American Options with Uncertainty of the Stock Prices: The Discrete-Time Model

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1. Introduction

A discrete-time mathematical model for American put option with uncertainty is presented, and the randomness and fuzziness are evaluated by both probabilistic expectation and λ-weighted possibilistic mean values.

2. Fuzzy stochastic processes

First we give some mathematical notations regarding fuzzy numbers. Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a σ-field and P is a non-atomic probability measure. $\mathbb{R}$ denotes the set of all real numbers, and let $\mathcal{C}(\mathbb{R})$ be the set of all non-empty bounded closed intervals. A 'fuzzy number' is denoted by its membership function $\tilde{a} : \mathbb{R} \mapsto [0,1]$ which is normal, upper-semicontinuous, fuzzy convex and has a compact support. Refer to Zadeh [12] regarding fuzzy set theory. $\mathcal{R}$ denotes the set of all fuzzy numbers. In this paper, we identify fuzzy numbers with its corresponding membership functions. The α-cut of a fuzzy number $\tilde{a}(\in \mathcal{R})$ is given by

$$\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\} \ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{a}_0 := \mathrm{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\},$$

where cl denotes the closure of an interval. In this paper, we write the closed intervals by

$$\tilde{a}_\alpha := [\tilde{a}^-_\alpha, \tilde{a}^+_\alpha] \quad \text{for } \alpha \in [0,1].$$

Hence we introduce a partial order $\succeq$, so called the 'fuzzy max order', on fuzzy numbers $\mathcal{R}$: Let $\tilde{a}, \tilde{b} \in \mathcal{R}$ be fuzzy numbers.

$$\tilde{a} \succeq \tilde{b} \quad \text{means that} \quad \tilde{a}^-_\alpha \geq \tilde{b}^-_\alpha \quad \text{and} \quad \tilde{a}^+_\alpha \geq \tilde{b}^+_\alpha \quad \text{for all } \alpha \in [0,1].$$

Then $(\mathcal{R}, \succeq)$ becomes a lattice. For fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{R}$, we define the maximum $\tilde{a} \vee \tilde{b}$ with respect to the fuzzy max order $\succeq$ by the fuzzy number whose α-cuts are

$$(\tilde{a} \vee \tilde{b})_\alpha = [\max\{\tilde{a}^-_\alpha, \tilde{b}^-_\alpha\}, \max\{\tilde{a}^+_\alpha, \tilde{b}^+_\alpha\}], \quad \alpha \in [0,1]. \tag{2.1}$$

An addition, a subtraction and a scalar multiplication for fuzzy numbers are defined as follows: For $\tilde{a}, \tilde{b} \in \mathcal{R}$ and $\lambda \geq 0$, the addition and subtraction $\tilde{a} \pm \tilde{b}$ of $\tilde{a}$ and $\tilde{b}$ and the scalar multiplication $\lambda\tilde{a}$ of λ and $\tilde{a}$ are fuzzy numbers given by

$$(\tilde{a}+\tilde{b})_\alpha := [\tilde{a}^-_\alpha + \tilde{b}^-_\alpha, \tilde{a}^+_\alpha + \tilde{b}^+_\alpha], \quad (\tilde{a}-\tilde{b})_\alpha := [\tilde{a}^-_\alpha - \tilde{b}^+_\alpha, \tilde{a}^+_\alpha - \tilde{b}^-_\alpha]$$

$$\text{and} \quad (\lambda\tilde{a})_\alpha := [\lambda\tilde{a}^-_\alpha, \lambda\tilde{a}^+_\alpha] \quad \text{for } \alpha \in [0,1].$$

A fuzzy-number-valued map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a 'fuzzy random variable' if the maps $\omega \mapsto \tilde{X}^-_\alpha(\omega)$ and $\omega \mapsto \tilde{X}^+_\alpha(\omega)$ are measurable for all $\alpha \in [0,1]$, where $\tilde{X}_\alpha(\omega) = [\tilde{X}^-_\alpha(\omega), \tilde{X}^+_\alpha(\omega)] = \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ (see [10]). Next we need to introduce expectations of fuzzy random variables in order to describe an optimal stopping model in the next section. A fuzzy random variable $\tilde{X}$ is called integrably bounded if both $\omega \mapsto \tilde{X}^-_\alpha(\omega)$ and $\omega \mapsto \tilde{X}^+_\alpha(\omega)$ are integrable for all $\alpha \in [0,1]$. Let $\tilde{X}$ be an integrably bounded fuzzy random variable. The expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number (see [7])

$$E(\tilde{X})(x) := \sup_{\alpha\in[0,1]} \min\{\alpha, 1_{E(\tilde{X})_\alpha}(x)\}, \quad x \in \mathbb{R}, \tag{2.2}$$

where closed intervals $E(\tilde{X})_\alpha := \left[\int_\Omega \tilde{X}^-_\alpha(\omega)\,\mathrm{d}P(\omega), \int_\Omega \tilde{X}^+_\alpha(\omega)\,\mathrm{d}P(\omega)\right]$ $(\alpha \in [0,1])$.

