On Egoroff's theorems on finite monotone non-additive measure space ⋆

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Abstract

In this note, we give four versions of Egoroff's theorem in non-additive measure theory by using the condition (E), the pseudo-condition (E) of set function and the duality relations between the conditions. These conditions offered are not only sufficient but also necessary for the four kinds of Egoroff's theorem respectively.

Keywords: Non-additive measure; Condition (E); Pseudo-condition (E); Egoroff's theorem

1 Introduction

Egoroff's theorem is one of the most important theorem in classical measure theory. It is stated that almost everywhere convergence implies almost uniform convergence on a finite measure space ([1]). The researches on the theorem in non-additive measure theory were made by Wang and Klir ([12]), Li ([3,4]), Li and Yasuda ([2,5,7]), and Murofushi *et al.*([10]). These results faithfully contribute to non-additive measure theory. In [4] Li introduced the concept of *condition (E)* of set function and proved an essential result: a necessary

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and sufficient condition that Egoroff's theorem remain valid for monotone set function is that the monotone set function fulfil condition (E). In [10] Murofushi *et al.* defined the concept of *Egoroff codition* and proved that it is a necessary and sufficient condition for Egoroff's theorem with respect to non-additive measures. Hence the two concepts are equivalent to each other.

In this paper, Egoroff's theorems in the sense of pseudo-convergence in non-additive measure theory are discussed. We will introduce the concept of *pseudo-condition (E)* of a set function. Three pseudo-versions of Egoroff's theorem on finite monotone non-additive measure spaces by using the condition (E) and the pseudo-condition (E) of set function and the duality relations between the conditions are given. These conditions employed are not only sufficient but also necessary for the different kinds of Egoroff's theorem respectively. In our discussion the set functions considered are only monotone without the assumption of continuity from above and below, therefore the previous results we obtained in [3] are generalized and Egoroff's theorem on finite non-additive measure space are formulated and developed in full generality.

2 Preliminaries

Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X. Unless stated otherwise, all the subsets mentioned are supposed to belong to $\mathcal{F}$.

Definition 2.1 A monotone non-additive measure on a measurable space $(X, \mathcal{F})$ is an extended real valued set function $\mu : \mathcal{F} \to [0, +\infty]$ satisfying the following conditions:

(1) $\mu(\emptyset) = 0$;

(2) $\mu(A) \leq \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{F}$ (monotonicity).

When μ is a monotone non-additive measure, the triple $(X, \mathcal{F}, \mu)$ is called a monotone measure space ([11]).

A monotone non-additive measure μ is called *finite*, if $\mu(X) < \infty$; *order-continuous* [11], if $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $A_n \searrow \emptyset$; *strongly order-continuous* [6], if $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $A_n \searrow A$ and $\mu(A) = 0$; *continuous at* X, if $\lim_{n\to\infty} \mu(A_n) = \mu(X)$ whenever $A_n \nearrow X$; *strongly continuous at* X, if $\lim_{n\to\infty} \mu(A_n) = \mu(H)$ whenever $A_n \nearrow H$ and $\mu(H) = \mu(X)$.

When μ is finite, we define the conjugate $\overline{\mu}$ of μ by

$$\overline{\mu}(A) = \mu(X) - \mu(X \setminus A), \quad A \in \mathcal{F}.$$

Obviously, the conjugate $\overline{\mu}$ of a monotone non-additive measure μ is also a monotone non-additive measure, and it holds that $\overline{\overline{\mu}} = \mu$.

From the duality principle of non-additive measure ([8,9]), we know that: (1) The order-continuity is dual to the continuity at X; (2) The strong order-continuity is dual to the strong continuity at X.

