ファジイ確率変数の最適化問題とファイナンスへの応用

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1. Introduction and notations

This paper deals with an American put option model in financial engineering which is based on Black-Scholes stochastic model under uncertaity. The theory of option pricing in a financial market has been developing on the basis of the famous Black-Scholes log-normal stochastic differential models. When we sell or buy stocks by Internet in a financial market, there sometimes exists a difference between the actual prices and the theoretical value which derived from Black-Scholes methods, and it is not easy to predict the future actual prices. The difficulty comes from not only randomness of financial stochastic systems but also uncertainty which we cannot represent by only probability theory. When the market are changing rapidly, the losses/errors often become bigger between the decision maker's expected price and the actual price. Mathematical modeling of stochastic systems in decision-making has many applications to engineering, economics, etc., and in general, one of the conditions that stochastic modeling works successfully is stability of systems. When we deal with systems like financial markets, fuzzy logic works well because the markets contain the uncertain factors which are different from probabilistic essence and in which there exists a difficulty to identify actual price values exactly. In this paper, probability is applied as the uncertainty such that something occurs or not with probability, and fuzziness is applied as the uncertainty such that we cannot specify the exact values because of a lack of knowledge regarding the present stock market.

By introducing fuzzy logic to the log-normal stochastic processes for the financial market, we present a new model with uncertainty of both randomness and fuzziness in output, which is a reasonable and natural extension of the original log-normal stochastic processes in Black-Scholes model. To valuate the American put option, we need to deal with optimal stopping in log-normal stochastic processes (Elliott and Kopp [1], Karatzas and Shreve [4], Ross [8] and so on). In this paper, we discuss an optimal stopping model regarding log-normal stochastic processes with fuzziness, and the optimal stopping times mean exercise times for American option in the financial markets. In order to describe an optimal stopping model with fuzziness, we need to extend real-valued random variables in probability theory to 'fuzzy random variables', which are random variables with fuzzy number values. We introduce a 'fuzzy stochastic process' by fuzzy random variables to define prices in American put option, and we evaluate the randomness and fuzziness by probabilistic expectation and linear ranking functions from the viewpoint of Yoshida et al. [11].

We derive an optimality equation for the optimal stopping model and gives a method to solve the stopping problem without loss of worthy information contained in uncertainty like randomness and fuzziness. This paper deals with an American option model with uncertainty by an approach of dynamic programming.

In the next section, we introduce a fuzzy stochastic process by fuzzy random variables to define prices in American put option with uncertainty. The prices are called 'fuzzy prices' in this paper. The randomness and fuzziness in the fuzzy stochastic process are evaluated by both probabilistic expectation and linear ranking functions. In Section 3, this paper formulates an American option model with uncertainty. In Section 4, we consider the optimal expected price in the American put option and discuss writer's/seller's optimal expected prices, and we give an optimal exercise time for the American put option. We show, by dynamic programming, that the optimal fuzzy price is a solution of an optimality equation under a reasonable assumption. Finally, a numerical example is given to illustrate our idea.

In the remainder of this section, we describe notations regarding bond price processes and stock price processes. We consider American put option in a finance model where there is no arbitrage opportunities ([1, 4]). Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a σ-field of Ω and P is a non-atomic probability measure. $\mathbb{R}$ denotes the set of all real numbers. Let μ be the appreciation rate and let σ be the volatility ($\mu \in \mathbb{R}$, $\sigma > 0$). Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{M}, P)$. $\{\mathcal{M}_t\}_{t\geq 0}$ denotes a family of nondecreasing right-continuous complete sub-σ-fields of $\mathcal{M}$ such that $\mathcal{M}_t$ is generated by $B_s (0 \leq s \leq t)$. We consider two assets, a bond price $\{R_t\}_{t\geq 0}$ and a stock price $\{S_t\}_{t\geq 0}$, where the bond price process $\{R_t\}_{t\geq 0}$ is riskless and the stock price process $\{S_t\}_{t\geq 0}$ is risky. Let r ($r \geq 0$) be the instantaneous interest rate, i.e. interest factor, on a bond. Let a bond price process $\{R_t\}_{t\geq 0}$ satisfy the ordinary differential equation:

$$\mathrm{d}R_t = rR_t\mathrm{d}t, \quad t \geq 0, \tag{1.1}$$

with $R_0 = 1$, and then it follows

$$R_t = e^{rt}, \quad t \geq 0. \tag{1.2}$$

Let a stock price process $\{S_t\}_{t\geq 0}$ satisfy the log-normal stochastic differential equation: S_0 is a positive constant, and

$$\mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}B_t, \quad t \geq 0. \tag{1.3}$$