In the rest of this section, we introduce stopping times for fuzzy stochastic processes. Let T $(T > 0)$ be an 'expiration date' and let $\mathbb{T} := \{0, 1, 2, \cdots, T\}$ be the time space. Let a 'fuzzy stochastic process' $\{\tilde{X}_t\}_{t=0}^T$ be a sequence of integrably bounded fuzzy random variables such that $E(\max_{t\in} \tilde{X}^+_{t,0}) < \infty$, where $\tilde{X}^+_{t,0}(\omega)$ is the right-end of the 0-cut of the fuzzy number $\tilde{X}_t(\omega)$. For $t \in \mathbb{T}$, $\mathcal{M}_t$ denotes the smallest σ-field on Ω generated by all random variables $\tilde{X}^-_{s,\alpha}$ and $\tilde{X}^+_{s,\alpha}$ $(s = 0, 1, 2, \cdots, t; \alpha \in [0,1])$. We call $(\tilde{X}_t, \mathcal{M}_t)_{t=0}^\infty$ a fuzzy stochastic process. A map $\tau : \Omega \mapsto \mathbb{T}$ is called a 'stopping time' if

$$\{\omega \in \Omega \mid \tau(\omega) = t\} \in \mathcal{M}_t \quad \text{for all } t = 0, 1, 2, \cdots, T.$$

Then, the following lemma is trivial from the definitions ([11]).

Lemma 2.1. *Let τ be a stopping time. We define*

$$\tilde{X}_\tau(\omega) := \tilde{X}_t(\omega) \quad \textit{if } \tau(\omega) = t \quad \textit{for } t = 0, 1, 2, \cdots, T \textit{ and } \omega \in \Omega.$$

Then, $\tilde{X}_\tau$ is a fuzzy random variable.

3. American put option with uncertainty of stock prices

In this section, we formulate American put option with uncertainty of stock prices by fuzzy random variables. Let $\mathbb{T} := \{0, 1, 2, \cdots, T\}$ be the time space with an expiration date T $(T > 0)$ similarly to the previous section, and take a probability space $\Omega := \mathbb{R}^{T+1}$. Let r $(r > 0)$ be an interest rate of a bond price, which is riskless asset, and put a discount rate $\beta = 1/(1+r)$. Define a 'stock price process' $\{S_t\}_{t=0}^T$ as follows: An initial stock price S_0 is a positive constant and stock prices are given by

$$S_t := S_0 \prod_{s=1}^{t} (1 + Y_s) \quad \text{for } t = 1, 2, \cdots, T, \tag{3.1}$$

where $\{Y_t\}_{t=1}^T$ is a uniform integrable sequence of independent, identically distributed real random variables on $[r-1, r+1]$ such that $E(Y_t) = r$ for all $t = 1, 2, \cdots, T$. The σ-fields $\{\mathcal{M}_t\}_{t=0}^T$ are defined as follows: $\mathcal{M}_0$ is the completion of $\{\emptyset, \Omega\}$ and $\mathcal{M}_t (t = 1, 2, \cdots, T)$ denote the complete σ-fields generated by $\{Y_1, Y_2 \cdots Y_t\}$.

We consider a finance model where the stock price process $\{S_t\}_{t=0}^T$ takes fuzzy values. Now we give fuzzy values by triangular fuzzy numbers for simplicity. Let $\{a_t\}_{t=0}^T$ be an $\mathcal{M}_t$-adapted stochastic process such that $0 < a_t(\omega) \leq S_t(\omega)$ for $\omega \in \Omega$. A 'stock price process with fuzzy values' are represented by a sequence of fuzzy random variables $\{\tilde{S}_t\}_{t=0}^T$:

$$\tilde{S}_t(\omega)(x) := L((x - S_t(\omega))/a_t(\omega)) \tag{3.2}$$

for $t \in \mathbb{T}$, $\omega \in \Omega$ and $x \in \mathbb{R}$, where $L(x) := \max\{1 - |x|, 0\}$ $(x \in \mathbb{R})$ is the triangle shape function. Hence, $a_t(\omega)$ is a spread of triangular fuzzy numbers $\tilde{S}_t(\omega)$ and corresponds to the amount of fuzziness in the process. Then, $a_t(\omega)$ should be an increasing function of the stock price $S_t(\omega)$ (see Assumption S in the next section).

