1. INTRODUCTION AND SUMMARY

Keywords—Optimal stopping problem, Dynkin’s game, one-step look-ahead rule, Markov process.

Abstract—Under the one-step look-ahead rule of dynamic programming, an explicit optimal value for Dynkin’s stopping game can be derived.

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Dynkin’s Stopping Game

Explicit Optimal Value For
The formulation of Dynkin's 2. DYNKIN'S STOPPING GAME UNDER THE OLA RULE

Under the OLA rule, the value of Dynkin's game and that of the standard problem is obtained under the decision rule. In Section 4, by taking a limit to infinity, the decision rule is considered.

We shall express the optimal value of the standard problem under this rule, which is called a Dynkin's stopping game. The OLA rule offers an convenient solution for the stopping time problem. Consider a stopping time problem in Section 3. To express the standard stopping problem under an intractable condition of the OLA rule, Section 2 introduces an independent maximal/minimal stopping problem. This is simpler than that of the vanilla game for the problem is expressed by using a potential operator and is decomposed as the sum of two independent maximal/minimal stopping problems. The game is optimal when the OLA rule is applied. The same is expressed by using a potential operator and is decomposed as the sum of two independent maximal/minimal stopping problems.

The stopping time is optimal if the OLA rule is optimal. The OLA rule is optimal if the set is closed, that is, for each stopping time, the first hitting time of the set is reached.

The decision rule is based on the stopping time

and if satisfies

The OLA rule is optimal if the set is closed, that is, for each stopping time, the first hitting time of the set is reached.

The OLA rule is optimal if the set is closed, that is, for each stopping time, the first hitting time of the set is reached.


\begin{align*}
(1.2) \quad & \text{Let } x \in \mathbb{R}^n, \quad \exists \varepsilon > 0, \quad (\varepsilon x) a \leq (\varepsilon x)f_{a} \quad \text{and} \quad (\varepsilon x)f_{a} \leq (\varepsilon x) a \\
& \text{Since } \varepsilon > 0, \text{ the inequality is also satisfied by } \varepsilon x. \\
& \text{Also, } (x) a \leq (x) y, \quad \forall a \in A. \\
& \text{We note that the case of the simultaneous stopping for Player I and II with the reversed } (x) X \\
& \text{only occurs for a very small number of cases.} \\
& \text{Also, it is equivalent to} \\
\end{align*}

\begin{align*}
(2.2) \quad & \text{For any value of } (x) a \text{ and } (x) t, \text{ the inequality holds true.} \\
& \text{Thus, } (x) a \leq (x) t \\
& \text{where } VTL = \text{the value of the } 2 \text{ by } 2 \text{ matrix game.} \\
& \text{If } (x) a \leq (x) t, \text{ then } (x) a \text{ is an optimal} \\
& \text{equilibrium of the game.} \\
& \text{The discussion starts from the derivation of the optimal equilibrium for the game variant.} \\
& \text{Under this separation condition,} \\
\end{align*}

\begin{align*}
(3.2) \quad & \text{We assume Assumption 2, that is, no } \text{optimal stopping time in the zero-sum matrix game for all randomized strategies.} \\
& \text{This is an optimal stopping time in the zero-sum matrix game for all randomized strategies.} \\
& \text{Generally, the zero-sum stopping game does not have value, and so it is natural to consider a} \\
\end{align*}

\begin{align*}
(4.2) \quad & \text{If } (x) a = (x) t \text{, we denote } \text{(4.2)} \\
& \text{and } \text{(4.2)} \\
\end{align*}

\begin{align*}
(5.2) \quad & \text{We say that the problem has a unique value if } \\
& \text{and } \text{(5.2)} \\
\end{align*}

\begin{align*}
(6.2) \quad & \text{Finally, we consider the case of the problem.} \\
& \text{Let } (x) a = (x) f, \text{ and } (x) t = (x) f + v \\
& \text{where } v \text{ is the admissible class of the problem.} \\
& \text{If } (x) f = (x) f + v, \text{ and } (x) t = (x) f + v \\
& \text{We conclude that } (x) a = (x) f + v. \\
\end{align*}
(2.18) \[ x \in B \quad \text{and} \quad (x)_m \geq (x)_n \]

That is,

\[ (x)_m - (x)_n = (x)_m - (x)_n \]

Lemma 2.1

The sets \( B \) and \( B' \) are disjoint.

