Approximation of fuzzy neural networks by using Lusin’s theorem

Jun Li; Commun. Univ. of China, China
Jianzeng Li; Commun. Univ. of China, China
Masami Yasuda; Chiba University, Japan

September 23, 2006
Abstract

In this note, we study an approximation property of regular fuzzy neural network (RFNN). It is shown that any fuzzy-valued measurable function can be approximated by the four-layer RFNN in the sense of fuzzy integral norm for the finite sub-additive fuzzy measure on $\mathbb{R}$.

Keywords:
Fuzzy measure; Lusin’s theorem; Approximation; Regular fuzzy neural network
Introduction
In neural network theory, the learning ability of a neural network is closely related to its approximating capabilities, so it is important and interesting to study the approximation properties of neural networks.
The studies on this matter were undertaken by many authors and a great number of important results were obtained.

- J.G Attali, G. Pages, Approximation of functions by a multilayer perceptron: a new approach, *Neural Networks* 10(1997) 1069-1081,
The similar approximation problems in fuzzy environment were investigated by

- J.J Buckley, Y. Hayashi, Fuzzy input-output controllers are universal approximators, *Fuzzy Sets and Systems* 58(1993) 273-278,
- J.J Buckley, Y. Hayashi, Can fuzzy neural nets approximate continuous fuzzy function, *Fuzzy Sets and Systems* 61(1994) 43-51,
In P. Liu (2001) proved that continuous fuzzy-valued function can be closely approximated by a class of regular fuzzy neural networks (RFNNs) with real input and fuzzy-valued output. In this note, by using Lusin’s theorem on fuzzy measure space, we show that such RFNNs is pan-approximator for fuzzy-valued measurable function. That is, any fuzzy-valued measurable function can be approximated by the four-layer RFNNs in the sense of fuzzy integral norm for the finite sub-additive measure on $\mathbb{R}$. 
Preliminaries
Preliminaries 1/3

We suppose that $(X, \rho)$ is a metric space, and that $\mathcal{O}$ and $\mathcal{C}$ are the classes of all open and closed sets in $(X, \rho)$, respectively, and $\mathcal{B}$ is Borel $\sigma$-algebra on $X$, i.e., it is the smallest $\sigma$-algebra containing $\mathcal{O}$. A set function $\mu : \mathcal{B} \rightarrow [0, +\infty)$ is called a fuzzy measure (Narukawa/Murofushi(2004)), if it satisfies the following properties:

(FM1) $\mu(\emptyset) = 0$;
(FM2) $A \subset B$ implies $\mu(A) \leq \mu(B)$.

A fuzzy measure $\mu$ is called null-additive (Wang/Klir(1992)), if for any $E, F \in \mathcal{B}$ and $\mu(F) = 0$ imply $\mu(E \cup F) = \mu(E)$; sub-additive (Pap(1995)), if for any $E, F \in \mathcal{B}$ we have $\mu(E \cup F) \leq \mu(E) + \mu(F)$. 
In this paper, we always assume that $\mu$ is a finite, sub-additive and continuous fuzzy measure on $B$.
Consider a nonnegative real-valued measurable function $f$ on $A$ and the fuzzy integral of $f$ on $A$ with respect to $\mu$, which is denoted by

$$(S) \int_A f \, d\mu$$

$$\triangleq \sup_{0 \leq \alpha < +\infty} [\alpha \wedge \mu(\{x : f(x) \geq \alpha\} \cap A)]$$
Theorem 0.1 (Lusin’s theorem (Li/Yasuda(2004), Song/Li(2003)))

Let \((X, \rho)\) be metric space and \(\mu\) be null additive fuzzy measure on \(\mathcal{B}\). If \(f\) is a real-valued measurable function on \(E \in \mathcal{B}\), then, for every \(\epsilon > 0\), there exists a closed subset \(F_\epsilon \in \mathcal{B}\) such that \(f\) is continuous on \(F_\epsilon\) and \(\mu(E - F_\epsilon) < \epsilon\).
Approximation in fuzzy mean by RFNN
In this section, we study an approximation property of the four-layer RFNNs to fuzzy-valued measurable function in the sense of fuzzy integral norm for fuzzy measure on $\mathbb{R}$. 