It is known ([1]) that there exists an equivalent probability measure Q such that $\{S_t/R_t\}_{t\geq 0}$ is a martingale under Q, by setting $\mathrm{d}Q/\mathrm{d}P|_{\mathcal{M}_t} = \exp\left(((r-\mu)/\sigma)B_t - \frac{1}{2}((r-\mu)/\sigma)^2 t\right)$, $t \geq 0$. Under Q, $W_t := B_t - ((r-\mu)/\sigma)t$ is a standard Brownian motion and it holds that $\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t$. By Ito's formula, we have

$$S_t = S_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma W_t\right), \quad t \geq 0. \tag{1.4}$$

In this paper, we present option models where a stock price process S_t takes fuzzy values using fuzzy random variables, whose mathematical notations are introduced in the next section.

2. Fuzzy stochastic processes

Fuzzy random variables, which take values in fuzzy numbers, were first studied by Puri and Ralescu [7] and have been studied by many authors. It is known that the fuzzy random variable is one of the successful hybrid notions of randomness and fuzziness. First we introduce fuzzy numbers. Let $\mathcal{C}(\mathbb{R})$ be the set of all non-empty bounded closed intervals. A fuzzy number is denoted by its membership function $\tilde{a} : \mathbb{R} \mapsto [0,1]$ which is normal, upper-semicontinuous, fuzzy convex and has a compact support. Refer to Zadeh [12] regarding fuzzy set theory. In this paper, we identify fuzzy numbers with its corresponding membership functions. $\mathcal{R}$ denotes the set of all fuzzy numbers. The α-cut of a fuzzy number $\tilde{a}(\in \mathcal{R})$ is given by

$$\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\} \ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{a}_0 := \mathrm{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\},$$

where cl denotes the closure of an interval. We write the closed intervals as

$$\tilde{a}_\alpha := [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+] \quad \text{for } \alpha \in [0,1].$$

We also use a metric δ_∞ on $\mathcal{R}$ defined by

$$\delta_\infty(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \delta(\tilde{a}_\alpha, \tilde{b}_\alpha) \quad \text{for fuzzy numbers } \tilde{a}, \tilde{b} \in \mathcal{R},$$

where δ is the Hausdorff metric on $\mathcal{C}(\mathbb{R})$([6]). Hence we introduce a partial order $\succeq$, so called the fuzzy max order, on fuzzy numbers $\mathcal{R}$([5]): Let $\tilde{a}, \tilde{b} \in \mathcal{R}$ be fuzzy numbers.

$$\tilde{a} \succeq \tilde{b} \quad \text{means that} \quad \tilde{a}_\alpha^- \geq \tilde{b}_\alpha^- \quad \text{and} \quad \tilde{a}_\alpha^+ \geq \tilde{b}_\alpha^+ \quad \text{for all } \alpha \in [0,1].$$

Then $(\mathcal{R}, \succeq)$ becomes a lattice. For fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{R}$, we define the maximum $\tilde{a} \vee \tilde{b}$ with respect to the fuzzy max order $\succeq$ by the fuzzy number whose α-cuts are

$$(\tilde{a} \vee \tilde{b})_\alpha = [\max\{\tilde{a}_\alpha^-, \tilde{b}_\alpha^-\}, \max\{\tilde{a}_\alpha^+, \tilde{b}_\alpha^+\}], \quad \alpha \in [0,1]. \tag{2.1}$$

An addition, a subtraction and a scalar multiplication for fuzzy numbers are defined as follows: For $\tilde{a}, \tilde{b} \in \mathcal{R}$ and $\lambda \geq 0$, the addition and subtraction $\tilde{a} \pm \tilde{b}$ of $\tilde{a}$ and $\tilde{b}$ and the scalar multiplication $\lambda\tilde{a}$ of λ and $\tilde{a}$ are fuzzy numbers given by

$$(\tilde{a} + \tilde{b})_\alpha := [\tilde{a}_\alpha^- + \tilde{b}_\alpha^-, \tilde{a}_\alpha^+ + \tilde{b}_\alpha^+], \quad (\tilde{a} - \tilde{b})_\alpha := [\tilde{a}_\alpha^- - \tilde{b}_\alpha^+, \tilde{a}_\alpha^+ - \tilde{b}_\alpha^-]$$

$$\text{and} \quad (\lambda\tilde{a})_\alpha := [\lambda\tilde{a}_\alpha^-, \lambda\tilde{a}_\alpha^+] \quad \text{for } \alpha \in [0,1].$$