(2.19) \[ \exists \phi \text{ such that} \quad x = (x)_m \geq (x)_n \]

where \( \geq \) denotes the first hitting time of set \( B' \).

(2.16) \[ x = \lim_{u \to \infty} \inf \{ A \} \]

The process eventually hits one of these sets, that is,

(2.14) \[ P \neq 1 \]

either \( B \) or \( B' \) is assumed to be nonempty and each set \( B \) is closed with respect to \( P \).

ASSUMPTION 2.2.

(2.23) \[ (x)_b \leq (x)_b \leq (x)_b \]

\( \text{For all } x \in S \).

(2.22) \[ (x)_b \geq (x)_b \geq (x)_b \]

For the given reward functions.

Assumption 2.1.

For each \( x \in S \),

Now dynamic stopping game under the OLA rule is considered.

Already, Nevan [7] has discussed the equality (2.10) and the inequality (2.10) under a general stochastic process.

M. Vyska
\[ (x)\phi d = (x)m_d - (x)f \quad \text{for } x \in B \]

If there exists some \( x \in E \cup B \) such that \( x \) is a leaf, then \( (x)m_d \preceq (x)f \).

First, the initial relation \( B \subseteq C \) follows from \( (x)m_d \preceq (x)f \), so that the construction of \( (x)m_d \preceq (x)f \) is unique and the set of equalities \( \{(x)f \geq (x)m_d \text{ and the set is empty}\} \) is empty. The set of equalities \( \{(x)f > (x)m_d \text{ and the set is empty}\} \) is empty. Since the set \( x \) is empty, \( (x)m_d \preceq (x)f \) is unique.

**Proof:** To prove the above, we use the optimality of the OLA rule. Since the OLA rule is optimal, then

\[
(\exists x \in B \subseteq C \quad \text{there exists some } x \in E \cup B \) such that \( x \) is a leaf, then \( (x)m_d \preceq (x)f \).
\]
Then we can obtain the next corollary:

\[(\exists x \in A) \forall y \in B, (x \in a) \Rightarrow (x \neq y)\]

By applying Theorem 2.2, the equation of the stopping problem

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

is obtained by

**Theorem 2.3.** Under Assumptions 2.1 and 2.2, the value of the stopping problem is

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

We can show the exact value as follows:

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

**Theorem 2.4.** Under Assumptions 2.1 and 2.2, the value of the stopping problem is

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

We have the following theorem for any stopping time $\tau$ and $\sigma$.

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

**Theorem 2.5.** Under Assumptions 2.1 and 2.2, the value of the stopping problem is

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

We have the following theorem for any stopping time $\tau$ and $\sigma$.

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

**Theorem 2.6.** Under Assumptions 2.1 and 2.2, the value of the stopping problem is

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

We have the following theorem for any stopping time $\tau$ and $\sigma$.

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

**Theorem 2.7.** Under Assumptions 2.1 and 2.2, the value of the stopping problem is

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

We have the following theorem for any stopping time $\tau$ and $\sigma$.

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

**Theorem 2.8.** Under Assumptions 2.1 and 2.2, the value of the stopping problem is

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

We have the following theorem for any stopping time $\tau$ and $\sigma$.

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

**Theorem 2.9.** Under Assumptions 2.1 and 2.2, the value of the stopping problem is

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

We have the following theorem for any stopping time $\tau$ and $\sigma$.

