psi(X_n)] &\leq \varphi(x), \qquad x \in B_1, \\ \limsup_n E^x[\varphi(X_n)] &\geq \psi(x), \qquad x \in B_2. \end{aligned} \tag{2.16}$$

We shall discuss the problem under Assumption 2.1 throughout the paper, but Assumption 2.2 is tentative for considering the OLA rule in this section. The set B_i, $i = 1, 2$ means the stopping region of the OLA rule for each player, and Assumption 2.1(2), 2.2(3) implies the simultaneous stopping decision does not occur for the OLA rule. So the stopping regions for each player are disjoint, and a receivable reward $\chi(x)$ in the formulation does not appear. Intuitively, we note that (2.12) requires that the reward for one player is disengaged from the stopping region of the opposite player.

LEMMA 2.1.

(1) *The sets* B_1 *and* B_2 *are disjoint.*

(2) *Let* $w(x) = E^x[R(\nu_{B_1}, \nu_{B_2})]$, $x \in S$. *Then,*

$$w(x) - Pw(x) = (\varphi - P\varphi)^+(x) - (\psi - P\psi)^-(x), \qquad x \in S; \tag{2.17}$$

that is,

$$w(x) = \begin{cases} \varphi(x), & x \in B_1, \\ Pw(x), & x \in C, \\ \psi(x), & x \in B_2. \end{cases} \tag{2.18}$$

Proof.

(1) If there exist some $x \in B_1 \cap B_2$, the inequality

$$P\varphi(x) \le \varphi(x) < \psi(x) \le P\psi(x)$$

must hold simultaneously. Since B_1 and B_2 are closed by Assumption 2.2(1), $X_0 = x$ implies $X_1 \in B_1 \cap B_2$ a.e. P^x, $x \in S$. So $E^{X_1}[\varphi(X_2)] \le \varphi(X_1) < \psi(X_1) \le E^{X_1}[\psi(X_2)]$. Repeating this, we have, for each $n, E^x[\varphi(X_n)] \le \varphi(x) < \psi(x) \le E^x[\psi(X_n)], x \in B_1 \cap B_2$. So $\limsup_n E^x[\varphi(X_n)] \le \varphi(x) < \psi(x) \le \liminf_n E^x[\psi(X_n)]$, for $x \in B_1 \cap B_2$. But this contradicts (2.16). Hence, the sets B_1 and B_2 must be disjoint.

(2) Since B_1 and B_2 are the stopping region of each player, if $x \in B_1$, then $\nu_{B_1} = 0$ and $0 < \nu_{B_2} = \infty$ a.e. P^x, $x \in B_1$. So we have $w(x) = \varphi(x)$, $x \in B_1$. Similarly $w(x) = \psi(x)$, $x \in B_2$ and $w(x) = Pw(x)$, $x \in C$. To show the relation (2.17), note that $Pw(x) = P\varphi(x)$, $x \in B_1$ and $Pw(x) = P\psi(x)$, $x \in B_2$ by the closedness of Assumption 2.2(1). Therefore, the conclusion (2.17) follows easily.

(3) The result is immediate from the closedness of sets by Assumption 2.2(1).

Remark. In the case of the one-player problem, the finiteness of the hitting time (1.9) is assumed to obtain the expression (1.6). But, for this game version, we are not in this situation because $\nu(B_1) = \infty$ a.e. P^x in $x \in B_2$ and $\nu(B_2) = \infty$ a.e. P^x in $x \in B_1$.

Let us consider two standard stopping problems:

$$\overline{\varphi}(x) = \sup_{0 \le \tau < \infty} E^x[\varphi(X_\tau)], \tag{2.19}$$

$$\underline{\psi}(x) = \inf_{0 \le \sigma < \infty} E^x[\psi(X_\sigma)], \qquad x \in S; \tag{2.20}$$

then, each value function is obtained by the OLA rule. Define $C_1 = \{x \in C; \nu(B_1) < \infty \text{ a.e. } P^x\}$ and $C_2 = \{x \in C; \nu(B_2) < \infty \text{ a.e. } P^x\}$. Since the OLA rule is the least criterion of considering one-period-after and $0 \le \nu(B_1) < \infty$ a.e. P^x, $x \in B_1 \cup C_1$, the optimal strategy exists in this region and the value is $E^x[\varphi(X_{\nu(B_1)})] = \varphi(x) + \mathbb{N}(P\varphi - \varphi)^+(x)$. If X_n is not in B_1, it would be foolish to stop at such a state and forego. Hence, we have

$$\overline{\varphi}(x) = \begin{cases} \varphi(x) + \mathbb{N}(P\varphi - \varphi)^+(x), & x \in B_1 \cup C_1, \\ \limsup_n E^x[\varphi(X_{n \wedge \nu(B_1)})], & \text{otherwise.} \end{cases} \tag{2.21}$$