Let $\mathbf{F}$ be the class of all finite real-valued measurable functions on $(X, \mathcal{F}, \mu)$, and let $f, f_n \in \mathbf{F}$ $(n = 1, 2, \ldots)$. We say that $\{f_n\}$ *converges almost everywhere to* f *on* X , and denote it by $f_n \xrightarrow{a.e.} f$, if there is subset $E \subset X$ such that $\mu(E) = 0$ and $f_n \to f$ on $X \setminus E$; $\{f_n\}$ *converges pseudo-almost everywhere to* f *on* X, and denote it by $f_n \xrightarrow{p.a.e.} f$, if there is a subset $F \subset X$ such that $\mu(X \setminus F) = \mu(X)$ and $f_n \to f$ on $X \setminus F$; $\{f_n\}$ *converges almost uniformly to* f *on* X, and denote it by $f_n \xrightarrow{a.u.} f$, if for any $\epsilon > 0$ there is a subset $E_\epsilon \in \mathcal{F}$ such that $\mu(X \setminus E_\epsilon) < \epsilon$ and f_n converges to f uniformly on E_ϵ; $\{f_n\}$ *converges to* f *pseudo-almost uniformly on* X and, denote it by $f_n \xrightarrow{p.a.u.} f$, if there exists $\{F_k\} \subset \mathcal{F}$ with $\lim_{k \to +\infty} \mu(X \setminus F_k) = \mu(X)$ such that f_n converges to f on $X \setminus F_k$ uniformly for any fixed $k = 1, 2, \ldots$ (cf. [12]).

Proposition 2.2 *Let* μ *be a finite monotone non-additive measure. Then*

(1) $f_n \xrightarrow{a.e.} f[\mu]$ *iff* $f_n \xrightarrow{p.a.e.} f[\overline{\mu}]$;

(2) $f_n \xrightarrow{a.u.} f[\mu]$ *iff* $f_n \xrightarrow{p.a.u.} f[\overline{\mu}]$.

3 Condition (E) of set function

In [4] we introduced the concept of condition (E) of set function and shown a version of Egoroff's theorem in non-additive measure theory. Now we propose a new structural — pseudo-condition (E) — it plays an important role in establishing pseudo-versions of Egoroff's theorem on non-additive measure spaces.

Definition 3.1 A set function $\mu : \mathcal{F} \to [0, +\infty]$ is said to fulfil *condition (E)* (resp. *pseudo-condition (E)*), if for every double sequence $\{E_n^{(m)}\} \subset \mathcal{F}$ $(m, n \in N)$ satisfying the conditions: for any fixed $m = 1, 2, \ldots$,

$$E_n^{(m)} \searrow E^{(m)} \ (n \to \infty) \ \text{ and } \ \mu\left(\bigcup_{m=1}^{+\infty} E^{(m)}\right) = 0$$

there exist increasing sequences $\{n_i\}_{i \in N}$ and $\{m_i\}_{i \in N}$ of natural numbers, such that

$$\lim_{k\to+\infty}\mu\left(\bigcup_{i=k}^{+\infty}E_{n_i}^{(m_i)}\right)=0 \qquad (\text{resp.} \quad \lim_{k\to+\infty}\overline{\mu}\left(\bigcup_{i=k}^{+\infty}E_{n_i}^{(m_i)}\right)=0\ ).$$

It is easy to prove the following results (The first conclusion has been shown in [7,10]).

Proposition 3.2 *Let μ be a finite monotone non-additive measure. Then*

(1) *If μ fulfils condition (E), then it is strongly order continuous.*

(2) *If $\overline{\mu}$ fulfils condition (E), then μ is strongly continuous at X.*

(3) *If μ fulfils pseudo-condition (E), then it is continuous at X and $\overline{\mu}\ll\mu$, i.e., for any $N\in\mathcal{F}$, $\mu(N)=0$ implies $\overline{\mu}(N)=0$.*

(4) *If $\overline{\mu}$ fulfils pseudo-condition (E), then μ is order continuous and $\mu\ll\overline{\mu}$, i.e., for any $N\in\mathcal{F}$, $\overline{\mu}(N)=0$ implies $\mu(N)=0$.*

The *condition (E)* and *pseudo-condition (E)* are independent of each other.