A fuzzy-number-valued map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a 'fuzzy random variable' if the maps $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are measurable for all $\alpha \in [0,1]$, where $\tilde{X}_\alpha(\omega) =$

$[\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] = \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ (see [10]). Next we need to introduce expectations of fuzzy random variables in order to describe an optimal stopping model in the next section. A fuzzy random variable $\tilde{X}$ is called integrably bounded if both $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are integrable for all $\alpha \in [0,1]$. Let $\tilde{X}$ be an integrably bounded fuzzy random variable. The expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number (see [7])

$$E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{E(\tilde{X})_\alpha}(x)\}, \quad x \in \mathbb{R}, \tag{2.2}$$

where closed intervals $E(\tilde{X})_\alpha := \left[\int_\Omega \tilde{X}_\alpha^-(\omega)\,\mathrm{d}P(\omega), \int_\Omega \tilde{X}_\alpha^+(\omega)\,\mathrm{d}P(\omega)\right]$ $(\alpha \in [0,1])$.

Now, we introduce a continuous-time fuzzy stochastic process by fuzzy random variables. Let $\{\tilde{X}_t\}_{t \geq 0}$ be a family of integrably bounded fuzzy random variables such that $E(\sup_{t \geq 0} \tilde{X}_{t,0}^+) < \infty$, where $\tilde{X}_{t,0}^+(\omega)$ is the right-end of the 0-cut of the fuzzy number $\tilde{X}_t(\omega)$. We assume that the map $t \mapsto \tilde{X}_t(\omega) (\in \mathcal{R})$ is continuous on $[0, \infty)$ for almost all $\omega \in \Omega$. $\{\mathcal{M}_t\}_{t \geq 0}$ is a family of nondecreasing sub-σ-fields of $\mathcal{M}$ which is right continuous, i.e. $\mathcal{M}_t = \bigcap_{r: r > t} \mathcal{M}_r$ for all $t \geq 0$, and fuzzy random variables $\tilde{X}_t$ are $\mathcal{M}_t$-adapted, i.e. random variables $\tilde{X}_{r,\alpha}^-$ and $\tilde{X}_{r,\alpha}^+$ $(0 \leq r \leq t; \alpha \in [0,1])$ are $\mathcal{M}_t$-measurable. And $\mathcal{M}_\infty$ denotes the smallest σ-field containing $\bigcup_{t \geq 0} \mathcal{M}_t$. We call $(\tilde{X}_t, \mathcal{M}_t)_{t \geq 0}$ a 'fuzzy stochastic process'.

3. American put option with uncertainty of stock prices

In this section, we introduce American put option with fuzzy prices and we discuss its properties. Let $\{a_t\}_{t \geq 0}$ be an $\mathcal{M}_t$-adapted stochastic process such that the map $t \mapsto a_t(\omega)$ is continuous on $[0, \infty)$ and $0 < a_t(\omega) \leq S_t(\omega)$ for almost all $\omega \in \Omega$. We give a fuzzy stochastic process of the stock price process $\{\tilde{S}_t\}_{t \geq 0}$ by the following fuzzy random variables:

$$\tilde{S}_t(\omega)(x) := L((x - S_t(\omega))/a_t(\omega)) \tag{3.1}$$

for $t \geq 0$, $\omega \in \Omega$ and $x \in \mathbb{R}$, where $L(x) := \max\{1 - |x|, 0\}$ $(x \in \mathbb{R})$ is the triangle type shape function(Fig.3.1) and $\{S_t\}_{t \geq 0}$ is defined by (1.3). Hence, $a_t(\omega)$ is a spread of triangular fuzzy numbers $\tilde{S}_t(\omega)$ and corresponds to the amount of fuzziness in the process. Then $a_t(\omega)$ should be an increasing function of the stock price $S_t(\omega)$ since the fuzziness in the process depends on the volatility σ and stock price $S_t(\omega)$ in (1.3) (see Assumption S in the next section). The α-cuts of (3.1) are

$$\tilde{S}_{t,\alpha}(\omega) = [\tilde{S}_{t,\alpha}^-(\omega), \tilde{S}_{t,\alpha}^+(\omega)] = [S_t(\omega) - (1-\alpha)a_t(\omega), S_t(\omega) + (1-\alpha)a_t(\omega)]. \tag{3.2}$$