Similarly,

$$\underline{\psi}(x) = \begin{cases} \psi(x) - \mathbb{N}(P\psi - \psi)^-(x), & x \in B_2 \cup C_2, \\ \liminf_n E^x[\psi(X_{n \wedge \nu(B_2)})], & \text{otherwise.} \end{cases} \tag{2.22}$$

By Assumption 2.2(1), we note that $\overline{\varphi}(x) = \limsup_n E^x[\varphi(X_n)]$ for $x \in B_2$ and $\underline{\psi}(x) = \liminf_n E^x[\psi(X_n)]$ for $x \in B_1$ hold.

Theorem 2.2. *Under Assumption 2.1 and 2.2,*

$$w(x) = E^x[R(\nu_{B_1}, \nu_{B_2})], \qquad x \in S \tag{2.23}$$

satisfies the optimality equation (2.7) and the inequality (2.10). The OLA rule is optimal; that is, the stopping times $\nu(B_i), i = 1, 2$ are the optimal strategy for each player.

Proof. To prove that $w(x), x \in S$ satisfies (2.7), it is enough to show that the set B_1 equals $\{x \in S; \varphi(x) \ge Pw(x)\}$, the set C equals $\{x \in S; \varphi(x) < Pw(x) < \psi(x)\}$ and the set B_2 equals $\{x \in S; \psi(x) \le Pw(x)\}$ by the comparison of (2.9) and (2.17).

First, the inclusive relation $B_1 \subset \{x \in S; \psi(x) \ge Pw(x)\}$ is clear because $Pw(x) = P\varphi(x)$, $x \in B_1$. Inversely, if $x \in S$ such that $\varphi(x) \ge Pw(x)$, then

$$w(x) - \varphi(x) \le w(x) - Pw(x) = \begin{cases} 0, & \text{for } x \in C, \\ \psi(x) - P\psi(x) \le 0, & \text{for } x \in B_2. \end{cases}$$

On $B_2 = \{x \in S; \psi(x) \geq P\psi(x)\}$, it is apparent that $w(x) - \varphi(x) \leq 0$ is contradictory to $w(x) = \psi(x) > \varphi(x)$, $x \in B_2$ by Assumption 2.1(2). On $C = \{x \in S; \varphi(x) < P\varphi(x) < P\psi(x) < \psi(x)\}$, it holds that $w(x) = Pw(x)$, as we have seen already in Lemma 2.1(2). Generally, $\varphi(x) \leq w(x) \leq \psi(x)$, $x \in S$ by the definition, so $P\varphi(x) \leq Pw(x) \leq P\psi(x)$, $x \in S$. Hence, the claim that $w(x) - \varphi(x) \leq 0$, $x \in C$ contradicts $\varphi(x) < P\varphi(x) \leq Pw(x) = w(x)$. This concludes that if $x \in \{\varphi(x) \geq Pw(x)\}$, it never occurs that $x \in C$ nor $x \in B_2$, but only the rest case $x \in B_1$ occurs. Therefore, the set $\{\varphi(x) \geq Pw(x)\}$ equals B_1. Similar arguments could be applied to the set B_2. For the set C, it is immediate from the definition and the claim for B_1 and B_2.

Because $w(x) = \varphi(x) = \overline{\varphi}(x)$, $x \in B_1$, $w(x) = Pw(x)$, $\overline{\varphi}(x) \leq P\overline{\varphi}(x)$, $x \in C$ and $w(x) = \psi(x) \leq \limsup_n E^x[\varphi(X_n)]$, $x \in B_2$ all hold, the inequality (2.10), $w(x) \leq \overline{\varphi}(x)$, $x \in S$ can be obtained. The proof for another side of the inequality is similar.

