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Optimal Stopping Problems in a Stochastic and Fuzzy System

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In a stochastic and fuzzy environment, two kinds of stopping models are discussed and compared. The optimal fuzzy stopping times are given under the assumptions of monotonicity and regularity for stopping rules. Also, we find that fuzzy stopping times are favored in a comparison between fuzzy and classical stopping models.

Key Words: fuzzy stopping; stopping model; fuzzy stochastic system; optimal stopping times.

1. INTRODUCTION

The fuzzy random variable, which is a "fuzzy-number-valued" extension of classical random variables, was first studied by Puri and Ralescu [7] and has been discussed by many authors (Stojaković [10], Puri and Ralescu [8], etc.). It is one of the successful notions combining randomness and fuzziness.

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This paper deals with a new optimal stopping problem for "fuzzy stochastic systems" given by a sequence of fuzzy random variables. Classical stopping problems for a sequence of "real-valued" random variables have been studied by many authors, and their applications are well known in various fields (Presman and Sonin [6], Chow et al. [1], Shiryayev [9]). On the other hand, stopping models for dynamic fuzzy systems have been studied by Yoshida [12–14].

We discuss two kinds of optimization by stopping times regarding fuzzy random variables: One is by "classical" stopping times and the other is by "fuzzy" stopping times. Fuzzy stopping times as defined by Kurano *et al.* [4] are introduced in dynamic fuzzy systems. In comparisons of these stopping models in a numerical example, we find that the fuzzy stopping model is better than the classical one, and it will be concluded that the fuzzification of stopping times is effective in the case of fuzzy stochastic systems.

In Section 2, the notations and definitions of fuzzy random variables are given. Next, in Section 3, we discuss a stopping model by classical stopping times and we give an optimal stopping time for the model. In this paper, we estimate the randomness and fuzziness of fuzzy random variables using expectations and scalarization functions, respectively. In Section 4, fuzzy stopping times and their stopping model are introduced, and an optimal fuzzy stopping time for the model is given under the assumptions of monotonicity and regularity for stopping rules. Finally, in Section 5, we compare these two models using a numerical example.

2. FUZZY RANDOM VARIABLES

In this section, we give some notations for fuzzy random variables. Let $(\Omega, \mathscr{M}, P)$ be a probability space, where $\mathscr{M}$ is a σ-field and P is a nonatomic probability measure. Let $\mathbf{R}$ and $\mathbf{N}$ be the set of all real numbers and the set of all nonnegative integers, respectively. $\mathscr{B}$ denotes the Borel σ-field of $\mathbf{R}$ and $\mathscr{I}$ denotes the set of all bounded closed sub-intervals of $\mathbf{R}$. A fuzzy number is denoted by its membership function $\tilde{a}$: $\mathbf{R} \mapsto [0,1]$ which is normal, upper-semicontinuous, and fuzzy convex and has compact support. Refer to Zadeh [15] for the theory of fuzzy sets. $\mathscr{R}$ denotes the set of all fuzzy numbers. The α-cut of a fuzzy number $\tilde{a}(\in \mathscr{R})$ is given by

$$\tilde{a}_\alpha := \{x \in \mathbf{R} \mid \tilde{a}(x) \geq \alpha\} \quad (\alpha \in (0,1]) \qquad \text{and}$$
$$\tilde{a}_0 := \mathrm{cl}\{x \in \mathbf{R} \mid \tilde{a}(x) > 0\},$$

where cl denotes the closure of an interval. In this paper, we write the closed intervals by

$$\tilde{a}_\alpha := [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+] \qquad \text{for } \alpha \in [0,1].$$

A map $\tilde{X}: \Omega \mapsto \mathscr{R}$ is called a fuzzy random variable if

$$\{(\omega, x) \in \Omega \times \mathbf{R} \mid \tilde{X}(\omega)(x) \geq \alpha\} \in \mathscr{M} \times \mathscr{B} \qquad \text{for all } \alpha \in [0,1]. \quad (2.1)$$

The condition (2.1) is also written as

$$\{(\omega, x) \in \Omega \times \mathbf{R} \mid x \in \tilde{X}_\alpha(\omega)\} \in \mathscr{M} \times \mathscr{B} \qquad \text{for all } \alpha \in [0,1], \quad (2.2)$$

where $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] := \{x \in \mathbf{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ is the α-cut of the fuzzy number $\tilde{X}(\omega)$ for $\omega \in \Omega$. We can find some equivalent conditions [8, 10]. However, in this paper, we adopt a simple equivalent condition in the following lemma.