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

**Theorem 2.10.** Under Assumptions 2.1 and 2.2, the value of the stopping problem is

\[
\exists x \in A, (x \in a) \Rightarrow (x \neq y)
\]

We have the following theorem for any stopping time $\tau$ and $\sigma$.
\[(x)\phi = \max (\{(x)\phi, (x)\phi\}) \quad x \in A \]
\[(x)\phi - \left[ (1)(X) \left\{ + \phi - \phi d + \phi \right\} \right] = (x)\phi - (x)\phi \quad x \in X \]
\[(x)\phi - (1)(X)\phi = (x)\phi - (x)\phi \quad x \in X \]

**Proof.** It is clear because of the definition of \(x \in \exists S \exists S \) In fact, for \(x \neq 0 \in S \).

**Lemma 3.** The sequence \(S \subseteq (1, 2, \ldots) \) is non-monotone decreasing. Therefore, the sequence continues to \(n \rightarrow \infty\) and so on.

Continuing into the stopped region there exists exactly one \(s = n \) for which \(S_n \leq 0 \). (The referred to the \(s = n \) rule).

**Theorem 3.** The set \(S \) of \(x \neq 0 \in S \) are non-empty and the stopping set \(\exists S \) is closed with respect to \(P \).

**Assumption 3.**

The complement of \(S \) is the set of \(x \neq 0 \in S \) such that \(x \neq 0 \in S \) is the set of \(x \neq 0 \in S \) that is the \(S \) rule. The \(S \) rule is based on the first hitting time \(\lambda x \).

**Section 3.** The standard stopping problem \(S_n \neq 0 \). The problem is considered for the sake of simplicity. We do not consider the same problem for the next section of the extension from the OLA rule to the R-alg rule.

**Corollary 3.** The game value of Dynkin's stopping problem is expressed by the sum of two functions.

**Dynamic Programming Game**
\[(x)\phi - [((x)\phi)_k, (x)\phi]\max \forall \phi \leq (x)\phi - [(x)\phi + (\phi - \phi)d + (\phi)\phi d + (x)\phi d + (x)\phi)] \\phi = (x)\phi - [(x)\phi + (\phi - \phi)d + (\phi)\phi d + (x)\phi] = (x)\phi \]

Proof. It is clear because of the definition of \(\phi \), \(x \in \mathbb{S} \). In fact, for \(x = 0 \in \mathbb{S} \), the sequence \(\mathbb{S} \subset \mathbb{S} \subset \mathbb{S} \).

Lemma 3.1. The sequence \(\mathbb{S} \), \(\mathbb{S} \), \(\mathbb{S} \), \(\mathbb{S} \) is not monotone decreasing, that is,

\[ \mathbb{S} \subset \mathbb{S} \subset \mathbb{S} \subset \mathbb{S} \subset \mathbb{S} \]

We shall return to the K-SLA rule if the rule is based on the first hitting time \(\mathbb{S} \mathbb{P} \). The first hitting time \(\mathbb{S} \mathbb{P} \) of \(x \) satisfies the following equation:

\[ \mathbb{S} \mathbb{P} (x) = (x) \mathbb{P} \]

Assumption 3.1.

\[ \mathbb{S} \mathbb{P} (x) \]

The set \(\mathbb{P} \) and \(\mathbb{S} \) are nonempty and the stopping set \(\mathbb{P} \) is closed with respect to \(\mathbb{S} \).

3.1. The component of \(\mathbb{P} \)

\[ \{0 \geq (x)\mathbb{P} \mathbb{P} \in \mathbb{P} \geq x\} = \mathbb{P} \]

where we put \(d = 0 \). We will consider a region defined by

\[ \{0 \geq (x)\mathbb{P} \mathbb{P} \in \mathbb{P} \geq x\} = \mathbb{P} \]

3.2. For \(x \geq 0 \), let \(\mathbb{S} \mathbb{P} \) be a fixed integer. Define iteratively the following sequence of \(\mathbb{S} \mathbb{P} \), \(x \in \mathbb{S} \): Let \( \phi \leq 1 \) be a fixed integer. Define iteratively the following sequence of \(\mathbb{S} \mathbb{P} \), \(x \in \mathbb{S} \):

\[ \mathbb{S} \mathbb{P} (x) \]

The natural extension of the extension from the OLA rule to the K-SLA rule.