Let K be a strike price $(K > 0)$. We define the fuzzy price process by a fuzzy stochastic process $\{\tilde{P}_t\}_{t \geq 0}$:

$$\tilde{P}_t(\omega) := e^{-rt}(1_{\{K\}} - \tilde{S}_t(\omega)) \vee 1_{\{0\}} \quad \text{for } t \geq 0,\ \omega \in \Omega, \tag{3.3}$$

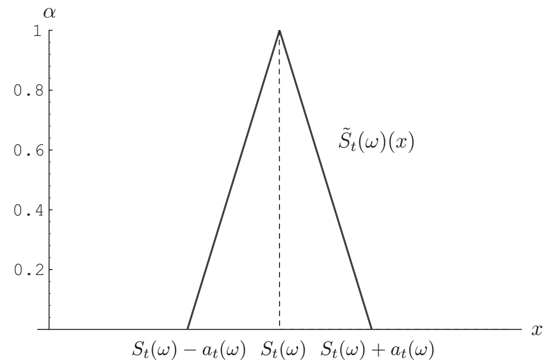

Fig. 3.1. Fuzzy random variable $\tilde{S}_t(\omega)(x)$.

where $\vee$ is given by (2.1), and $1_{\{K\}}$ and $1_{\{0\}}$ denote the crisp numbers K and zero respectively. Their α-cuts are

$$\tilde{P}_{t,\alpha}(\omega) = [\max\{e^{-rt}(K - \tilde{S}^+_{t,\alpha}(\omega)), 0\}, \max\{e^{-rt}(K - \tilde{S}^-_{t,\alpha}(\omega)), 0\}].$$

Then, we obtain $E(\sup_{t\geq 0} \sup_{\alpha\in[0,1]} \tilde{P}^+_{t,\alpha}) \leq K < \infty$ since $\tilde{S}^-_{t,\alpha}(\omega) \geq 0$ for $t \geq 0$, $\alpha \in [0,1]$ and $\omega \in \Omega$.

In this paper, we deal with a model with the time space $\mathbb{T} = [0,T]$, where T is a positive constant and is called an expiration date. A map $\tau : \Omega \mapsto \mathbb{T}$ is called a stopping time if

$$\{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{M}_t \quad \text{for all } t \in \mathbb{T}.$$

An exercise time in American put option is given by a stopping time τ with values in $[0,T]$. For an exercise time τ, we define

$$\tilde{P}_\tau(\omega) := \tilde{P}_t(\omega) \quad \text{if } \tau(\omega) = t \quad \text{for } t \in \mathbb{T},\ \omega \in \Omega. \tag{3.4}$$

Then, $\tilde{P}_\tau$ is a fuzzy random variable. The expectation of the fuzzy random variable $\tilde{P}_\tau$ is a fuzzy number(see (2.2))

$$E(\tilde{P}_\tau)(x) := \sup_{\alpha\in[0,1]} \min\{\alpha, 1_{E(\tilde{P}_\tau)_\alpha}(x)\}, \quad x \in \mathbb{R}, \tag{3.5}$$

where $E(\tilde{P}_\tau)_\alpha = \left[\int_\Omega \tilde{P}^-_{\tau,\alpha}(\omega)\,\mathrm{d}P(\omega), \int_\Omega \tilde{P}^+_{\tau,\alpha}(\omega)\,\mathrm{d}P(\omega)\right]$. In American put option, we must maximize the expected values (3.5) of the price process by stopping times τ, and we need to evaluate the fuzzy numbers (3.5) since the fuzzy max order (2.1) on $\mathcal{R}$ is a partial order and not a linear order. In this paper, we consider the following estimation regarding the price process $\{\tilde{P}_t\}_{t=0}^T$ of American put option. Let $g : \mathcal{C}(\mathbb{R}) \mapsto \mathbb{R}$ be a map such that