LEMMA 2.1 (Wang and Zhang [11, Theorems 2.1 and 2.2]). *For a map $\tilde{X}: \Omega \mapsto \mathscr{R}$, the following are equivalent*:

(i) *$\tilde{X}$ is a fuzzy random variable.*

(ii) *The maps $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are measurable for all $\alpha \in [0,1]$.*

Now we introduce expectations of fuzzy random variables for the description of stopping models for fuzzy stochastic systems. A fuzzy random variable $\tilde{X}$ is called integrably bounded if $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are integrable for all $\alpha \in [0,1]$. Let $\tilde{X}$ be an integrably bounded fuzzy random variable. We put closed intervals

$$E(\tilde{X})_\alpha := \left[\int_\Omega \tilde{X}_\alpha^-(\omega)\,\mathrm{d}P(\omega), \int_\Omega \tilde{X}_\alpha^+(\omega)\,\mathrm{d}P(\omega)\right], \qquad \alpha \in [0,1]. \quad (2.3)$$

Since the map $\alpha \mapsto E(\tilde{X})_\alpha$ is left-continuous by the monotone convergence theorem, the expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number [3, Lemma 3]

$$E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{E(\tilde{X})_\alpha}(x)\} \qquad \text{for } x \in \mathbf{R}, \quad (2.4)$$

where 1_D is the classical indicator function of a set D.

3. A STOPPING MODEL

In this section, we deal with a stopping model for fuzzy stochastic systems. Let $\{\tilde{X}_n\}_{n=0}^{\infty}$ be a sequence of integrably bounded fuzzy random variables such that $E(\sup_n \tilde{X}_{n,0)}^{+}) < \infty$, where $\tilde{X}_{n,0}^{+}(\omega)$ is the right end of the 0-cut of the fuzzy number $\tilde{X}_n(\omega)$ for $n = 0, 1, 2, \ldots$. For $n = 0, 1, 2, \ldots, \mathscr{M}_n$ denotes the smallest σ-field on Ω generated by all random variables $\tilde{X}_{k,\alpha}^{-}$ and $\tilde{X}_{k,\alpha}^{+}$ $(k = 0, 1, 2, \ldots, n;\ \alpha \in [0, 1])$, and $\mathscr{M}_\infty$ denotes the smallest σ-field generated by $\bigcup_{n=0}^{\infty} \mathscr{M}_n$. Then $\{\tilde{X}_n, \mathscr{M}_n, n \in \mathbf{N}\}$ is called a "fuzzy stochastic system." A map $\tau: \Omega \mapsto \mathbf{N} \cup \{\infty\}$ is said to be a stopping time if

$$\{\omega \mid \tau(\omega) = n\} \in \mathscr{M}_n \qquad \text{for all } n = 0, 1, 2, \ldots . \tag{3.1}$$

Then we have the following lemma.

LEMMA 3.1. *For a finite stopping time τ, we define*

$$\tilde{X}_\tau(\omega) := \tilde{X}_n(\omega), \qquad \omega \in \{\tau = n\} \qquad \textit{for } n = 0, 1, 2, \ldots . \tag{3.2}$$

Then $\tilde{X}_\tau$ is a fuzzy random variable.

Proof. This lemma follows from the definitions. ∎

Let $g: \mathscr{I} \mapsto \mathbf{R}$ be a σ-additively homogeneous map; that is, g satisfies

$$g\left(\sum_{n=0}^{\infty} c_n\right) = \sum_{n=0}^{\infty} g(c_n) \tag{3.3}$$

for bounded closed intervals $\{c_n\}_{n=0}^{\infty} \subset \mathscr{I}$ such that $\sum_{n=0}^{\infty} c_n \in \mathscr{I}$ and

$$g(\lambda c) = \lambda g(c) \tag{3.4}$$

for bounded closed intervals $c \in \mathscr{I}$ and real numbers $\lambda \geq 0$, where the operation on closed intervals is defined ordinarily as $\sum_{n=0}^{\infty} c_n := \mathrm{cl}\{\sum_{n=0}^{\infty} x_n \mid x_n \in c_n,\ n = 0, 1, 2, \ldots\}$ and $\lambda c := \{\lambda x \mid x \in c\}$. Weighting functions, which satisfy (3.3) and (3.4), are used for the evaluation of fuzzy numbers (Fortemps and Roubens [2]). Let τ be a finite stopping time. Now we consider the definition and evaluation of the fuzzy random variable $\tilde{X}_\tau$. Let $\omega \in \Omega$. From (3.2), the α-cut of the fuzzy number $\tilde{X}_\tau(\omega)$ must be a closed interval $\tilde{X}_{\tau(\omega),\alpha}(\omega)$. Therefore, from the definition (2.3), the expectation is given by the closed interval

$$E\left(\tilde{X}_{\tau,\alpha}\right). \tag{3.5}$$

Using the scalarization function g in (3.3) and (3.4), we give its estimation by

$$g\left(E\left(\tilde{X}_{\tau,\alpha}\right)\right). \tag{3.6}$$

Therefore the evaluation of the fuzzy random variable $\tilde{X}_\tau$ is given by the integral

$$\int_0^1 g\left(E\left(\tilde{X}_{\tau,\alpha}\right)\right) \mathrm{d}\alpha. \tag{3.7}$$