Approximation 2/10

Let \( \mathcal{F}_0(\mathbb{R}) \) be the set of all bounded fuzzy numbers, i.e., for \( \tilde{A} \in \mathcal{F}_0(\mathbb{R}) \), the following conditions hold:

(i) \( \forall \alpha \in (0, 1], \tilde{A}_\alpha \triangleq \{ x \in \mathbb{R} \mid \tilde{A}(x) \geq \alpha \} \) is the closed interval of \( \mathbb{R} \);
(ii) The support \( \text{Supp}(\tilde{A}) \triangleq \text{cl}\{ x \in \mathbb{R} \mid \tilde{A}(x) > 0 \} \subset \mathbb{R} \) is a bounded set;
(iii) \( \{ x \in \mathbb{R} \mid \tilde{A}(x) = 1 \} \neq \emptyset \).

For simplicity, \( \text{supp}(\tilde{A}) \) is also written as \( \tilde{A}_0 \). Obviously, \( \tilde{A}_0 \) is a bounded and closed interval of \( \mathbb{R} \). For \( \tilde{A} \in \mathcal{F}_0(\mathbb{R}) \), let \( \tilde{A}_\alpha = [a_\alpha^-, a_\alpha^+] \) for each \( \alpha \in [0, 1] \) and we denote

\[
|\tilde{A}| \triangleq \bigvee_{\alpha \in [0, 1]} (|a_\alpha^-| \lor |a_\alpha^+|).
\]
**Proposition 0.1**  \textit{Liu(2001)} Assume $\tilde{A}$, $\tilde{A}_1$, $\tilde{A}_2 \in \mathcal{F}_0(\mathbb{R})$, and $\tilde{W}_i, \tilde{V}_i \in \mathcal{F}_0(\mathbb{R})$ ($i = 1, 2, \ldots, n$).

Then

1. $d(\tilde{A} \cdot \tilde{A}_1, \tilde{A} \cdot \tilde{A}_2) \leq |\tilde{A}| \cdot d(\tilde{A}_1, \tilde{A}_2)$,

2. $d\left(\sum_{i=1}^{n} \tilde{W}_i, \sum_{i=1}^{n} \tilde{V}_i\right) \leq \sum_{i=1}^{n} d(\tilde{W}_i, \tilde{V}_i)$. 
For \( \tilde{A}, \tilde{B} \in \mathcal{F}_0(\mathbb{R}) \), define metric \( d(\tilde{A}, \tilde{B}) \) between \( \tilde{A} \) and \( \tilde{B} \) by

\[
d(\tilde{A}, \tilde{B}) \triangleq \sup_{\alpha \in [0,1]} d_H(\tilde{A}_\alpha, \tilde{B}_\alpha)
\]

where \( d_H \) means Hausdorff metric: for \( A, B \subset \mathbb{R} \),

\[
d_H(A, B) \triangleq \max \left\{ \sup_{x \in A} \inf_{y \in B} (|x - y|), \sup_{y \in B} \inf_{x \in A} (|x - y|) \right\}.
\]
Approximation 5/10

It is known that $(\mathcal{F}_0(\mathbb{R}), d)$ is a completely separable metric space (Diamond/Kloeden(1994)).