$$g([x,y]) := \lambda x + (1-\lambda)y, \quad [x,y] \in \mathcal{C}(\mathbb{R}), \tag{3.6}$$

where λ is a constant satisfying $0 \leq \lambda \leq 1$. This scalarization is used for the evaluation of fuzzy numbers (Fortemps and Roubens [3]), and λ is called a pessimistic-optimistic indicator. We call g a linear ranking function and we evaluate fuzzy numbers $\tilde{a}$ by

$$\int_0^1 g(\tilde{a}_\alpha)\,\mathrm{d}\alpha, \tag{3.7}$$

where $\tilde{a}_\alpha$ is the α-cut of fuzzy numbers $\tilde{a}$. When we apply a linear ranking function g to (3.5), its evaluation follows

$$\int_0^1 g(E(\tilde{P}_\tau)_\alpha)\,\mathrm{d}\alpha. \tag{3.8}$$

Now we analyze (3.8) by α-cuts technique of fuzzy numbers. The α-cuts of fuzzy random variables (3.2) are

$$\tilde{S}_{t,\alpha}(\omega) = [S_t(\omega) - (1-\alpha)a_t(\omega), S_t(\omega) + (1-\alpha)a_t(\omega)], \quad \omega \in \Omega, \tag{3.9}$$

and so we put

$$\tilde{S}^{\pm}_{t,\alpha}(\omega) := S_t(\omega) \pm (1-\alpha)a_t(\omega), \quad \omega \in \Omega \tag{3.10}$$

for $t \in \mathbb{T}$ and $\alpha \in [0,1]$. Therefore, the α-cuts of (3.3) are

$$\tilde{P}_{t,\alpha}(\omega) = [\tilde{P}^-_{t,\alpha}(\omega), \tilde{P}^+_{t,\alpha}(\omega)] := [e^{-rt}\max\{K - \tilde{S}^+_{t,\alpha}(\omega), 0\}, e^{-rt}\max\{K - \tilde{S}^-_{t,\alpha}(\omega), 0\}], \tag{3.11}$$

and we obtain $E(\max_{t\in\mathbb{T}}\sup_{\alpha\in[0,1]}\tilde{P}^+_{t,\alpha}) \le K < \infty$ since $\tilde{S}^-_{t,\alpha}(\omega) \ge 0$, where $E(\cdot)$ is the expectation with respect to some risk-neutral equivalent martingale measure([1],[9]). For a stopping time τ, the expectation of the fuzzy random variable $\tilde{P}_\tau$ is a fuzzy number whose α-cut is a closed interval

$$E(\tilde{P}_\tau)_\alpha = E(\tilde{P}_{\tau,\alpha}) = [E(\tilde{P}^-_{\tau,\alpha}), E(\tilde{P}^+_{\tau,\alpha})] \quad \text{for } \alpha \in [0,1], \tag{3.12}$$

where $\tilde{P}_{\tau(\omega),\alpha}(\omega) = [\tilde{P}^-_{\tau(\omega),\alpha}(\omega), \tilde{P}^+_{\tau(\omega),\alpha}(\omega)]$ is the α-cut of fuzzy number $\tilde{P}_\tau(\omega)$. Using the linear ranking function g, from (3.7) the evaluation of the fuzzy random variable $\tilde{P}_\tau$ is given by the integral

$$\int_0^1 g(E(\tilde{P}_{\tau,\alpha}))\,\mathrm{d}\alpha. \tag{3.13}$$

Put the value (3.13) by $\boldsymbol{P}(\tau)$. Then, from (2.2), the terms (3.8) and (3.13) coincide:

$$\boldsymbol{P}(\tau) = \int_0^1 g(E(\tilde{P}_{\tau,\alpha}))\,\mathrm{d}\alpha = \int_0^1 g(E(\tilde{P}_\tau)_\alpha)\,\mathrm{d}\alpha. \tag{3.14}$$

Therefore $\boldsymbol{P}(\tau)$ means an evaluation of the expected price of American put option when τ is an exercise time. Further, we have the following equality.