LEMMA 3.2. *For a finite stopping time τ, it holds that*

$$\int_0^1 g\left(E\left(\tilde{X}_{\tau,\alpha}\right)\right) \mathrm{d}\alpha = \int_0^1 E\left(g\left(\tilde{X}_{\tau,\alpha}\right)\right) \mathrm{d}\alpha = E\left(\int_0^1 g\left(\tilde{X}_{\tau,\alpha}(\cdot)\right) \mathrm{d}\alpha\right). \tag{3.8}$$

Proof. The properties (3.3) and (3.4) of g imply

$$g\left(E\left(\tilde{X}_{\tau,\alpha}\right)\right) = E\left(g\left(\tilde{X}_{\tau,\alpha}\right)\right).$$

Therefore

$$\int_0^1 g\left(E\left(\tilde{X}_{\tau,\alpha}\right)\right) \mathrm{d}\alpha = \int_0^1 E\left(g\left(\tilde{X}_{\tau,\alpha}\right)\right) \mathrm{d}\alpha.$$

Also, by Fubini's theorem, we have

$$\int_0^1 E\left(g\left(\tilde{X}_{\tau,\alpha}\right)\right) \mathrm{d}\alpha = E\left(\int_0^1 g\left(\tilde{X}_{\tau,\alpha}(\cdot)\right) \mathrm{d}\alpha\right).$$

These complete the proof of this lemma. ∎

From Lemma 3.2, for a finite stopping time τ, we define a random variable

$$G_\tau(\omega) := \int_0^1 g\left(\tilde{X}_{\tau,\alpha}(\omega)\right) \mathrm{d}\alpha, \qquad \omega \in \Omega. \tag{3.9}$$

Note that since $g(\tilde{X}_{\tau,\alpha}(\omega))$ is a real number and the function $\alpha \mapsto g(\tilde{X}_{\tau,\alpha}(\omega))$ is left-continuous on $(0,1]$, (3.9) is well defined. Thus, the expectation $E(G_\tau)$ is the estimation (3.7) of the fuzzy stochastic system stopped at a finite stopping time τ. In this approach, for a stopping time which is not necessarily finite, we can also define a random variable

$$G_\tau(\omega) := \begin{cases} \int_0^1 g\left(\tilde{X}_{\tau,\alpha}(\omega)\right) \mathrm{d}\alpha & \text{if } \tau(\omega) < \infty \\ \int_0^1 \limsup_{n\to\infty} g\left(\tilde{X}_{n,\alpha}(\omega)\right) \mathrm{d}\alpha & \text{if } \tau(\omega) = \infty \end{cases} \quad \text{for } \omega \in \Omega. \tag{3.10}$$

Now we present the following optimal stopping problem for fuzzy stochastic systems.

Problem 1. Find a finite stopping time τ^* such that $E(G_{\tau^*}) \geq E(G_\tau)$ for all finite stopping times τ.

Then τ^* is called an "optimal stopping time." To consider this stopping problem, we define random variables

$$Z_n := \operatorname*{ess\,sup}_{\tau:\text{stopping times},\, \tau \geq n} E(G_\tau | \mathscr{M}_n) \tag{3.11}$$

for $n = 0, 1, 2, \ldots$. Refer to [5, Proposition 6-1-1; 1, Chaps. 1–6] regarding the definition of the essential supremum. Define a stopping time

$$\sigma^*(\omega) := \inf\{n \mid G_n(\omega) = Z_n(\omega)\}, \qquad \omega \in \Omega, \tag{3.12}$$

where the infimum of the empty set is understood to be $+\infty$. Then we immediately obtain the following result from Chow *et al*. [1, Theorems 4.1 and 4.5].

THEOREM 3.1. *If* $P(\sigma^* < \infty) = 1$, *then* σ^* *is an optimal stopping time for Problem* 1.

4. A "FUZZY" STOPPING MODEL

In this section, we introduce a "fuzzy" stopping time for the fuzzy stochastic system $\{\tilde{X}_n, \mathscr{M}_n, n \in \mathbf{N}\}$ defined in Section 3, and we discuss a fuzzy stopping model.

DEFINITION 4.1. A map $\tilde{\tau}$: $\mathbf{N} \times \Omega \mapsto [0, 1]$ is called a fuzzy stopping model if it satisfies the following:

(i) For each $n = 0, 1, 2, \ldots$, the map $\omega \mapsto \tilde{\tau}(n, \omega)$ is $\mathscr{M}_n$-measurable.

(ii) For almost all $\omega \in \Omega$, the map $n \mapsto \tilde{\tau}(n, \omega)$ is non-increasing.

(iii) For almost all $\omega \in \Omega$, there exists an integer n_0 such that $\tilde{\tau}(n, \omega) = 0$ for all $n \geq n_0$.