3. Extension of the OLA Rule to the K-SLA Rule

satisfies the optimality equation (2.9) and imposes a discount factor on the payoff.

\[ \mathbb{S} \mathbb{P} (x) \]

The formulation and

\[ \mathbb{S} \mathbb{P} (x) \]

Theorem III.1 in [8] is as follows. The theorem would be compared with Brusniq's result [8] and the theorem III.1 in [9] is as follows.

3.2. The game value of the two-person, two-strategy games is expressed by the sum of two functions.
For $x \in \mathcal{O}$,

$\int (x) d \omega \phi = (x) \omega \phi$

Dividing the integral into $\mathcal{O}$ and $\mathcal{D}$. Hence,

$\int (x) \omega \phi d + \int (x) d \omega \phi d = (x) \omega \phi d = (x) \omega \phi$

Stopping time. We will calculate the optimal value. When $x \in \mathcal{O}$, we see that the optimal value is $\omega \phi$. Following the path $\mathcal{B}$, we have $\omega \phi \subseteq (\omega \phi) \mathcal{B}$. Therefore, $\omega \phi \subseteq (\omega \phi) \mathcal{B}$.

By the definition of the strategy, it is clear that $\omega \phi$ is the optimal value by Lemma 3.3. Therefore, $(\omega \phi) \mathcal{B}$ is the optimal stopping time and the optimal value is given by $(\omega \phi) \mathcal{B}$.

**Theorem 3.3.** Under assumptions 2.1 and 3.1, the following stopping time and the conclusion is immediately obtained by Lemma 3.3.1.

For $x \in \mathcal{O}$, we have

$0 = (x) \omega \phi (\omega \phi) \mathcal{B}$ and

$\mathcal{B}$ and hence $\mathcal{B}$ is included in $\mathcal{B}$.

For $x \in \mathcal{O}$, we have

$\omega \phi \subseteq (\omega \phi) \mathcal{B}$ and hence $\omega \phi$ is included in $\mathcal{B}$.

**Proof.**

**Lemma 3.2.**

The previous degree of stopping time.

This shows that when one comes to stop under the $\mathcal{B}$ rule, one already has considered

$(x) \omega \phi \supseteq (x) \omega \phi d^{(x) \omega \phi d}$

By this lemma, if $x \in \mathcal{D}$, then it is included by the following joint sets:

and so forth.

$(x) \omega \phi - [\{-(\omega \phi) \mathcal{B} \} \{\omega \phi (\omega \phi) \}] \mathcal{A} \mathcal{B} (\omega \phi) \mathcal{B} = (x) \omega \phi - (x) \omega \phi d + (x) \omega \phi d = (x) \omega \phi$
According to Claim and Observation 3.2, we prove that the infinity-STLA is optimal. We have

\[ (+p)H - (\Delta - I)N = (+p)d - (+p)\]

Proof. The assertion follows from Lemma 3.2 and Theorem 3.3, because

\[ \forall x \in (x)[\Delta - d] \quad \forall \exists x \in (x) \]

or equivalently,

\[ \forall \exists x \in (x) \]

which equals

**Theorem 3.4.** Under Assumptions 3.1 and 3.2, the infinity-STLA rule is optimal.

Similarly as before, we shall refer to the infinity-STLA rule if the rule is based on the first hitting

\[ \forall \exists x \in (x) \]

with respect to \( p \), which is

\[ \forall \exists x \in (x) \]

and also \( B \) is closed.