To prove the latter part of the theorem, it suffices to show that $\overline{v}(x) \leq w(x)$ for each $x \in S$. Because the alternative discussion implies that $\underline{v}(x) \geq w(x)$ and $w(x) = \overline{v}(x) = \underline{v}(x)$, we will show that, for some σ, $\sup_\tau E^x[R(\tau, \sigma)] \leq w(x)$, $x \in S$.

Define

$$\begin{aligned} \tau^* &= \inf\{n \geq 0; w(X_n) \leq \varphi(X_n)\}, \\ \sigma^* &= \inf\{n \geq 0; w(X_n) \geq \psi(X_n)\}, \end{aligned} \tag{2.24}$$

and ∞ if there exists no such n. Clearly, $\tau^* = \nu(B_1)$, $\sigma^* = \nu(B_2)$ and $\tau^* \wedge \sigma^* < \infty$ a.e. P^x, $x \in S$ by Assumption 2.2.

Since $w(x)$ satisfies (2.18), $\{w(X_{n\wedge\sigma^*}); n \geq 0\}$ is a super-Martingale with respect to $\{\mathcal{F}_n\}$ by following the discussion of [1, Chapter 3]. We have, for any stopping time $\tau < \infty$, $w(x) \geq E^x[w(X_{\tau\wedge\sigma^*})]$ by using Doob's optional sampling theorem. For $x \in S$ such that $\sigma^* < \infty$ a.e. P^x, it holds that $E^x[R(\tau, \sigma^*)] \leq w(x)$ provided $0 \leq \tau < \infty$, and that $E^x[R(\infty, \sigma^*)] = E^x[\psi(\sigma^*)] = \inf_\sigma E^x[\psi(\sigma)] \leq w(x)$. If $\sigma^* = \infty$, then $\tau^* < \infty$ by the assumption. In this case, $\sup_\tau E^x[R(\tau, \infty)] = \sup_\tau E^x[\varphi(X_\tau)] = E^x[\varphi(X_{\tau^*})] = w(x)$. The state of $\sigma^* \wedge \tau^* < \infty$ a.e. P^x covers S by Assumption 2.2(2). Therefore, being combined with these cases, $\sup_\tau E^x[R(\tau, \sigma^*)] \leq w(x)$ is shown for all $x \in S$. Analogously, since $\{w(X_{n\wedge\tau^*}); n \geq 0\}$ become a sub-Martingale, and so $w(x) \leq \inf_{0\leq\sigma<\infty} E^x[R(\tau^*, \sigma)] = \underline{v}(x)$, $x \in S$. Thus, we obtain that $w(x)$, $x \in S$ is the game value. ∎

We can show the explicit game value as follows.

THEOREM 2.3. *Under Assumptions 2.1 and 2.2, the game value of Dynkin's stopping problem is given by*

$$v(x) = \begin{cases} \varphi(x), & x \in B_1, \\ \mathbb{N}_C[P_{B_1}\varphi + P_{B_2}\psi](x), & x \in C, \\ \psi(x), & x \in B_2. \end{cases} \tag{2.25}$$

PROOF. By Theorem 2.2, $w(x) = E^x[R(\nu_{B_1}, \nu_{B_2})]$ satisfies the optimality equation (2.8) and the inequality (2.10). Hence, the game has value and its value $v(x)$ equals $w(x)$ for all $x \in S$. The explicit form of $w(x)$ is easily obtained from (2.18) in Lemma 2.1. ∎

If we define two functions as

$$v_1(x) = \begin{cases} \varphi(x), & x \in B_1, \\ \mathbb{N}_C[P_{B_1}\varphi](x), & x \in C, \\ 0, & x \in B_2, \end{cases} \tag{2.26}$$

$$v_2(x) = \begin{cases} \psi(x), & x \in B_2, \\ \mathbb{N}_C[P_{B_2}\psi](x), & x \in C, \\ 0, & x \in B_1. \end{cases} \tag{2.27}$$

Then, we can obtain the next corollary.

COROLLARY 2.4. *The game value of Dynkin's stopping problem is expressed by the sum of two functions*

$$v(x) = v_1(x) + v_2(x), \qquad x \in S. \tag{2.28}$$

We note that, from (2.26) and (2.27), $v_1(x) = Pv_1(x)$, $x \in B_2 \cup C$ and $v_2(x) = Pv_2(x)$, $x \in B_1 \cup C$, respectively.