Definition 4.1 is similar to the idea of fuzzy stopping times given for dynamic fuzzy systems by Kurano *et al*. [4]. Regarding the membership grade of fuzzy stopping times, $\tilde{\tau}(n, \omega) = 0$ means "to stop time at n" and $\tilde{\tau}(n, \omega) = 1$ means "to continue at time n," respectively. We can easily check the following lemma regarding the properties of fuzzy stopping times (see [4]).

LEMMA 4.1. (i) *Let $\tilde{\tau}$ be a fuzzy stopping time. Define a map $\tilde{\tau}_\alpha$: $\Omega \mapsto$ $\mathbf{N}$ by*

$$\tilde{\tau}_\alpha(\omega) := \inf\{n \mid \tilde{\tau}(n, \omega) < \alpha\}, \quad \omega \in \Omega \qquad \textit{for } \alpha \in (0,1], \quad (4.1)$$

where the infimum of the empty set is understood to be $+\infty$. Then we have

(a) $\{\tilde{\tau}_\alpha \leq n\} \in \mathscr{M}_n$ *for* $n = 0, 1, 2, \ldots$;

(b) $\tilde{\tau}_\alpha(\omega) \leq \tilde{\tau}_{\alpha'}(\omega)$ *a.a.* $\omega \in \Omega$ *if* $\alpha \geq \alpha'$;

(c) $\lim_{\alpha' \uparrow \alpha} \tilde{\tau}_{\alpha'}(\omega) = \tilde{\tau}_\alpha(\omega)$ *a.a.* $\omega \in \Omega$ *if* $\alpha > 0$;

(d) $\tilde{\tau}_0(\omega) := \lim_{\alpha \downarrow 0} \tilde{\tau}_\alpha(\omega) < \infty$ *a.a.* $\omega \in \Omega$.

(ii) *Let* $\{\tilde{\tau}_\alpha\}_{\alpha \in [0,1]}$ *be maps* $\tilde{\tau}_\alpha$: $\Omega \mapsto \mathbf{N}$ *satisfying* (a), (b), *and* (d) *above. Define a map* $\tilde{\tau}$: $\mathbf{N} \times \Omega \mapsto [0, 1]$ *by*

$$\tilde{\tau}(n, \omega) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{\{\tilde{\tau}_\alpha > n\}}(\omega)\}$$

$$\textit{for } n = 0, 1, 2, \ldots \quad \textit{and} \quad \omega \in \Omega. \quad (4.2)$$

Then $\tilde{\tau}$ is a fuzzy stopping time.

Now we consider the estimation of the fuzzy stochastic system stopped at a fuzzy stopping time $\tilde{\tau}$. Let $\omega \in \Omega$. A fuzzy stopping time $\tilde{\tau}$ is called finite if $\tilde{\tau}_0(\omega) := \lim_{\alpha \downarrow 0} \tilde{\tau}_\alpha(\omega) < \infty$ for almost all $\omega \in \Omega$. Let $\tilde{\tau}$ be a finite fuzzy stopping time. From Lemma 4.1(i.a), its α-cut is $\tilde{X}_{\tilde{\tau}_\alpha, \alpha}(\omega) := \tilde{X}_{\tilde{\tau}_\alpha(\omega), \alpha}(\omega)$, where $\tilde{\tau}_\alpha(\omega)$ is a "classical" stopping time given by (4.1). Therefore, similarly to (3.9), we define a random variable

$$G_{\tilde{\tau}}(\omega) := \int_0^1 g\left(\tilde{X}_{\tilde{\tau}_\alpha, \alpha}(\omega)\right) \mathrm{d}\alpha, \qquad \omega \in \Omega. \quad (4.3)$$

Note that the function $\alpha \mapsto g(\tilde{X}_{\tilde{\tau}_\alpha, \alpha}(\omega))$ is left-continuous on $0 < \alpha \leq 1$, so that (4.3) is well defined. Therefore the expectation $E(G_{\tilde{\tau}})$ is the evaluation of the fuzzy random variable $\tilde{X}_{\tilde{\tau}}$. In this section, we discuss the following problem.

Problem 2. Find a fuzzy stopping time $\tilde{\tau}^*$ such that $E(G_{\tilde{\tau}^*}) \geq E(G_{\tilde{\tau}})$ for all fuzzy stopping times $\tilde{\tau}$.