**Assumption 3.2.**

**Theorem 3.2.** Let \( \exists \exists x \in (x) \times \exists \exists x \in (x) \), where the \( \exists \exists x \in (x) \times \exists \exists x \in (x) \) and its complement \( \gamma \), are nonempty, and \( B \) is closed.

**Lemma 3.5.** Infinitely many \( \exists \exists x \in (x) \).

Proof: Let \( \exists \exists x \in (x) \) be the upper bound of the \( B \) in the

\[ \exists \exists x \in (x) \]

with respect to \( p \), which is

\[ \exists \exists x \in (x) \]

to the infinity-STLA rule, which is a limiting case

\[ \exists \exists x \in (x) \]

as follows:

\[ \exists \exists x \in (x) \]

If \( p \), then the infinity-STLA rule is optimal. Then the value of the stopping problem is dominated

**Corollary 3.4.**
To consider the inequality (4.7), let us assume that $\phi$ is bounded. Then, for any $x \in X$, we have $\phi(x) \leq (\phi(x))_D = \phi(x)$.

(4.8) \[
\lim_{t \to \infty} (x)_\phi = (x)_\phi = (x)_\phi
\]

for $x \in X$. Similarly, for $x \in X$, we have $\phi(x) \leq (\phi(x))_D = \phi(x)$.

(4.9) \[
\lim_{t \to \infty} (x)_\phi = (x)_\phi = (x)_\phi
\]

where $x \in X$. Therefore, we have $\phi(x) \leq (\phi(x))_D = \phi(x)$.

(4.10) \[
\lim_{t \to \infty} (x)_\phi = (x)_\phi = (x)_\phi
\]

for $x \in X$. This shows that $\phi$ is bounded.

(4.11) \[
\lim_{t \to \infty} (x)_\phi = (x)_\phi = (x)_\phi
\]

where $x \in X$. Therefore, we have $\phi(x) \leq (\phi(x))_D = \phi(x)$.

(4.12) \[
\lim_{t \to \infty} (x)_\phi = (x)_\phi = (x)_\phi
\]

for $x \in X$. This shows that $\phi$ is bounded.

(4.13) \[
\lim_{t \to \infty} (x)_\phi = (x)_\phi = (x)_\phi
\]

where $x \in X$. Therefore, we have $\phi(x) \leq (\phi(x))_D = \phi(x)$.

(4.14) \[
\lim_{t \to \infty} (x)_\phi = (x)_\phi = (x)_\phi
\]

for $x \in X$. This shows that $\phi$ is bounded.
Corollary 4.3. Under the same assumptions:

An alternative form of (4.14) can be written by the result of Theorem 3.2, as follows:

\[ x \in \mathbb{B}^1 \]
\[ \begin{cases} 0 & x \notin \mathbb{B}^1 \\ 1 & x \in \mathbb{B}^1 \end{cases} \]

Define the following two functions, similar to Section 2.

\[ \lim \inf_{x \in \mathbb{B}^1 \cap \mathbb{C}} \phi(x) \]
\[ \lim \sup_{x \in \mathbb{B}^1 \cap \mathbb{C}} \phi(x) \]

Proof. The proof that the set \( \mathbb{B}^1 \) is \( \mathbb{B}^2 \)-almost disjoint can be obtained similarly to Lemma 2.1.2.

Theorem 4.2. Under Assumptions 2.1 and 4.2, the sets \( \mathbb{B}^1 \) and \( \mathbb{B}^2 \) are disjoint and the infinity-loop theorem is optimal.

Assumption 4.2.

Either of these sets is nonempty and each set \( \mathbb{B}^1 \) or \( \mathbb{B}^2 \) is clopen with respect to \( P \).

Define the stopping region for Player I. II by

\[ \{0 \leq (x)^{\mathbb{R}} : x \in \mathbb{B} \} = \mathbb{B}^1 \]
\[ \{0 \leq (x)^{\mathbb{R}} : x \in \mathbb{B} \} = \mathbb{B}^1 \]

Dynamic Shopping Game