This would be compared with Bismut's result [8]. Theorem III.1 in [8] is as follows. The simultaneous equation

$$\begin{aligned} u_1(x) &= Pu_1(x) + (\varphi - u_2 - Pu_1)^+(x), \\ u_2(x) &= Pu_2(x) - (\psi - u_1 - Pu_2)^-(x), \qquad x \in S \end{aligned} \tag{2.29}$$

has the unique solution under Assumption 2.1, and imposing a discount factor on the payoff of the formulation, and

$$u(x) = u_1(x) + u_2(x), \qquad x \in S \tag{2.30}$$

satisfies the optimality equation (2.8) and $\varphi(x) \le u(x) \le \psi(x)$, $x \in S$.

3. EXTENSION OF THE OLA RULE TO THE k-SLA RULE

The natural requirement of the extension from the OLA rule to the k-Step ($k \ge 1$), the Look Ahead rule [9] is considered. For the sake of simplicity, we do not treat the game problem, but the standard stopping problem (1.1) in this section. The game variant is discussed in the next section.

Let $k \ge 1$ be a fixed integer. Define iteratively the following sequence of $d_i(x)$, $x \in S$:

$$d_i(x) = (P\varphi - \varphi)(x) + P(d_{i-1})^+(x), \qquad i = 1, 2, \dots, k, \tag{3.1}$$

where we put $d_0(x) = 0$. We will consider a region defined by

$$\begin{aligned} B^k &= \{x \in S; d_k(x) \le 0\}, \\ C^k &= \text{ the complement of } B^k. \end{aligned} \tag{3.2}$$

ASSUMPTION 3.1.

(1) *The set B^k and C^k are nonempty and the stopping set B^k is closed with respect to P; that is,*

$$P(x, B^k) = 1, \qquad x \in B^k. \tag{3.3}$$

(2) *The first hitting time $\nu(B^k)$ satisfies*

$$\nu(B^k) < \infty \text{ a.e. } P^x, \qquad X_0 = x \in S. \tag{3.4}$$

We shall refer to the *k-SLA rule* if the rule is based on the first hitting time $\nu(B^k)$. The procedure is as follows. First, one starts by considering the OLA (that is, the 1-SLA) rule. If it reaches into the stopping region, one switches to the 2-SLA rule and considers whether to continue or stop, and so on.

LEMMA 3.1. *The sequence B^i, $i = 1, 2, \dots, k$ is monotone decreasing; that is,*

$$B^1 \supset B^2 \supset \cdots \supset B^k. \tag{3.5}$$

PROOF. It is clear because of the definition of $d_k(x)$, $x \in S$. In fact, for $X_0 = x \in S$,

$$\begin{aligned} d_1(x) &= P\varphi(x) - \varphi(x) = E^x[\varphi(X_1)] - \varphi(x), \\ d_2(x) &= P\varphi(x) + P(P\varphi - \varphi)^+(x) - \varphi(x) = E^x\left[\left\{\varphi + (P\varphi - \varphi)^+\right\}(X_1)\right] - \varphi(x) \\ &= E^x\left[\max\left\{\varphi(X_1), \mathbf{E}^{X_1}[\varphi(X_2)]\right\}\right] - \varphi(x), \end{aligned}$$

COROLLARY 2.4. *The game value of Dynkin's stopping problem is expressed by the sum of two functions*

$$v(x) = v_1(x) + v_2(x), \qquad x \in S. \tag{2.28}$$

We note that, from (2.26) and (2.27), $v_1(x) = Pv_1(x)$, $x \in B_2 \cup C$ and $v_2(x) = Pv_2(x)$, $x \in B_1 \cup C$, respectively.

This would be compared with Bismut's result [8]. Theorem III.1 in [8] is as follows. The simultaneous equation

$$\begin{aligned} u_1(x) &= Pu_1(x) + (\varphi - u_2 - Pu_1)^+(x), \\ u_2(x) &= Pu_2(x) - (\psi - u_1 - Pu_2)^-(x), \qquad x \in S \end{aligned} \tag{2.29}$$

has the unique solution under Assumption 2.1, and imposing a discount factor on the payoff of the formulation, and

$$u(x) = u_1(x) + u_2(x), \qquad x \in S \tag{2.30}$$

satisfies the optimality equation (2.8) and $\varphi(x) \le u(x) \le \psi(x)$, $x \in S$.