In Problem 2, $\tilde{\tau}^*$ is called an "optimal fuzzy stopping time." On the other hand, by Fubini's theorem, we have

$$E(G_{\tilde{\tau}}) := E\left(\int_0^1 g\left(\tilde{X}_{\tilde{\tau}_\alpha, \alpha}(\cdot)\right) \mathrm{d}\alpha\right) = \int_0^1 E\left(g\left(\tilde{X}_{\tilde{\tau}_\alpha, \alpha}\right)\right) \mathrm{d}\alpha \qquad (4.4)$$

for a fuzzy stopping time $\tilde{\tau}$. For a fuzzy stopping time which is not necessarily finite, we can also define a random variable

$$G_{\tilde{\tau}}(\omega) := \int_{\alpha^\infty}^1 g\left(\tilde{X}_{\tilde{\tau}_\alpha, \alpha}(\omega)\right) \mathrm{d}\alpha + \int_0^{\alpha^\infty} \limsup_{n \to \infty} g\left(\tilde{X}_{n, \alpha}(\omega)\right) \mathrm{d}\alpha \qquad \text{for } \omega \in \Omega,$$

where

$$\alpha^\infty := \begin{cases} \inf\{\alpha \mid \tilde{\tau}_\alpha(\omega) < \infty\} & \text{if } \tilde{\tau}_1(\omega) < \infty \\ 1 & \text{if } \tilde{\tau}_1(\omega) = \infty. \end{cases}$$

In order to analyze Problem 2, we first need to discuss the following subproblem induced from (4.4).

Problem 3. Let $\alpha \in [0, 1]$. Find a stopping time τ^* such that $E(g(\tilde{X}_{\tau^*, \alpha})) \geq E(g(\tilde{X}_{\tau, \alpha}))$ for all stopping times τ.

In Problem 3, τ^* is called an "α-optimal stopping time." Let $\alpha \in [0, 1]$. We define a sequence of subsets $\{\Lambda_n\}_{n=0}^\infty$ of Ω by

$$\Lambda_n := \left\{\omega \,|\, g\left(\tilde{X}_{n, \alpha}(\omega)\right) \geq E\left(g\left(\tilde{X}_{n+1, \alpha}\right) | \mathscr{M}_n\right)(\omega)\right\} \qquad \text{for } n = 0, 1, 2, \dots .$$

This paper deals with the following monotone case (Chow *et al.* [1]).

Assumption A (Monotone case). The following conditions hold almost surely:

$$\Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots \qquad \text{and} \qquad \bigcup_{n=0}^\infty \Lambda_n = \Omega.$$

In order to characterize α-optimal stopping times, we let

$$\gamma_n^\alpha := \operatorname*{ess\,sup}_{\tau:\text{ stopping times}, \tau \geq n} E\left(g\left(\tilde{X}_{\tau, \alpha}\right) | \mathscr{M}_n\right) \qquad \text{for } n = 0, 1, 2, \dots . \qquad (4.5)$$

We define a stopping time $\sigma_\alpha^*: \Omega \mapsto \mathbf{N}$ by

$$\sigma_\alpha^*(\omega) := \inf\left\{n \mid g\left(\tilde{X}_{n,\alpha}(\omega)\right) = \gamma_n^\alpha(\omega)\right\} \tag{4.6}$$

for $\omega \in \Omega$ and $\alpha \in [0,1]$, where the infimum of the empty set is understood to be $+\infty$. Then the next lemma is as given by Chow *et al*. [1, Theorems 4.1 and 4.5].

LEMMA 4.2. *Let* $\alpha \in [0,1]$. *Suppose Assumption* A *holds*. *Then*, *the following* (i) *and* (ii) *hold*:

(i) $\gamma_n^\alpha(\omega) = \max\{g(\tilde{X}_{n,\alpha}(\omega)), \gamma_{n+1}^\alpha(\omega)\}$ *a.a.* $\omega \in \Omega$ *for* $n = 0, 1, 2, \ldots$;

(ii) σ_α^* *is* α*-optimal and* $E(\gamma_0^\alpha) = E(g(\tilde{X}_{\sigma_\alpha^*,\alpha}))$.

In order to construct an optimal fuzzy stopping time from the α-optimal stopping time $\{\sigma_\alpha^*\}_{\alpha \in [0,1]}$, we need the following regularity condition.

Assumption B (Regularity). The map $\alpha \mapsto \sigma_\alpha^*(\omega)$ is non-increasing for almost all $\omega \in \Omega$.

Under Assumption B, we can define a map $\tilde{\sigma}^*: \mathbf{N} \times \Omega \mapsto [0,1]$ by

$$\tilde{\sigma}^*(n, \omega) := \sup_{\alpha \in [0,1]} \min\left\{\alpha, 1_{\{\sigma_\alpha^* > n\}}(\omega)\right\}$$

$$\text{for } n = 0, 1, 2, \ldots \quad \text{and} \quad \omega \in \Omega. \tag{4.7}$$

We put the α-cut (4.1) of $\tilde{\sigma}^*(n, \omega)$ by $\tilde{\sigma}_\alpha^*(\omega)$. Then we note that $\tilde{\sigma}_\alpha^*(\omega)$ and $\sigma_\alpha^*(\omega)$ may not equal at most countable many $0 < \alpha \le 1$.

