2

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ON THE STOCHASTIC OPTIMIZATION OF LINEAR SYSTEMS

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1. Introduction

We note here that the maximum principle of Pontryagin gives the necessary and sufficient condition in the following linear stochastic systems;

(1.1) $$dx(t) = A(t)\alpha(x(s),\ s\leq t)dt + h(t,\ u(t))dt + B(t)d\xi(t) \qquad T \geq t \geq 0$$

where T is fixed, α is a continuous linear functional and $\xi(t)$ is Brownian motion.

In the systems of ordinary differential equations, many authers considered such optimization problems. For examples, L.S. Pontryagin, et al [1], M.N. Oğuzstöreli [2], D.H. Chang and E.B. Lee [3], etc.. Concerning with the stochastic optimal control problem of a functional type, W.H. Fleming and M. Nishio [4] proved the existence of an optimal control. On the stochastic maximum principle, it is known by J.H. Kushner [5], [6], [7] that it is necessary for optimal in the non-linear Markovian optimization. This principle can be extended to more general non-linear systems. This argument is outlined by W.H. Fleming [8]. The proceeding papers are for a necessary condition, but we also concider a sufficient condition. This result of ours neither include the other, nor be included. On the existence of this optimal control problem, it could be proved, by the same method of [4], an ε-optimal control exists.

2. Definitions and Formulation

Let $T>0$, $t_0\geq 0$ be fixed constants. Given a stochastic process $x(t)=(x_1(t), \cdots, x_l(t))$, $t \in [-t_0, T]$, $\mathfrak{B}(x(\cdot))$ denotes the least Borel field generated by $\{x(t), t \in [-t_0, T]\}$. For Borel fields $\mathfrak{B}_1$, $\mathfrak{B}_2$, the leat Borel field which contains $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is denoted by $\mathfrak{B}_1 \vee \mathfrak{B}_2$.

We define the processes $\pi_s x(t)$, $\sigma_s x(t)$, $t \in [-t_0, 0]$ for a given stochastic process $x(t)$, $t \in [-t_0, T]$ in the following.

(2.1) $$\pi_s x(t) = x(s+t) \text{ for } t\in[-t_0, 0],\ s\in[0, T]$$

(2.2) $$\sigma_s x(t) = x(s-t) \text{ for } t\in[-t_0, 0],\ s\in[-t_0, T-t_0]$$

$(\Omega, \mathfrak{B}, P; \mathfrak{B}_t)$ denotes that $(\Omega, \mathfrak{B}, P)$ is a probability space and $\mathfrak{B}_t$, $t \in [0, T]$ is a increasing Borel field. $\xi(t)$, $t \in [0, T]$ is Brownian motion defined on it.

(AI) Let $x_-(t)$, $t \in [0, t_0]$ be a stochastic process with continuous paths. Also assume that

$$Ex_-^2(t) < \infty \qquad \text{for each } t \in [0, -t_0]$$

and independent to above Brownian motion $\xi(t)$.

We will define an admissible controller. A stochastic process $u(t)=u(t,\omega)$, $t\in[0,T]$, $\omega\in\Omega$ is an admissible controller, if

(AII) (i) it is measurable in the pair (t,ω),
(ii) as a function of $\omega\in\Omega$, $\mathfrak{B}_t\vee\mathfrak{B}(x_-(\cdot))$-measurable for each fixed t in $[0,T]$,
(iii) and if $u(t,\omega)\in K$ where K is the control region provided it is a compact subset in Euclidean space.

(AIII) Let $A(t)$, $B(t)$ be continuos matrix functions defined on $[0,T]$,

$$\alpha(f)=\int_{-t_0}^{0} f(s)\Gamma(ds) \tag{2.3}$$

where Γ is a probability measure on $[-t_0,0]$. The function $h(t,u)$ defined on $([0,T],K)$ is continuous in each t and u, respectively.

We will determine a response $x(t)$ for an admissible controller $u(t)$. The response is defined by the solution of the following stochastic differential equation;

$$dx(t)=A(t)\,\alpha(\pi_t x)dt+h(t,u(t))dt+B(t)d\xi(t)\quad t>0 \tag{2.4}$$

with $x(t)=x_-(t)$, $-t_0\leq t\leq 0$.

To be more precise, $x(t)$ is called a solution of (2.4) for an admissible controller $u(t)$, if
(i) $x(t)$ is $\mathfrak{B}_t\vee\mathfrak{B}(x_-(\cdot))$-measurable with continuous paths for almost all ω,
(ii) $x(t)=x_-(t)\quad -t_0\leq t\leq 0$,
(iii) and if $x(t)$ satisfies, with probability 1,

$$x(t)=x(0)+\int_0^t A(s)\alpha(\pi_s x)ds+\int_0^t h(s,u(s))ds+\int_0^t B(s)d\xi(s) \tag{2.5}$$

where $t>0$, $d\xi(s)$ is a stochastic integral of Brownian motion.