Let K $(K > 0)$ be a 'strike price'. The 'price process' $\{\tilde{P}_t\}_{t=0}^T$ of American put option under uncertainty is represented by

$$\tilde{P}_t(\omega) := \beta^t (1_{\{K\}} - \tilde{S}_t(\omega)) \vee 1_{\{0\}} \quad \text{for } t = 0, 1, 2, \cdots, T, \tag{3.3}$$

where $\vee$ is given by (2.1), and $1_{\{K\}}$ and $1_{\{0\}}$ denote the crisp number K and zero respectively. An 'exercise time' in American put option is given by a stopping time τ with values in $\mathbb{T}$. For an exercise time τ, we define

$$\tilde{P}_\tau(\omega) := \tilde{P}_t(\omega) \quad \text{if } \tau(\omega) = t \quad \text{for } t = 0, 1, 2, \cdots, T, \text{ and } \omega \in \Omega. \tag{3.4}$$

Then, from Lemma 2.1, $\tilde{P}_\tau$ is a fuzzy random variable. The expectation of the fuzzy random variable $\tilde{P}_\tau$ is a fuzzy number(see (2.2))

$$E(\tilde{P}_\tau)(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{E(\tilde{P}_\tau)_\alpha}(x)\}, \quad x \in \mathbb{R}, \tag{3.5}$$

where $E(\tilde{P}_\tau)_\alpha = \left[\int_\Omega \tilde{P}^-_{\tau,\alpha}(\omega)\,\mathrm{d}P(\omega), \int_\Omega \tilde{P}^+_{\tau,\alpha}(\omega)\,\mathrm{d}P(\omega)\right]$. In American put option, we must maximize the expected values (3.5) of the price process by stopping times τ, and we need to evaluate the fuzzy numbers (3.5) since the fuzzy max order (2.1) on $\mathcal{R}$ is a partial order and not a linear order. In this paper, we consider the following estimation regarding the price process $\{\tilde{P}_t\}_{t=0}^T$ of American put option. Let $g : \mathcal{C}(\mathbb{R}) \mapsto \mathbb{R}$ be a map such that

$$g([x, y]) := \lambda x + (1 - \lambda) y, \quad [x, y] \in \mathcal{C}(\mathbb{R}), \tag{3.6}$$

where λ is a constant satisfying $0 \leq \lambda \leq 1$. This scalarization is used for the evaluation of fuzzy numbers, and λ is called a 'pessimistic-optimistic index' and means the pessimistic degree in decision making. We call g a 'λ-weighting function' and we evaluate fuzzy numbers $\tilde{a}$ by "λ-weighted possibilistic mean value'

$$\int_0^1 2\alpha g(\tilde{a}_\alpha)\,\mathrm{d}\alpha, \tag{3.7}$$

where $\tilde{a}_\alpha$ is the α-cut of fuzzy numbers $\tilde{a}$. (see Carlsson and Fullér [1], Goetshel and Voxman [4]) When we apply a λ-weighting function g to (3.5), its evaluation follows

$$\int_0^1 2\alpha g(E(\tilde{P}_\tau)_\alpha)\,\mathrm{d}\alpha. \tag{3.8}$$

Now we analyze (3.8) by α-cuts technique of fuzzy numbers. The α-cuts of fuzzy random variables (3.2) are

$$\tilde{S}_{t,\alpha}(\omega) = [S_t(\omega) - (1-\alpha)a_t(\omega), S_t(\omega) + (1-\alpha)a_t(\omega)], \quad \omega \in \Omega, \tag{3.9}$$

and so

$$\tilde{S}^{\pm}_{t,\alpha}(\omega) = S_t(\omega) \pm (1-\alpha)a_t(\omega), \quad \omega \in \Omega \tag{3.10}$$

for $t \in \mathbb{T}$ and $\alpha \in [0,1]$. Therefore, the α-cuts of (3.3) are

$$\tilde{P}_{t,\alpha}(\omega) = [\tilde{P}^-_{t,\alpha}(\omega), \tilde{P}^+_{t,\alpha}(\omega)] := [\beta^t \max\{K - \tilde{S}^+_{t,\alpha}(\omega), 0\}, \beta^t \max\{K - \tilde{S}^-_{t,\alpha}(\omega), 0\}], \tag{3.11}$$