THEOREM 4.1. *Suppose Assumptions* A *and* B *hold*. *Then* $\tilde{\sigma}^*$ *is an optimal fuzzy stopping time for Problem* 2. *Further it holds that*

$$\tilde{\sigma}_\alpha^*(\omega) := \min\{n \mid \tilde{\sigma}^*(n, \omega) < \alpha\}, \quad \omega \in \Omega \qquad \textit{for } \alpha \in (0,1]. \tag{4.8}$$

Proof. From Assumption B and Lemma 4.1, $\tilde{\sigma}^*$ is a fuzzy stopping time. Lemma 4.2 implies

$$E(G_{\tilde{\tau}}) \le \int_0^1 \sup_\tau E\left(g\left(\tilde{X}_{\tau,\alpha}\right)\right) \mathrm{d}\alpha = \int_0^1 E(\gamma_0^\alpha) \, \mathrm{d}\alpha = \int_0^1 E\left(g\left(\tilde{X}_{\sigma_\alpha^*,\alpha}\right)\right) \mathrm{d}\alpha \tag{4.9}$$

for all fuzzy stopping times $\tilde{\tau}$. Since $\tilde{\sigma}_\alpha^*(\omega) \neq \sigma_\alpha^*(\omega)$ holds only at most countable $0 < \alpha \le 1$, we have

$$E\left(\int_0^1 g\left(\tilde{X}_{\sigma_\alpha^*,\alpha}\right) \mathrm{d}\alpha\right) = E\left(\int_0^1 g\left(\tilde{X}_{\tilde{\sigma}_\alpha^*,\alpha}\right) \mathrm{d}\alpha\right). \tag{4.10}$$

By (4.9), (4.10), and Fubini's theorem, we obtain

$$E(G_{\tilde{\tau}}) \le \int_0^1 E\Big(g\big(\tilde{X}_{\sigma_\alpha^*,\alpha}\big)\Big)\,\mathrm{d}\alpha = E\Big(\int_0^1 g\big(\tilde{X}_{\tilde{\sigma}_\alpha^*,\alpha}\big)\,\mathrm{d}\alpha\Big)$$

$$= \int_0^1 E\Big(g\big(\tilde{X}_{\tilde{\sigma}_\alpha^*,\alpha}\big)\Big)\,\mathrm{d}\alpha = E(G_{\tilde{\sigma}^*}). \tag{4.11}$$

Therefore $\tilde{\sigma}^*$ is optimal for Problem 2. Finally, (4.8) is trivial from Lemma 4.1. ▮

The following result implies a comparison between the optimal values of the "classical" stopping model (Problem 1) and the "fuzzy" stopping model (Problem 2). Then we find that the fuzzy stopping model is more favorable than the classical one.

COROLLARY 4.1. *It holds that*

$$E(G_{\tau^*}) \le E(G_{\tilde{\sigma}^*}), \tag{4.12}$$

where τ^* *is the optimal stopping time and* $\tilde{\sigma}^*$ *is the optimal fuzzy stopping time.*

Proof. For all stopping times τ, we have

$$E(G_\tau) = E\Big(\int_0^1 g\big(\tilde{X}_{\tau,\alpha}\big)\,\mathrm{d}\alpha\Big) \le \int_0^1 \sup_\tau E\Big(g\big(\tilde{X}_{\tau,\alpha}\big)\Big)\,\mathrm{d}\alpha = E(G_{\tilde{\sigma}^*}).$$

Therefore this corollary holds. ▮

In the next section, we compare the optimal values of Corollary 4.1 through a numerical example.

5. A NUMERICAL EXAMPLE

An example is given to compare the results of the stopping models in Sections 3 and 4. Let $\{Y_n\}_{n=0}^{\infty}$ be a uniform integrable sequence of independent, identically distributed real random variables. Let c and d be constants satisfying $0 < d < 3c$. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence given by $a_n := d(n+1)$ for $n = 0, 1, 2, \ldots$. Put

$$M_n := \max\{Y_0, Y_1, Y_2, \ldots, Y_n\} - c(n+1) \qquad \text{for } n = 0, 1, 2, \ldots. \tag{5.1}$$

Hence we take a sequence of fuzzy random variables $\{\tilde{X}_n\}_{n=0}^{\infty}$ as

$$\tilde{X}_n(\omega)(x) := \begin{cases} L((M_n(\omega) - x)/a_n) & \text{if } x \leq M_n(\omega) \\ L((x - M_n(\omega))/a_n) & \text{if } x \geq M_n(\omega) \end{cases} \tag{5.2}$$

for $n = 0, 1, 2, \ldots, \omega \in \Omega$, and $x \in \mathbf{R}$, where the shape function $L(x) := \max\{1 - |x|, 0\}$ for $x \in \mathbf{R}$ (see Fig. 5.1). Then their α-cuts are

$$\tilde{X}_{n,\alpha}(\omega) = [M_n(\omega) - (1-\alpha)a_n, M_n(\omega) + (1-\alpha)a_n], \qquad \omega \in \Omega$$

for $n = 0, 1, 2, \ldots$ and $\alpha \in [0, 1]$.