3. EXTENSION OF THE OLA RULE TO THE k-SLA RULE

The natural requirement of the extension from the OLA rule to the k-Step ($k \ge 1$), the Look Ahead rule [9] is considered. For the sake of simplicity, we do not treat the game problem, but the standard stopping problem (1.1) in this section. The game variant is discussed in the next section.

Let $k \ge 1$ be a fixed integer. Define iteratively the following sequence of $d_i(x)$, $x \in S$:

$$d_i(x) = (P\varphi - \varphi)(x) + P(d_{i-1})^+(x), \qquad i = 1, 2, \ldots, k, \tag{3.1}$$

where we put $d_0(x) = 0$. We will consider a region defined by

$$\begin{aligned} B^k &= \{x \in S; d_k(x) \le 0\}, \\ C^k &= \text{ the complement of } B^k. \end{aligned} \tag{3.2}$$

ASSUMPTION 3.1.

(1) *The set B^k and C^k are nonempty and the stopping set B^k is closed with respect to P; that is,*

$$P(x, B^k) = 1, \qquad x \in B^k. \tag{3.3}$$

(2) *The first hitting time $\nu(B^k)$ satisfies*

$$\nu(B^k) < \infty \text{ a.e. } P^x, \qquad X_0 = x \in S. \tag{3.4}$$

We shall refer to the *k-SLA rule* if the rule is based on the first hitting time $\nu(B^k)$. The procedure is as follows. First, one starts by considering the OLA (that is, the 1-SLA) rule. If it reaches into the stopping region, one switches to the 2-SLA rule and considers whether to continue or stop, and so on.

LEMMA 3.1. *The sequence B^i, $i = 1, 2, \ldots, k$ is monotone decreasing; that is,*

$$B^1 \supset B^2 \supset \cdots \supset B^k. \tag{3.5}$$

PROOF. It is clear because of the definition of $d_k(x)$, $x \in S$. In fact, for $X_0 = x \in S$,

$$\begin{aligned} d_1(x) &= P\varphi(x) - \varphi(x) = E^x[\varphi(X_1)] - \varphi(x), \\ d_2(x) &= P\varphi(x) + P(P\varphi - \varphi)^+(x) - \varphi(x) = E^x\left[\left\{\varphi + (P\varphi - \varphi)^+\right\}(X_1)\right] - \varphi(x) \\ &= E^x\left[\max\left\{\varphi(X_1), \mathbf{E}^{X_1}[\varphi(X_2)]\right\}\right] - \varphi(x), \end{aligned}$$

and

$$\begin{aligned} d_3(x) &= P\varphi(x) + P(d_2)^+(x) - \varphi(x) \\ &= E^x\left[\max\left\{\varphi(X_1), \mathbf{E}^{X_1}\left[\max\left\{\varphi(X_2), \mathbf{E}^{X_2}[\varphi(X_3)]\right\}\right]\right\}\right] - \varphi(x), \end{aligned}$$

and so forth.

By this lemma, if $x \in B^k$, then it is included by the following joint sets:

$$P\varphi(x) \le \varphi(x), \qquad P^2\varphi(x) \le \varphi(x), \dots, P^k\varphi(x) \le \varphi(x). \tag{3.6}$$

This shows that, when one comes to stop under the k-SLA rule, one already has been considering the previous degree of stopping rules.

LEMMA 3.2.

(1)
$$\mathbb{N}(P\varphi - \varphi)^+(x) < \infty, \qquad \text{for } x \in S. \tag{3.7}$$

(2)
$$\mathbb{N}[(d_k)^+ - P(d_{k-1})^+](x) < \infty, \qquad \text{for } x \in S. \tag{3.8}$$

PROOF.

(1) By Lemma 3.1, we have that $B^1 \supset B^k$, and hence, $\nu(B^1) \le \nu(B^k)$ a.e. P^x, $x \in S$. Assumptions 3.1(2) and 2.1(1) imply that $\mathbb{N}_{C^1}[P_{B^1}\varphi](x) < \infty$ and $\lim_{n\to\infty}(P_{C^1})^n\varphi(x) = 0$ for $x \in S$.

(2) From the definition of (3.1), we have

$$(d_i)^+(x) - P(d_{i-1})^+(x) \le (P\varphi - \varphi)^+(x), \qquad x \in S, \quad i = 1, 2, \dots, k.$$

The conclusion is immediately obtained by Lemma 3.2(1).