and we obtain $E(\max_{t\in} \sup_{\alpha\in[0,1]} \tilde{P}^+_{t,\alpha}) \leq K < \infty$ since $\tilde{S}^-_{t,\alpha}(\omega) \geq 0$, where $E(\cdot)$ is the expectation with respect to some risk-neutral equivalent martingale measure([2],[6]). For a stopping time τ, the expectation of the fuzzy random variable $\tilde{P}_\tau$ is a fuzzy number whose α-cut is a closed interval

$$E(\tilde{P}_\tau)_\alpha = E(\tilde{P}_{\tau,\alpha}) = [E(\tilde{P}^-_{\tau,\alpha}), E(\tilde{P}^+_{\tau,\alpha})] \quad \text{for } \alpha \in [0,1], \tag{3.12}$$

where $\tilde{P}_{\tau(\omega),\alpha}(\omega) = [\tilde{P}^-_{\tau(\omega),\alpha}(\omega), \tilde{P}^+_{\tau(\omega),\alpha}(\omega)]$ is the α-cut of fuzzy number $\tilde{P}_\tau(\omega)$. Using the λ-weighting function g, from (3.7) the evaluation of the fuzzy random variable $\tilde{P}_\tau$ is given by the integral

$$\int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha}))\,\mathrm{d}\alpha. \tag{3.13}$$

Put the value by $\boldsymbol{P}(\tau)$. Then, from (2.2), the terms (3.8) and (3.13) coincide:

$$\boldsymbol{P}(\tau) = \int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha}))\,\mathrm{d}\alpha = \int_0^1 2\alpha g(E(\tilde{P}_\tau)_\alpha)\,\mathrm{d}\alpha. \tag{3.14}$$

Therefore $\boldsymbol{P}(\tau)$ means an evaluation of the expected price of American put option when τ is an exercise time. Further, we have the following equality.

Lemma 3.1. *For a stopping time τ ($\tau \le T$), it holds that*

$$\boldsymbol{P}(\tau) = \int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha}))\, d\alpha = \int_0^1 2\alpha E(g(\tilde{P}_{\tau,\alpha}))\, d\alpha = E\left(\int_0^1 2\alpha g(\tilde{P}_{\tau,\alpha}(\cdot))\, d\alpha\right). \quad (3.15)$$

We put the 'optimal expected price' by

$$\boldsymbol{V} := \sup_{\tau:\tau\le T} \boldsymbol{P}(\tau) = \sup_{\tau:\tau\le T} \int_0^1 2\alpha g(E(\tilde{P}_{\tau,\alpha}))\, \mathrm{d}\alpha. \quad (3.16)$$

In the next section, this paper discusses the following optimal stopping problem regarding American put option with fuzziness.

Problem P. Find a stopping time $\tau^*(\tau^* \le T)$ and the optimal expected price $\boldsymbol{V}$ such that

$$\boldsymbol{P}(\tau^*) = \boldsymbol{V}, \quad (3.17)$$

where $\boldsymbol{V}$ is given by (3.16).

Then, τ^* is called an 'optimal exercise time'.

4. The optimal expected price and the optimal exercise time

In this section, we discuss the optimal fuzzy price $\boldsymbol{V}$ and the optimal exercise time τ^*, by using dynamic programming approach. Now we introduce an assumption.

Assumption S. The stochastic process $\{a_t\}_{t=0}^T$ is represented by

$$a_t(\omega) := cS_t(\omega), \quad t = 0, 1, 2, \cdots, T, \ \omega \in \Omega,$$

where c is a constant satisfying $0 < c < 1$.

Assumption S is reasonable since $a_t(\omega)$ means a size of fuzziness and it should depend on the volatility and the stock price $S_t(\omega)$ because one of the most difficulties is estimation of the actual volatility ([8, Sect.7.5.1]). In this model, we represent by c the fuzziness of the volatility, and we call c a 'fuzzy factor' of the process. From now on, we suppose that Assumption S holds. For a stopping time τ ($\tau \le T$), we define a random variable

$$\Pi_\tau(\omega) := \int_0^1 2\alpha g(\tilde{P}_{\tau,\alpha}(\omega))\, \mathrm{d}\alpha, \quad \omega \in \Omega. \quad (4.1)$$

From Lemma 3.1, $\boldsymbol{P}(\tau) = E(\Pi_\tau)$ is the evaluated price of American put option when τ is an exercise time. Then we have the following representation about (4.1).