Lemma 3.1. *For a stopping time τ ($\tau \le T$), it holds that*

$$\boldsymbol{P}(\tau) = \int_0^1 g(E(\tilde{P}_{\tau,\alpha}))\,d\alpha = \int_0^1 E(g(\tilde{P}_{\tau,\alpha}))\,d\alpha = E\left(\int_0^1 g(\tilde{P}_{\tau,\alpha}(\cdot))\,d\alpha\right). \tag{3.15}$$

We put the 'optimal expected price' by

$$\boldsymbol{V} := \sup_\tau \boldsymbol{P}(\tau) = \sup_\tau \int_0^1 g(E(\tilde{P}_{\tau,\alpha}))\,\mathrm{d}\alpha. \tag{3.16}$$

In the next section, this paper discusses the following optimal stopping problem regarding American put option with fuzziness.

Problem P. Find a stopping time τ^* ($\tau^* \leq T$) and the optimal expected price $\boldsymbol{V}$ such that

$$\boldsymbol{P}(\tau^*) = \boldsymbol{V}, \tag{3.17}$$

where $\boldsymbol{V}$ is given by (3.16).

Then, τ^* is called an 'optimal exercise time'.

4. The optimal expected price and the optimal exercise time

In this section, we discuss the optimal fuzzy price $\boldsymbol{V}$ and the optimal exercise time τ^*, by using dynamic programming approach. Now we introduce an assumption.

Assumption S. The stochastic process $\{a_t\}_{t\in\mathbb{T}}$ is represented by

$$a_t(\omega) := cS_t(\omega), \quad t \in \mathbb{T},\ \omega \in \Omega,$$

where c is a constant satisfying $0 < c < 1$.

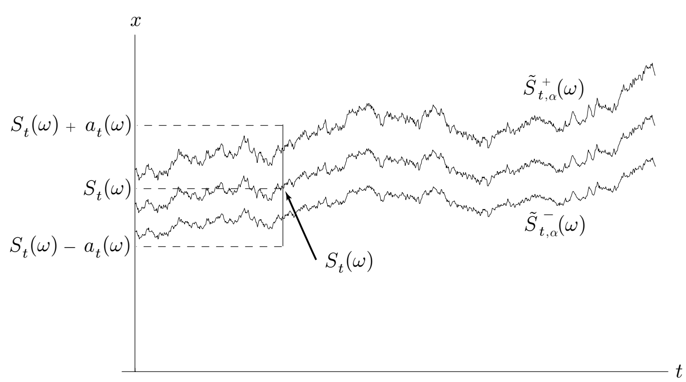

Fig. 4.1. Fuzzy stock price $\tilde{S}_t(\omega)(x)$ and Assumption S.

Since (1.3) can be written as

$$\mathrm{d}\log S_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t, \quad t \geq 0, \tag{4.1}$$

one of the most difficulties is estimation of the actual volatility σ ([8, Sect.7.5.1]). Therefore, Assumption S is reasonable since $a_t(\omega)$ corresponds to a size of fuzziness(Figs.3.1 and 4.1) and so it is reasonable that $a_t(\omega)$ should depend on the fuzziness of the volatility σ and the stock price $S_t(\omega)$ of the term $\sigma S_t(\omega)$ in (1.3). In this model, we represent by c the fuzziness of the volatility σ, and we call c a fuzzy factor of the process. From now on, we suppose that Assumption S holds. By putting $b^{\pm}(\alpha) := 1 \pm (1-\alpha)c$ ($\alpha \in [0,1]$), from (3.2) we have

$$\begin{aligned} \tilde{S}^{\pm}_{t,\alpha}(\omega) &= S_t(\omega) \pm (1-\alpha)a_t(\omega) \\ &= b^{\pm}(\alpha)S_t(\omega), \quad \omega \in \Omega \end{aligned} \tag{4.2}$$

for $t \in \mathbb{T}$ and $\alpha \in [0,1]$. Then, from (3.11) and (4.2), we have the fuzzy price process:

$$\tilde{P}^{\pm}_{\tau,\alpha}(\omega) = e^{-r\tau(\omega)} \max\{K - b^{\mp}(\alpha)S_\tau(\omega), 0\}, \quad \omega \in \Omega. \tag{4.3}$$

For a stopping time τ ($\tau \le T$), we define a random variable

$$\Pi_\tau(\omega) := \int_0^1 g(\tilde{P}_{\tau,\alpha}(\omega))\,\mathrm{d}\alpha, \quad \omega \in \Omega. \tag{4.4}$$

From Lemma 3.1, $\boldsymbol{P}(\tau) = E(\Pi_\tau)$ is the evaluated price of American put option when τ is an exercise time. Then we have the following representation about (4.4).