THEOREM 3.3. *Under Assumptions 2.1 and 3.1, $\nu(B^k)$ is the optimal stopping time and the optimal value of (1.1) is given by*

$$v(x) = \varphi(x) + \mathbb{N}\left[(d_k)^+ - P(d_{k-1})^+\right](x), \qquad x \in S, \tag{3.9}$$

$$= \begin{cases} \varphi(x), & x \in B^k, \\ \mathbb{N}_{C^k}[P_{B^k}\varphi](x), & x \in C^k. \end{cases} \tag{3.10}$$

PROOF. Let $w(x) = E^x[\varphi(X_{\nu(B^k)})]$, $x \in S$. Immediately,

$$w(x) = \begin{cases} \varphi(x), & x \in B^k, \\ Pw(x), & x \in C^k \end{cases}$$

by the definition of the strategy. If $x \in B^k$, then (3.2) and Lemma 3.1 yield that $Pw(x) = P\varphi(x) \le \varphi(x)$. Therefore, $w(x)$, $x \in S$ satisfies the optimality equation (1.2). Let $\tau^* = \inf\{n \ge 0; w(X_n) \le \varphi(X_n)\} = \inf\{n \ge 0; X_n \in B^k\} = \nu(B^k)$. Following the Martingale system theory [1], we see that $w(x)$ is equal to the optimal value $v(x)$ and τ^* is the optimal stopping time. We will calculate the optimal value. When $x \in C^k$,

$$v(x) = Pv(x) = P_{B^k}\varphi(x) + P_{C^k}v(x),$$

dividing S of the integral P into B^k and C^k. Hence,

$$v(x) = \mathbb{N}_{C^k}[P_{B^k}\varphi](x), \qquad \text{for } x \in C_k.$$

Since $v(x) = \varphi(x)$ for $x \in B^k$, this proves (3.10). Next, we shall show (3.9). If $x \in B^k$, $v(x) = \varphi(x)$ and so (3.2) implies that $Pv(x) = P\varphi(x)$. And also, $(d_k)^+(x) = P(d_{k-1})^+(x) = 0$, $x \in B^k$ by Lemma 3.1 and (3.2). On the other hand, if $x \in C^k$, then $v(x) - Pv(x) = 0$ and $\varphi(x) - P\varphi(x) + (d_k)^+(x) - P(d_{k-1})^+(x) = 0$ from the definition of d_k. Thus, we have

$$v(x) - Pv(x) = \varphi(x) - P\varphi(x) + (d_k)^+(x) - P(d_{k-1})^+(x), \qquad x \in S.$$

Hence, (3.9) is proved by Lemma 3.2. ∎

COROLLARY 3.4.
If, for some $j \geq 1$, the j-SLA rule is optimal, then the value of its stopping problem is dominated as follows:

$$\varphi(x) \leq v(x) \leq \varphi(x) + \mathbb{N}(P\varphi - \varphi)^+(x), \tag{3.11}$$

for all x in S.

PROOF. Since $(d_j)^+(x) \leq (P\varphi - \varphi)^+(x) + P(d_{j-1})^+(x)$, $x \in S$, the upper bound could be obtained from (3.10). The lower bound is immediate by (1.1).

Note that, if $j = 1$, that is, the OLA rule is optimal, the upper bound holds with equality. This upper bound of the optimal value is consistant with the result of [14, Lemma 3.3]. In the conclusion of this section, we should like to discuss the infinity-SLA rule, which is a limiting case of tending k to infinity.

LEMMA 3.5. $\lim_{i\to\infty} d_i(x) = d^*(x)$, $x \in S$ *exists. It is integrable with respect to P and satisfies*

$$d^*(x) = (P\varphi - \varphi)(x) + P(d^*)^+(x), \qquad x \in S, \tag{3.12}$$

and also $B^* = \{x \in S; d^*(x) \leq 0\}$ is equal to $\bigcap_{k=1}^{\infty} B^k$.

PROOF. The sequence is shown to be monotone increasing by the induction and is dominated by $\mathbb{N}(P\varphi - \varphi)^+(x)$, which is integrable with respect to P. The assertion (3.12) holds by the Dominated Convergence Theorem. The last assertion is clear from Lemma 3.1.

ASSUMPTION 3.2.