Lemma 4.1. *For a stopping time τ ($\tau \le T$), it holds that*

$$\Pi_\tau(\omega) = \beta^{\tau(\omega)} f^P(S_\tau(\omega)), \quad \omega \in \Omega, \quad (4.2)$$

where f^P is a function on $(0,\infty)$ such that

$$f^P(y) := \begin{cases} K - y - \frac{1}{3}cy(2\lambda - 1) + \lambda\varphi^1(y) & \textit{if } 0 < y < K \\ (1-\lambda)\varphi^2(y) & \textit{if } y \geq K, \end{cases} \tag{4.3}$$

and

$$\varphi^1(y) := \frac{1}{(cy)^2}((-K+y+cy)\max\{0,-K+y+cy\}^2 - \frac{2}{3}\max\{0,-K+y+cy\}^3), \quad y > 0, \tag{4.4}$$

$$\varphi^2(y) := \frac{1}{(cy)^2}((K-y+cy)\max\{0,K-y+cy\}^2 - \frac{2}{3}\max\{0,K-y+cy\}^3), \quad y > 0. \tag{4.5}$$

Now we give an optimal stopping time for Problem P and we discuss an iterative method to obtain the optimal expected price $\boldsymbol{V}$ in (3.16). To analyze the optimal fuzzy price $\boldsymbol{V}$, we put

$$\boldsymbol{V}_t^P(y) = \sup_{\tau: t \leq \tau \leq T} E(\beta^{-t}\Pi_\tau | S_t = y) \tag{4.6}$$

for $t = 0, 1, 2, \cdots, T$ and an initial stock price y $(y > 0)$. Then we note that $\boldsymbol{V} = \boldsymbol{V}_0^P(y)$.

Theorem 4.1 (Optimality equation).

(i) *The optimal expected price $\boldsymbol{V} = \boldsymbol{V}_0^P(y)$ with an initial stock price y $(y > 0)$ is given by the following backward recursive equations (4.7) and (4.8):*

$$\boldsymbol{V}_t^P(y) = \max\{\beta E(\boldsymbol{V}_{t+1}^P(y(1+Y_1))), f^P(y)\}, \quad t = 0, 1, \cdots, T-1, \ y > 0, \tag{4.7}$$

$$\boldsymbol{V}_T^P(y) = f^P(y), \quad y > 0. \tag{4.8}$$

(ii) *Define a stopping time*

$$\tau^P(\omega) := \inf\{t \in \mathbb{T} \mid \boldsymbol{V}_0^P(S_t(\omega)) = f^P(S_t(\omega))\}, \quad \omega \in \Omega, \tag{4.9}$$

where the infimum of the empty set is understood to be T. Then, τ^P is an optimal exercise time for Problem P, and the optimal value of American put option is

$$\boldsymbol{V} = \boldsymbol{V}_0^P(y) = \boldsymbol{P}(\tau^P) \tag{4.10}$$

for an initial stock price $y > 0$.

5. A numerical example

Now we give a numerica example to illustrate our idea in Sections 3 and 4.

Example 5.1. We consider CRR type American put option model (see Ross [8, Sect.7.4]). Put an expiration date $T = 10$, an interest rate of a bond $r = 0.05$, a fuzzy factor $c = 0.05$, an initial stock price $y = 30$ and a strike price $K = 35$. Assume that

$\{Y_t\}_{t=1}^{T}$ is a uniform sequence of independent, identically distributed real random variables such that

$$Y_t := \begin{cases} e^{\sigma} - 1 & \text{with probability } p \\ e^{-\sigma} - 1 & \text{with probability } (1-p) \end{cases}$$

for all $t = 1, 2, \cdots, T$, where $\sigma = 0.25$ and $p = (1 + r - e^{-\sigma})/(e^{\sigma} - e^{-\sigma})$. Then we have $E(Y_t) = r$. The corresponding optimal exercise time is given by

$$\tau^P(\omega) = \inf\{t \in \mathbb{T} \mid \boldsymbol{V}_0^P(S_t(\omega)) = f^P(S_t(\omega))\}.$$

In the following Table, the optimal expected price $\boldsymbol{V} = \boldsymbol{V}_0^P(y)$ at initial stock price $y = 30$ changes with the pessimistic-optimistic index λ of the λ-weighting function g.

Table. The optimal expected price $= {}_0^P(y)$ at initial stock prices $y = 30$.

λ	1/3	1/2	2/3
$\boldsymbol{V}$	7.48169	7.39649	7.31130

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