Let a weighting function $g([x, y]) := (x + 2y)/3$ for $x, y \in \mathbf{R}$ satisfying $x \leq y$. Then g satisfies the properties (3.3) and (3.4), and we can easily check that

$$g(\tilde{X}_{n,\alpha}(\omega)) = M_n(\omega) + \frac{1-\alpha}{3}a_n, \qquad \omega \in \Omega$$

for $\alpha \in [0, 1]$, and so

$$G_n(\omega) = \int_0^1 g(\tilde{X}_{n,\alpha}(\omega))\, d\alpha = M_n(\omega) + \tfrac{1}{6}a_n, \qquad \omega \in \Omega.$$

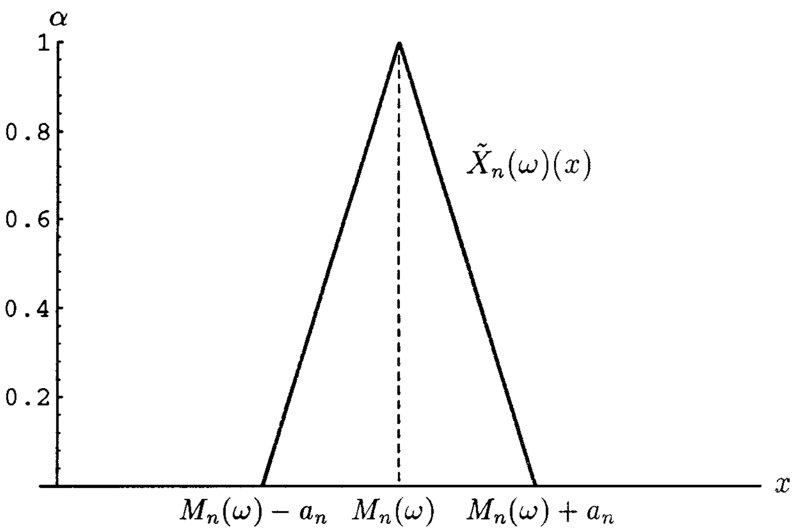

FIG. 1. Fuzzy random variable $\tilde{X}_n(\omega)(x)$.

Next we check Assumptions A and B. Let $\alpha, \alpha' \in [0,1]$ satisfy $\alpha' \le \alpha$ and let $\omega \in \Omega$. If $g(\tilde{X}_{n,\alpha'}(\omega)) = \gamma_n^{\alpha'}(\omega)$ for some n, then we have

$$\begin{aligned}
g\big(\tilde{X}_{n,\alpha}(\omega)\big) &= M_n(\omega) + \frac{1-\alpha}{3}a_n \\
&= M_n(\omega) + \frac{1-\alpha'}{3}a_n + \frac{\alpha'-\alpha}{3}a_n \\
&= g\big(\tilde{X}_{n,\alpha'}(\omega)\big) + \frac{\alpha'-\alpha}{3}a_n \\
&= \gamma_n^{\alpha'}(\omega) + \frac{\alpha'-\alpha}{3}a_n \\
&\ge E\Big(g\big(\tilde{X}_{\tau,\alpha'}\big)\,|\mathscr{M}_n\Big)(\omega) + \frac{\alpha'-\alpha}{3}E(a_\tau|\mathscr{M}_n)(\omega) \\
&= E\bigg(g\big(\tilde{X}_{\tau,\alpha'}\big) + \frac{\alpha'-\alpha}{3}a_\tau\,|\mathscr{M}_n\bigg)(\omega) \\
&= E\Big(g\big(\tilde{X}_{\tau,\alpha}\big)\,|\mathscr{M}_n\Big)(\omega) \qquad \text{a.a. } \omega \in \Omega
\end{aligned}$$

for all stopping times τ such that $\tau \ge n$. It follows $g(\tilde{X}_{n,\alpha}(\omega)) = \gamma_n^{\alpha}(\omega)$. Therefore we obtain $\tilde{\sigma}_\alpha^*(\omega) \le \tilde{\sigma}_{\alpha'}^*(\omega)$ for almost all $\omega \in \Omega$, and Assumption B is fulfilled. On the other hand, we have

$$\begin{aligned}
g\big(\tilde{X}_{n,\alpha}(\omega)\big) &= M_n(\omega) + \frac{1-\alpha}{3}a_n \\
&= \max\{Y_0(\omega), Y_1(\omega), Y_2(\omega), \dots, Y_n(\omega)\} \\
&\qquad - c(n+1) + \frac{1-\alpha}{3}d(n+1) \\
&= \max\{Y_0(\omega), Y_1(\omega), Y_2(\omega), \dots, Y_n(\omega)\} \\
&\qquad - \left(c - \frac{1-\alpha}{3}d\right)(n+1).
\end{aligned}$$