(1) *The set $B^* = \{x \in S; d^*(x) \leq 0\}$ and its complement C^* are nonempty, and B^* is closed with respect to P; that is,*

$$P(x, B^*) = 1, \qquad x \in B^*. \tag{3.13}$$

(2) *The first hitting time $\nu(B^*)$ satisfies*

$$\nu(B^*) < \infty \text{ a.e. } P^x, \qquad X_0 = x \in S. \tag{3.14}$$

Similarly as before, we shall refer to the *infinity-SLA rule* if the rule is based on the first hitting time $\nu(B^*)$.

THEOREM 3.6. *Under Assumptions 2.1 and 3.2, the infinity-SLA rule is optimal, its optimal value equals*

$$v(x) = \varphi(x) + (d^*)^+(x), \qquad x \in S, \tag{3.15}$$

or equivalently

$$v(x) = \begin{cases} \varphi(x), & x \in B^*, \\ \mathbb{N}_{C^*}[P_{B^*}\varphi](x), & x \in C^*, \end{cases} \tag{3.16}$$

and the stopping region is $B^* = \{x \in S; d^*(x) \leq 0\}$.

PROOF. The assertion follows from Lemma 3.5 and Theorem 3.3 because

$$\mathbb{N}(d^*)^+ - P(d^*)^+ = \mathbb{N}(I - P)(d^*)^+ = (d^*)^+.$$ ∎

According to Chow and Schechner [9], they claim that the infinity-SLA is optimal. We have confirmed this result under the assumption.

4. GAME VALUE FOR THE k-SLA AND INFINITY RULE

The result of the previous section is applied to the problem of Dynkin's stopping game, in which Player I is to maximize and Player II to minimize as already defined in Section 2. The optimal strategy of each player is defined by (2.4) and (2.5). The result of this section is an extension of the discussion in Section 2, which considers a policy from the OLA rule to the k-SLA and the infinity rule.

For the standard maximization problem of (2.19), the next sequence will be defined analogously to $d_i(x)$, $i = 1, 2, \ldots, k$ in (3.1) for the minimization of (2.20). We set $k \geq 1$ a fixed integer as before. Define $e_i(x)$, $x \in S$, $i = 1, 2, \ldots, k$ by

$$e_i(x) = (P\psi - \psi)(x) - P(e_{i-1})^-(x), \tag{4.1}$$

where we put $e_0(x) = 0$. Denote the stopping region for Player I and II by

$$\begin{aligned} B_1^k &= \{x \in S; d_k(x) \leq 0\}, \\ B_2^k &= \{x \in S; e_k(x) \geq 0\}, \end{aligned} \tag{4.2}$$

respectively, and C_k be the complement of $B_1^k \cup B_2^k$. We shall refer *k-SLA rule* of the game variant to the stopping rule based on the first hitting time of set B_1^k or B_2^k.

ASSUMPTION 4.1.

(1) *Either of B_1^k or B_2^k is assumed to be nonempty and each set B_i^k is closed with respect to P for $i = 1, 2$; that is,*

$$P(x, B_i^k) = 1, \qquad x \in B_i^k, \quad i = 1, 2. \tag{4.3}$$

(2)
$$\nu(B_1^k \cup B_2^k) < \infty \text{ a.e. } P^x, \qquad X_0 = x \in S. \tag{4.4}$$

(3) *We assume that*

$$\begin{aligned} \liminf_n E^x[\psi(X_n)] &\leq \varphi(x), \qquad x \in B_1^k, \\ \limsup_n E^x[\varphi(X_n)] &\geq \psi(x), \qquad x \in B_2^k. \end{aligned} \tag{4.5}$$

The result on the k-SLA rule for the stopping problem by the previous discussion would be as follows.