Example 3.3 Let $X=[0,1]$, $\mathcal{F}$ be the class of all Lebesgue measurable sets on $[0,1]$, and λ be Lebesgue's measure. Put

$$\mu_1(E)=\begin{cases}\frac{1}{2}\lambda(E) & \text{if } \lambda(E)<1,\\ 1 & \text{if } \lambda(E)=1.\end{cases}$$

Then μ_1 is a monotone non-additive measure. It is easy to verified that μ_1 fulfils the condition (E). Since μ_1 is not continuous at X, by Proposition 3.2, we know that μ_1 does not fulfil pseudo-condition (E). In fact, if we take $E_n=[0,1-\frac{1}{n}]\cup\{1\}$, $n=1,2,\cdots$, then $E_n\nearrow X$. But $\mu_1(E_n)=\frac{1}{2}(1-\frac{1}{n})$ $(n=1,2,\cdots)$, therefore $\mu_1(E_n)\longrightarrow\frac{1}{2}\neq\mu_1(X)$.

Example 3.4 Let $(X,\mathcal{F},\lambda)$ be the same Lebesgue measure space as Example 3.3, and μ_2 the monotone non-additive measure on $\mathcal{F}$ defined as

$$\mu_2(E)=\begin{cases}0 & \text{if } \lambda(E)=0,\\ 1 & \text{if } \lambda(E)>0.\end{cases}$$

Then μ_2 is not strongly order continuous, therefore it follows from Proposition 3.2 that μ_2 does not fulfil condition (E). But it is easy to verified that μ_2

fulfils the pseudo-condition (E).

Note 3.5 In Example 3.3 and 3.4 above, both μ_1 and μ_2 are null-additive.

4 Egoroff type theorem

Now we present the main results — four versions of Egoroff's theorem in finite monotone non-additive measure spaces. The first conclusion in Theorem 4.1 below has been shown in [7].

Theorem 4.1 (Egoroff's theorem) *Let μ be a finite monotone non-additive measure. Then,*

(1) *μ fulfils condition (E) iff for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$,*

$$f_n \xrightarrow{a.e.} f \implies f_n \xrightarrow{a.u.} f.$$

(2) *$\overline{\mu}$ fulfils condition (E) iff for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$,*

$$f_n \xrightarrow{p.a.e.} f \implies f_n \xrightarrow{p.a.u.} f.$$

(3) *μ fulfils pseudo-condition (E) iff for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$,*

$$f_n \xrightarrow{a.e.} f \implies f_n \xrightarrow{p.a.u.} f.$$

(4) *$\overline{\mu}$ fulfils pseudo-condition (E) iff for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$,*

$$f_n \xrightarrow{p.a.e.} f \implies f_n \xrightarrow{a.u.} f.$$

Proof. In [4] we have proved (1). From (1) and Proposition 2.2, we can obtain (2). From (3) and Proposition 2.2, we can obtain (4). Now we only prove (3).

Necessity: Suppose μ fulfills pseudo-condition (E) and $f_n \xrightarrow{a.e.} f$. Let D be the set of these points x in X at which $\{f_n(x)\}$ dose not converge to $f(x)$. If we denote

$$E_n^{(m)} = \bigcup_{j=n}^{+\infty} \left\{ x \in X : \ |f_j(x) - f(x)| \geq \frac{1}{m} \right\}$$

and $E^{(m)} = \bigcap_{n=1}^{+\infty} E_n^{(m)}$ for every $n, m = 1, 2, \ldots$, then we have

$$D = \bigcup_{m=1}^{+\infty} \bigcap_{n=1}^{+\infty} E_n^{(m)}.$$

Since $f_n \xrightarrow{a.e.} f$, $\mu(D) = 0$. Thus we obtain a double sequence $\{E_n^{(m)}\} \subset \mathcal{F}$ $(m, n \in N)$ satisfying the conditions: for any fixed $m = 1, 2, \ldots$,

$$E_n^{(m)} \searrow E^{(m)} \ (n \to \infty) \ \text{ and } \ \mu\left(\bigcup_{m=1}^{+\infty} E^{(m)}\right) = 0$$