Since $c - \frac{1-\alpha}{3}d \ge c - \frac{1}{3}d > 0$, this is the monotone case (Assumption A) from Chow *et al*. [1, Chaps. 3–5 (3.22)]. Then the finite α-optimal stopping times in Problem 3 are

$$\sigma_\alpha^*(\omega) = \inf\Big\{n \,|\, g\big(\tilde{X}_{n,\alpha}(\omega)\big) = \gamma_n^\alpha(\omega)\Big\} = \inf\{n \,|\, Y_n(\omega) \ge \beta(\alpha)\}, \qquad \omega \in \Omega, \quad (5.3)$$

where $\beta(\alpha)$ is a constant and is the unique solution $\beta = \beta(\alpha)$ of the equation

$$E\left((Y_0 - \beta)^+\right) = c - \frac{1-\alpha}{3}d$$

for $\alpha \in [0,1]$. Therefore the optimal fuzzy stopping time in Problem 2 is

$$\begin{aligned}\tilde{\sigma}^*(n,\omega) &= \sup_{\alpha\in[0,1]} \min\left\{\alpha, 1_{\{\sigma_\alpha^* > n\}}(\omega)\right\} \\ &= \sup\left\{\alpha \in [0,1] \mid Y_n(\omega) < \beta(\alpha)\right\}, \qquad (5.4)\end{aligned}$$

for $n = 0,1,2,\dots$ and $\omega \in \Omega$. We can similarly check that the finite optimal stopping time in Problem 1 is

$$\tau^*(\omega) = \inf\{n \mid G_n(\omega) = Z_n(\omega)\} = \inf\{n \mid Y_n(\omega) \geq \beta(1/2)\}, \qquad \omega \in \Omega. \quad (5.5)$$

Finally we compare the optimal expected values in the both models. From Chow *et al*. [1, Chaps. 3–5 (3.22)], the optimal expected value in Problem 1 is given by

$$E(G_{\tau^*}) = \int_0^1 E\left(g\left(\tilde{X}_{\tau^*,\alpha}\right)\right) \mathrm{d}\alpha = \beta(1/2).$$

On the other hand, we have

$$E\left(g\left(\tilde{X}_{\sigma_\alpha^*,\alpha}\right)\right) = \beta(\alpha)$$

for $\alpha \in [0,1]$. Therefore, from (4.11), we obtain the optimal expected value in Problem 2 as

$$E(G_{\tilde{\sigma}^*}) = \int_0^1 E\left(g\left(\tilde{X}_{\sigma_\alpha^*,\alpha}\right)\right) \mathrm{d}\alpha = \int_0^1 \beta(\alpha)\,\mathrm{d}\alpha.$$

For example, if $Y_0, Y_1, Y_2, \dots$ are independent and uniformly distributed on [0, 1], then

$$\beta(\alpha) = \begin{cases} \dfrac{1}{2} - c + \dfrac{1-\alpha}{3}d & \text{if } \dfrac{1}{2} - c + \dfrac{1-\alpha}{3}d < 0 \\ 1 - \sqrt{2\left(c - \dfrac{1-\alpha}{3}d\right)} & \text{if } \dfrac{1}{2} - c + \dfrac{1-\alpha}{3}d \geq 0. \end{cases}$$

Let $c = 2/3$ and $d = 1$. Then the optimal expected value for the "classical" stopping problem in Problem 1 is

$$E(G_{\tau^*}) = \beta(1/2) = 0. \tag{5.6}$$

On the other hand, we can easily check that the optimal expected values for the "fuzzy" stopping problem in Problem 2 are

$$E(G_{\tilde{\tau}^*}) = \beta(\alpha) = \begin{cases} \dfrac{1-2\alpha}{6} & \text{if } \dfrac{1}{2} < \alpha \le 1 \\ 1 - \sqrt{\dfrac{2(1+\alpha)}{3}} & \text{if } 0 \le \alpha \le \dfrac{1}{2} \end{cases}$$

and

$$E(G_{\tilde{\sigma}^*}) = \int_0^1 \beta(\alpha)\, d\alpha = \frac{11}{24} - \frac{1}{3\sqrt{3}} \approx 0.26588. \tag{5.7}$$

The difference between (5.6) and (5.7) implies the excellence of fuzzy stopping times rather than classical stopping times.

Finally we consider a special case. By letting d equal zero in this example, the fuzzy random variables $\tilde{X}_n$ are reduced to the classical random variables. Then we can easily check $E(G_{\tau^*}) = E(G_{\tilde{\sigma}^*})$. This shows that the fuzzification of stopping times is effective for "fuzzy" stochastic systems, but not for "classical" stochastic systems.

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