PROPOSITION 2.1 *If the above assumptions* (AI) ~ (AIII) *are satisfied, then the pathwise uniqueness holds for* (2.4), *that is, for any two solutions* $(x(t),\xi(t),u(t))$, $(x'(t),\xi'(t),u'(t))$ *defined on the same probability space* $(\Omega,\mathfrak{B},P;\mathfrak{B}_t)$, $\xi(t)=\xi'(t)$ and $u(t)=u'(t)$ *imply* $x(t)=x'(t)$.

Proof) By the usual iteration method, we can show the proposition. We omit the detail here.

Now we have arrived to define the cost criteria $C(u)$ of an admissible controller $u(t)$. Let $x(t)$ be a corresponding responce, and

$$C(u)=C(u(\cdot))=Eg(x(T))+E\int_0^T\{f^0(t,x(t))+h^0(t,u(t))\}dt \tag{2.6}$$

where E is a mathematical expectation. We assume these functions in (2.6) as follows.

(AIV) (i) $g(x)$ is a continuously differentiable convex function on $x\in R^l$,
(ii) $f^0(t,x)$, a gradient $f_x^0(t,x)$ of $f^0(t,x)$, $h^0(t,u)$ defined on $t\in[0,T]$, $x\in R^l$, $u\in R^m$, are continuous in (t,x), (t,u) respectively.

(AIV′) With the assumption (AIV), $f^0(t,x)$ is convex in x for each fixed t.

(AIV″) With the assumption (AIV), Jacobians $f_{xx}^0(t,x)$, $g_{xx}(x)$ are bounded functions in (t,x), x respectively.

Let us try to calculate the minimizing control problem where an initial stochastic process $x_-(t)$, $-t_0\leq t\leq 0$ and Brownian motion $\mathfrak{B}_t$, $\xi(t)$, $t\in[0,T]$ are fixed. We call an admissible controller $u^*(t)$ is optimal if, for any admissible controller $u(t)$, it holds that

$$C(u^*)\leq C(u)\,. \tag{2.7}$$

EXAMPLE (i) If a probability measure Γ in (2.3) is concentrated at 0, the system of (2.4) reduces to the following, so called, Markovian optimization (cf. [4]),

$$dx(t)=A(t)x(t)dt+h(t,u(t))dt+B(t)d\xi(t)\quad t>0. \tag{2.8}$$

(ii) If $\Gamma(-\theta)=1$, $0\leq\theta\leq t_0$, the system is the models with time delays,

$$dx(t)=A(t)x(t-\theta)dt+h(t,u(t))dt+B(t)d\xi(t)\quad t>0. \tag{2.9}$$

(iii) If $\Gamma(ds)=\gamma(s)ds$, i.e. absolutely continuous with respect to Lebesgue measure, the system (2.4) can be rewritten as

$$dx(t)=A(t)\left\{\int_{-t_0}^{0}x(t+s)\gamma(s)ds\right\}dt+h(t,u(t))dt+B(t)d\xi(t)\quad t>0\,. \tag{2.10}$$

3. A Condition for an Optimal Controller

For a given stochastic process $x(t)$ on $[-t_0,T]$, we define the coresponding adjoint stochastic process $\eta(t)$ which satisfies, with probability 1,

$$\eta(t)-\eta(s)=\int_s^t f_x(\tau,x(\tau))\,d\tau-\int_s^t\alpha(\sigma_\tau(\eta A\chi))d\tau \tag{3.1}$$

for each $s\leq t$ in $[0,T]$ and a terminal condition

$$\eta(T)=-g_x(x(T)) \tag{3.2}$$

where $\chi=\chi(t)$ in (3.1) is an indicator function of $[0,T]$. Note that $\sigma_\tau(\eta A\chi)$ can be extended to $\tau\in[0,T]$ because $\chi(t)=0$ if $t\bar{\in}[0,T]$.

LEMMA 3.1 *Let* $x(t)$ *be a given stochastic process and* $\eta(t)$ *be its adjoint stochastic process defined by* (3.1). *If a stochastic process* $z(t)$, $t\in[-t_0,T]$ *satisfies*

$$z(t)=\int_0^t A(s)\alpha(\pi_s z)ds+\int_0^t u_z(s)ds\,.\quad \textit{for } t>0, \tag{3.3}$$

where u_z *is a known stochastic process, it holds*

$$\eta(T)z(T)=\int_0^T f_x^0(s,x(s))z(s)ds+\int_0^T\eta(s)u_z(s)ds \tag{3.4}$$
$$+\int_0^T\eta(s)A(s)\alpha(\pi_s z)ds-\int_0^T\alpha(\sigma_s(\eta A\chi))z(s)ds\,.$$