Lemma 4.1. *For a stopping time τ ($\tau \le T$), it holds that*

$$\Pi_\tau(\omega) = e^{-r\tau(\omega)} f^P(S_\tau(\omega)), \quad \omega \in \Omega, \tag{4.5}$$

where f^P is a function on $(0,\infty)$ such that

$$f^P(y) := \begin{cases} K - y - \frac{1}{2}cy(2\lambda - 1) + \lambda\varphi^1(y) & \text{if } 0 < y < K, \\ (1-\lambda)\varphi^2(y) & \text{if } y \ge K, \end{cases} \tag{4.6}$$

and

$$\varphi^1(y) := \frac{1}{2cy}\max\{0, -K + y + cy\}^2, \quad y > 0, \tag{4.7}$$

$$\varphi^2(y) := \frac{1}{2cy}\max\{0, K - y + cy\}^2, \quad y > 0. \tag{4.8}$$

Next, to analyze the optimal fuzzy price $\boldsymbol{V}$, we introduce the follwowing stochastic process $\{Z_t\}_{t\in\mathbb{T}}$: Let $t \in \mathbb{T}$. Define

$$Z_t := \operatorname*{ess\,sup}_{\tau:\ \text{stopping times with values in } \mathbb{T}, \tau \ge t} E(\Pi_\tau | \mathcal{M}_t). \tag{4.10}$$

Refer to [9] regarding the essential supremum. Then Z_t are right continuous with respect to t since Π_t and $\mathcal{M}_t$ are right continuous with respect to t. The random variables Z_t is called Snell's envelope(Fakeev [2]). Hence, by using dynamic programming approach, we obtain the following optimality characterization for the stochastic process regarding the optimal fuzzy price $\boldsymbol{V}$ by random variables Z_t.

Lemma 4.2. *For $t \in \mathbb{T}$, the following (i) — (iii) hold:*

(i) *For almost all $\omega \in \Omega$, it holds that*

$$Z_t(\omega) \ge \Pi_t(\omega).$$

Particularly it holds that $\boldsymbol{V} = E(Z_0 | S_0 = y)$.

(ii) *For almost all* $\omega \in \Omega$, *it holds that*

$$Z_t(\omega) \geq E(Z_s|\mathcal{M}_t)(\omega), \quad s \in [t, \infty) \cap \mathbb{T}.$$

(iii) *For almost all* $\omega \in \Omega$ *satisfying* $Z_t(\omega) > \Pi_t(\omega)$, *there exists* $\varepsilon > 0$ *such that*

$$Z_t(\omega) = E(Z_s|\mathcal{M}_t)(\omega), \quad s \in [t, t+\varepsilon) \cap \mathbb{T}.$$

Now we give an optimal stopping time for Problem P. We define fuzzy price processes by

$$\boldsymbol{V}^P(y,t) = \sup_{\tau:\ \text{stopping times},\ t \leq \tau \leq T} E(e^{rt}\Pi_\tau | S_t = y) \tag{4.12}$$

for $y > 0$ and $t \in \mathbb{T}$. Then we note that $\boldsymbol{V}^P(y,0) = \boldsymbol{V}$, which is the optimal expected price defined in (3.16). Since the stock price process $\{S_t\}_{t\geq 0}$ is Markov from (3.1), the following theorem regarding the fuzzy price processes (4.12) holds by Lemma 4.2 and (4.5). Next, we define an operator

$$\mathcal{L} := \frac{1}{2}\sigma^2 y^2 \frac{\partial^2}{\partial y^2} + ry\frac{\partial}{\partial y} + \frac{\partial}{\partial t} \quad \text{on } [0,\infty) \times [0,T). \tag{4.13}$$

Then we obtain the following optimality equations.