THEOREM 4.1. *Under Assumptions 2.1 and 4.1, the sets B_1^k and B_2^k are disjoint. Further, the k-SLA rule is optimal and the game value is given by*

$$v(x) = \begin{cases} \varphi(x), & x \in B_1^k, \\ \mathrm{N}_{C^k}\left[P_{B_1^k}\varphi + P_{B_2^k}\psi\right](x), & x \in C^k, \\ \psi(x), & x \in B_2^k. \end{cases} \tag{4.6}$$

To consider the infinity-SLA rule of the game variant, we take the limit of k to infinity. Similar to Lemma 3.5, we see that the sequence $\{e_i(x); i \geq 1\}$ is monotone decreasing and bounded below. So

$$e^*(x) = \lim_{i \to \infty} e_i(x), \qquad x \in S \tag{4.7}$$

exists and satisfies that

$$e^*(x) = (P\psi - \psi)(x) - P(e^*)^-(x), \qquad x \in S. \tag{4.8}$$

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Define the stopping region for Player I, II by

$$\begin{aligned} B_1^* &= \{x \in S; d^*(x) \le 0\}, \\ B_2^* &= \{x \in S; e^*(x) \ge 0\}, \end{aligned} \tag{4.9}$$

respectively, and let C^* be the complement of $B_1^* \cup B_2^*$. *The infinity-SLA rule* of the Dynkin stopping problem is a stopping rule based on the first hitting time of set B_1^* or B_2^*.

ASSUMPTION 4.2.

(1) *Either of these sets is nonempty and each set B_i^*, $i = 1, 2$ is closed with respect to P; that is,*

$$P(x, B_i^*) = 1, \qquad x \in B_i^*. \tag{4.10}$$

(2) *The process eventually hits these sets; that is,*

$$\nu(B_1^* \cup B_2^*) < \infty \text{ a.e. } P^x, \qquad X_0 = x \in S. \tag{4.11}$$

(3) *We assume that*

$$\liminf_n E^x[\psi(X_n)] \le \varphi(x), \qquad x \in B_1^*, \tag{4.12}$$

$$\limsup_n E^x[\varphi(X_n)] \ge \psi(x), \qquad x \in B_2^*. \tag{4.13}$$

THEOREM 4.2. *Under Assumptions 2.1 and 4.2, the sets B_1^* and B_2^* are disjoint and the infinity-SLA rule is optimal. Its optimal value $v(x)$, $x \in S$ of the game variant problem is given by*

$$v(x) = \begin{cases} \varphi(x), & x \in B_1^*, \\ \mathbb{N}_{C^*}\left[P_{B_1^*}\varphi + P_{B_2^*}\psi\right](x), & x \in C^*, \\ \psi(x), & x \in B_2^*. \end{cases} \tag{4.14}$$

PROOF. The proof that the set B_i^*, $i = 1, 2$ is disjoint can be obtained similarly to Lemma 2.1(1). Also, the rest of the proof is easily obtained by combining the results in Sections 3 and 4. ∎

Let $C_1^* = \{x \in C^*; \nu(B_1^*) < \infty \text{ a.e. } P^x\}$ and $C_2^* = \{x \in C^*; \nu(B_2^*) < \infty \text{ a.e. } P^x\}$. Then we have, by similar discussion in Section 2,

$$\overline{\varphi}(x) = \begin{cases} \varphi(x) + (d^*)^+(x), & x \in B_1^* \cup C_1^*, \\ \limsup_n E^x\left[\varphi(X_{n\wedge\nu(B_1^*)})\right], & \text{otherwise}, \end{cases} \tag{4.15}$$

and

$$\underline{\psi}(x) = \begin{cases} \psi(x) - (e^*)^-(x), & x \in B_2^* \cup C_2^*, \\ \liminf_n E^x\left[\psi(X_{n\wedge\nu(B_2^*)})\right], & \text{otherwise}. \end{cases} \tag{4.16}$$

Define the following two functions, similar to Section 2.

$$v_1(x) = \begin{cases} \varphi(x), & x \in B_1^*, \\ \mathbb{N}_C\left[P_{B_1^*}\varphi\right](x), & x \in C^*, \\ 0, & x \in B_2^*, \end{cases} \tag{4.17}$$

$$v_2(x) = \begin{cases} \psi(x), & x \in B_2^*, \\ \mathbb{N}_C\left[P_{B_2^*}\psi\right](x), & x \in C^*, \\ 0, & x \in B_1^*. \end{cases} \tag{4.18}$$

An alternative form of (4.14) can be written, by the result of Theorem 3.6, as follows.

COROLLARY 4.3. *Under the same assumptions,*

$$v(x) = v_1(x) + v_2(x), \qquad x \in S, \tag{4.19}$$

where $v_i(x)$, $i = 1, 2$ are defined by (4.17) and (4.18), respectively.

What we want to claim is that the game value of Dynkin's problem decomposed into the sum of the two functions under the infinity rule.

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