Applying the condition (E) of μ to the double sequence $\{E_n^{(m)}\} \subset \mathcal{F}$ $(m, n \in N)$, then there exist increasing sequences $\{n_i\}_{i \in N}$ and $\{m_i\}_{i \in N}$ of natural numbers, such that

$$\lim_{k \to +\infty} \overline{\mu}\left(\bigcup_{i=k}^{+\infty} E_{n_i}^{(m_i)}\right) = 0,$$

that is,

$$\lim_{k \to +\infty} \mu\left(X \setminus \bigcup_{i=k}^{+\infty} E_{n_i}^{(m_i)}\right) = \mu(X).$$

Put $F_k = \bigcup_{i=k}^{+\infty} E_{n_i}^{(m_i)}$ $(k = 1, 2, \ldots)$, then $\lim_{k \to +\infty} \mu(X \setminus F_k) = \mu(X)$. Now, we prove that f_n converges to f on $X \setminus F_k$ uniformly for any fixed $k = 1, 2, \ldots$.

Since

$$\begin{aligned} X \setminus F_k &= X \setminus \bigcup_{i=k}^{+\infty} E_{n_i}^{(m_i)} \\ &= \bigcap_{i=k}^{+\infty} \bigcap_{j=n_i}^{+\infty} \left\{x \in X : \ |f_j(x) - f(x)| < \frac{1}{m_i}\right\}, \end{aligned}$$

then for any $i \geq k$,

$$X \setminus F_k \subset \bigcap_{j=n_i}^{+\infty} \left\{x \in X : \ |f_j(x) - f(x)| < \frac{1}{m_i}\right\}.$$

For any given $\sigma > 0$, we take i_0 $(\geq k)$ such that $\frac{1}{m_{i_o}} < \sigma$. Thus, as $j > n_{i_o}$, for any $x \in X \setminus F_k$,

$$|f_j(x) - f(x)| < \frac{1}{m_{i_o}} < \sigma.$$

This shows that $\{f_n\}$ converges to f on $X \setminus F_k$ uniformly.

Sufficiency: Suppose that for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$, $f_n \xrightarrow{a.e.} f$ implies $f_n \xrightarrow{p.a.u.} f$. Let $\{E_n^{(m)} \mid m, n \in N\} \subset \mathcal{F}$ be any given double sequence of sets and satisfy the conditions: for any fixed $m = 1, 2, \cdots$,

$$E_n^{(m)} \searrow E^{(m)} \ (n \to \infty) \ \text{ and } \ \mu\left(\bigcup_{m=1}^{+\infty} E^{(m)}\right) = 0.$$

We put

$$\hat{E}_n^{(m)} = \bigcup_{i=1}^{m} E_n^{(i)} = E_n^{(1)} \cup E_n^{(2)} \cdots \cup E_n^{(m)} \quad (m, n \in N)$$

and

$$\hat{E}^{(m)} = \bigcap_{n=1}^{+\infty} \hat{E}_n^{(m)} \quad (m = 1, 2, \cdots).$$

Then we obtain a double sequence $\{\hat{E}_n^{(m)}\} \subset \mathcal{F}$ $(m, n \in N)$ satisfying the properties: for any fixed $n \in N$, $\hat{E}_n^{(m)} \subset \hat{E}_n^{(m+1)}$, and for any fixed $m \in N$, $\hat{E}_n^{(m)} \searrow \hat{E}^{(m)}$ as $n \to \infty$, and from $\bigcup_{m=1}^{+\infty} \hat{E}^{(m)} = \bigcup_{m=1}^{+\infty} E^{(m)}$, it follows that $\mu(\bigcup_{m=1}^{+\infty} \hat{E}^{(m)}) = 0$.