Theorem 4.1 (Free boundary problem). *Suppose that Assumption S holds. Then, the fuzzy price* $\boldsymbol{V}^P(y,t)$ *satisfies the following equations:*

$$\mathcal{L}(e^{-rt}\boldsymbol{V}^P(y,t)) \leq 0 \quad \textit{in the sense of Schwartz distributions}, \tag{4.14}$$
$$\mathcal{L}(e^{-rt}\boldsymbol{V}^P(y,t)) = 0 \quad \textit{on } D, \tag{4.15}$$
$$\boldsymbol{V}^P(y,t) \geq f^P(y), \tag{4.16}$$
$$\boldsymbol{V}^P(y,T) = f^P(y), \tag{4.17}$$

where $D := \{(y,t) \in [0,\infty) \times [0,T) \mid \boldsymbol{V}^P(y,t) > f^P(y)\}$. *The corresponding optimal exercise time is*

$$\tau^*(\omega) = \inf\{t \in \mathbb{T} \mid \boldsymbol{V}^P(S_t(\omega),t) = f^P(S_t(\omega))\}, \qquad \omega \in \Omega. \tag{4.18}$$

5. A numerical example

Now we give a numerica example to illustrate our idea in Sections 3 and 4.

Example 5.1. Put an expiration date $T = 7$, an interest rate of a bond $r = 0.05$, a fuzzy factor $c = 0.05$, an initial stock price $y = 25$ and a strike price $K = 30$. If we take a linear ranking function g with $\lambda = 0.5$ in (3.6), Fig. 5.1 shows the corresponding optimal expected price $\boldsymbol{V}^P(y,0)$ for each initial stock price y.

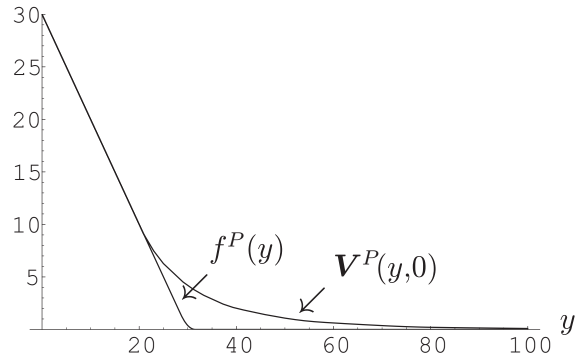

Fig. 5.1. The optimal expected price $\boldsymbol{V}^P(y,0)$ and the function $f^P(y)$.
($\lambda = 1/2, c = 0.05, K = 30$, y: initial stock price)

The optimal expected price $\boldsymbol{V}(y) = \boldsymbol{V}^P(y,0)$ at initial stock price $y = 25$ changes corresponding to the pessimistic-optimistic index λ in the definition (3.6) of the linear ranking function g(see Table 5.1).

Table 5.1. The optimal expected price $\boldsymbol{V}(y) = \boldsymbol{V}^P(y,0)$ and the index λ ($y = 25$).

λ	1/3	1/2	2/3
$\boldsymbol{V}(y)$	6.09478	5.95986	5.86189

In Table 5.1, the pessimistic-optimistic index $\lambda (0 \leq \lambda \leq 1)$ means the pessimistic degree in the writer's decision making.

References

[1] R.J.Elliott and P.E.Kopp, *Mathematics of Financial Markets* (Springer, New York, 1999).

[2] A.G.Fakeev, Optimal stopping rules for processes with continuous parameter, *Theory Probab. Appli.* **15** (1970) 324-331.

[3] P.Fortemps and M.Roubens, Ranking and defuzzification methods based on area compensation, *Fuzzy Sets and Systems* **82** (1996) 319-330.

[4] I.Karatzas and S.E.Shreve, *Methods of Mathematical Finance* (Springer, New York, 1998).

[5] G.J.Klir and B.Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications* (Prentice-Hall, London, 1995).

[6] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51** (1992) 83-88.

[7] M.L.Puri and D.A.Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.* **114** (1986) 409-422.

[8] S.M.Ross, *An Introduction to Mathematical Finance* (Cambridge Univ. Press, Cambridge , 1999).

[9] A.N.Shiryayev, *Optimal Stopping Rules* (Springer, New York, 1979).

[10] G.Wang and Y.Zhang, The theory of fuzzy stochastic processes, *Fuzzy Sets and Systems* **51** (1992) 161-178.

[11] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, Optimal stopping problems in a stochastic and fuzzy system, *J. Math. Analy. and Appl.* **246** (2000) 135-149.

[12] L.A.Zadeh, Fuzzy sets, *Inform. and Control* **8** (1965) 338-353.