In particular, $z(s)=0$, $s\in[-t_0,0]$ *implies*

$$\int_0^T\eta(s)A(s)\alpha(\pi_s(z))ds=\int_0^T\alpha(\sigma_s(\eta A\chi))z(s)ds\,. \tag{3.5}$$

Hence, (3.4) *is reduced to*

$$\eta(T)z(T)=\int_0^T f_x^0(s, x(s))z(s)ds+\int_0^T \eta(s)u_z(s)ds\,. \tag{3.6}$$

Proof) If we substitute $z(s)$ instead of t in (3.3) into $\int_0^T f_x^0(s,x(s))z(s)ds$, interchanging the order of an integration for Lebesgue measure implies the relation (3.4). If $z(s)=0$ for $s\in[-t_0, 0]$, (3.5) holds in case of the measure Γ is discete. Because the discrete measure on a complete metric space is dense in the sence of the weak topology, (3.5) holds if Γ is a probability measure.

THEOREM 3.1 *Let* $x^*(t)$ *be the coresponding response to an admissible controller* $u^*(t)$ *and* $\eta^*(t)$ *be its adjoint process. If* (AIV′), *and if* $u^*(t)$ *satisfies the following maximum principle, with probability* 1,

$$\begin{aligned}&-h^0(t,u^*(t))+E(\eta^*(t)/\mathfrak{B}_t)h(t,u^*(t))\\&=\max_{u\in K}\{-h^0(t,u)+E(\eta^*(t)/\mathfrak{B}_t)h(t,u)\}\quad \textit{for a.s. } t\,,\end{aligned} \tag{3.7}$$

then $u^*(t)$ *is the optimal controller*, where $E(\cdot/\mathfrak{B}_t)$ is a conditional expectation.

Under the assumption (AIV″), *an optimal controller is necessary to satisfy the above maximum principle.*

Proof) Let $y(t)$ be a response to an admissible controller $u(t)$. Noting $\eta^*(T)=-g_x(x^*(T))$, if we set $z(t)=x^*(t)-y(t)$, $u_z(t)=h(t,u^*(t))-h(t,u(t))$, we can obtain from lemma 3.1 that

$$\begin{aligned}&-g_x(x^*(T))(x^*(T)-y(T))\\&=\int_0^T f_x^0(s,x^*(s))(x^*(s)-y(s))ds\\&\quad+\int_0^T \eta^*(s)(h(s,u^*(s))-h(s,u(s)))ds\,.\end{aligned} \tag{3.8}$$

Combining (3.8) and the definition of the cost criteria $C(u^*)$, $C(u)$ in (2.6), since $u^*(s)$, $u(s)$, are $\mathfrak{B}_s$-measurable,

$$\begin{aligned}&C(u^*)-C(u)\\&=E[g(x^*(T))-g(y(T))-g_x(x^*(T))(x^*(T)-y(T))\\&\quad+\int_0^T\{f^0(s, x^*(s))-f^0(s, y(s))-f_x^0(s, x^*(s))(x^*(s)-y(s))\}\,ds\\&\quad+\int_0^T\{h^0(s,u^*(s))-h^0(s,u(s))-E(\eta^*(s)/\mathfrak{B}_s)(h(s,u^*(s))-h(s,u(s))\}ds\,].\end{aligned} \tag{3.9}$$

Since g, f are convex functions and $u^*(s)$ satisfies the maximum principle, we can prove cleary it is sufficient because of (3.9).

For the proof of the necessity, we owe much to the implicit function's lemma of Filippov type. Suppose that $u^*(t)$ fails to satisfy the maximum principle for some (t, ω)-set which measure is positive. Define a new controller $\tilde{u}(t)$ by

$$\begin{aligned}&-h^0(t,\tilde{u}(t))+E(\eta^*(t)/\mathfrak{B}_t)h(t,\tilde{u}(t))\\&=\max_{u\in K}[-h^0(t,u)+E(\eta^*(t)/\mathfrak{B}_t)h(t,u)]\quad T\geq t\geq 0\end{aligned} \tag{3.10}$$

where $\tilde{u}(t)$ can be chosen admissible as the help of the implicit function's lemma. We perturb the controller $u^*(t)$. On a small set where the maximum principle fails, we select $\tilde{u}(t)$ in place of $u^*(t)$. By such a perturbation, we have defined $u_\varepsilon(t)$ where an $\varepsilon>0$ denotes the measure of the above small set. The response $x_\varepsilon(t)$ to $u_\varepsilon(t)$ approximates $x^*(t)$ in L_2 and, since f_{xx}^0, g_{xx} are bounded, using the relation (3.9), we can prove

$$C(u^*)-C(u_\varepsilon)>0 \tag{3.11}$$

for a sufficiently small $\varepsilon>0$. This contradicts the optimality of $u^*(t)$. Q.E.D.

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