Now we construct a sequence $\{f_n\}_n \subset \mathbf{F}$: for every $n \in N$ we define

$$f_n(x) = \begin{cases} \dfrac{1}{m+1} & x \in \hat{E}_n^{(m+1)} - \hat{E}_n^{(m)} \quad m = 1, 2, \cdots \\ 1 & x \in \hat{E}_n^{(1)} \\ 0 & x \in X - \displaystyle\bigcup_{m=1}^{+\infty} \hat{E}_n^{(m)}. \end{cases}$$

It is similar to the proof of Theorem in [4], we can obtain $f_n \xrightarrow{a.e.} 0$ on X. Therefore, from the hypothesis, we have $f_n \xrightarrow{p.a.u.} 0$ on X. Thus, there exists a sequence $\{F_j\}_{j \in N}$ such that $\lim_{j \to +\infty} \mu(X \setminus F_j) = \mu(X)$ and f_n converges to f on $X \setminus F_j$ uniformly for any fixed $j = 1, 2, \ldots$ Without loss of generality, we can assume $F_1 \supset F_2 \supset \cdots$ (otherwise, we can take $\bigcap_{i=1}^{j} F_i$ instead of F_j). Thus for every $j \in N$, there exist $n_j \in N$ such that for any $x \in X \setminus F_j$, we have $|f_i(x)| < \frac{1}{j}$ whenever $i \geq n_j$. Therefore, for every $j \in N$, we have

$$X \setminus F_j \subset \bigcap_{i=n_j}^{+\infty} \left\{ x \mid |f_i(x)| < \frac{1}{j} \right\} = X \setminus \hat{E}_{n_j}^{(j)}.$$

Noting that $X \setminus F_1 \subset X \setminus F_2 \subset \cdots$, then for every $k \geq 1$,

$$X \setminus F_k = \bigcap_{j=k}^{+\infty}(X \setminus F_j) \subset \bigcap_{j=k}^{+\infty}(X \setminus \hat{E}_{n_j}^{(j)}) = X \setminus \bigcup_{j=k}^{+\infty} \hat{E}_{n_j}^{(j)}$$

and hence

$$\mu(X \setminus F_k) = \mu(\bigcap_{j=k}^{+\infty}(X \setminus F_j)) \leq \mu(X \setminus \bigcup_{j=k}^{+\infty} \hat{E}_{n_j}^{(j)}).$$

It follows from $\lim_{k\to+\infty} \mu(X \setminus F_k) = \mu(X)$ that

$$\lim_{k\to+\infty} \mu\left(X \setminus \bigcup_{j=k}^{+\infty} \hat{E}_{n_j}^{(j)}\right) = \mu(X).$$

Noting that for $m, n \in N, E_n^m \subset \hat{E}_n^m$, thus we have chosen a subsequence $\{E_{n_j}^{(j)}\}_{j\in N}$ of the double sequence $\{E_n^{(m)}\}$ such that

$$\lim_{k\to+\infty} \mu\left(X \setminus \bigcup_{j=k}^{+\infty} E_{n_j}^{(j)}\right) = \mu(X),$$

i.e.,

$$\lim_{k\to+\infty} \overline{\mu}\left(\bigcup_{i=k}^{+\infty} E_{n_i}^{(m_i)}\right) = 0.$$

This shows that $\overline{\mu}$ fulfils condition (E). $\square$

Combining Theorem 4.1 and Proposition 3.2, we can obtain necessary conditions of Egoroff's theorem.

Corollary 4.2 *Let μ be a finite monotone non-additive measure. Then,*

(1) *If for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$, $f_n \xrightarrow{a.e.} f$ implies $f_n \xrightarrow{a.u.} f$, then μ is strongly order continuous.*

(2) *If for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$, $f_n \xrightarrow{p.a.e.} f$ implies $f_n \xrightarrow{p.a.u.} f$, then μ is strongly continuous at X.*

(3) *If for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$, $f_n \xrightarrow{a.e.} f$ implies $f_n \xrightarrow{p.a.u.} f$, then μ is continuous at X and $\overline{\mu} \ll \mu$.*

(4) *If for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$, $f_n \xrightarrow{p.a.e.} f$ implies $f_n \xrightarrow{a.u.} f$, then μ is order continuous and $\mu \ll \overline{\mu}$.*

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