Let $T$ be a measurable set in $\mathbb{R}^n$, $(T, \mathcal{B} \cap T, \mu)$ finite fuzzy measure space. Let $\mathcal{L}(T)$ denote the set of all fuzzy-valued measurable function

$$\tilde{F} : T \to \mathcal{F}_0(\mathbb{R}).$$

For any $\tilde{F}_1, \tilde{F}_2 \in \mathcal{L}(T)$, $d(\tilde{F}_1, \tilde{F}_2)$ is measurable function on $(T, \mathcal{B} \cap T)$, we will write a fuzzy integral norm as

$$\triangle_S(\tilde{F}_1, \tilde{F}_2) \triangleq (S) \int_T d(\tilde{F}_1, \tilde{F}_2) d\mu.$$
Proposition 0.2 Let $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3 \in \mathcal{L}(T)$, then

$$\triangle_S(\tilde{F}_1, \tilde{F}_3) \leq 2(\triangle_S(\tilde{F}_1, \tilde{F}_2) + \triangle_S(\tilde{F}_2, \tilde{F}_3)).$$
Proof. From subadditivity of $\mu$, we have

$$\triangle_S(\tilde{F}_1, \tilde{F}_3) = (S) \int_T d(\tilde{F}_1, \tilde{F}_3) d\mu$$

$$= \bigvee_{\alpha \in [0,\infty)} \{ \alpha \wedge \mu(T \cap (d(\tilde{F}_1, \tilde{F}_3))_\alpha) \}$$

$$\leq \bigvee_{\alpha \in [0,\infty)} \{ \alpha \wedge \mu(T \cap (d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}} \cup d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})) \}$$

$$\leq \bigvee_{\alpha \in [0,\infty)} \{ \alpha \wedge [\mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}}) + \mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})] \}.$$
we have

\[ \triangle_S(\tilde{F}_1, \tilde{F}_3) \leq \bigvee_{\alpha \in [0, \infty)} \{ \alpha \land \mu(T \cap d(\tilde{F}_1, \tilde{F}_2) \frac{\alpha}{2} ) \}
\]

\[ + \alpha \land \mu(T \cap d(\tilde{F}_2, \tilde{F}_3) \frac{\alpha}{2} ) \}
\]

\[ \leq \bigvee_{\alpha \in [0, \infty)} [\alpha \land \mu(T \cap d(\tilde{F}_1, \tilde{F}_2) \frac{\alpha}{2} )]
\]

\[ + \bigvee_{\alpha \in [0, \infty)} [\alpha \land \mu(T \cap d(\tilde{F}_2, \tilde{F}_3) \frac{\alpha}{2} )] \]
\[
\leq \bigvee_{\alpha \in [0, \infty)} \left[ \frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2) \frac{\alpha}{2}) \right] \\
+ \frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2) \frac{\alpha}{2}) \\
+ \bigvee_{\alpha \in [0, \infty)} \left[ \frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3) \frac{\alpha}{2}) \right] \\
+ \frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3) \frac{\alpha}{2})
\]
\[ \leq \bigvee_{\alpha \in [0, \infty)} \left[ \frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2) \frac{\alpha}{2}) \right] \\
+ \bigvee_{\alpha \in [0, \infty)} \left[ \frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_1) \frac{\alpha}{2}) \right] \\
+ \bigvee_{\alpha \in [0, \infty)} \left[ \frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_3, \tilde{F}_2) \frac{\alpha}{2}) \right] \\
= 2 \left( \Delta_S(\tilde{F}_1, \tilde{F}_2) + \Delta_S(\tilde{F}_2, \tilde{F}_3) \right). \]
Definition 0.1 (Liu(2001)) A fuzzy-valued function $\tilde{\Phi} : T \rightarrow \mathcal{F}_0(\mathbb{R})$ is called a fuzzy-valued simple function, if there exist $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_m \in \mathcal{F}_0(\mathbb{R})$, such that $\forall \ x \in T$,

$$
\tilde{\Phi}(x) = \sum_{k=1}^{m} \tilde{A}_k \cdot \chi_{T_k}(x)
$$

where $T_k \in \mathcal{B} \cap T$ ($k = 1, 2, \ldots, m$), $T_i \cap T_j = \emptyset$ ($i \neq j$) and $T = \bigcup_{k=1}^{m} T_k$. 
Immediately, if $S(T)$ denotes the set of all fuzzy-valued simple functions, then $S(T) \subseteq \mathcal{L}(T)$.

Similar to the proof of Proposition 0.2 and by using subadditivity of $\mu$, we can obtain the following proposition.

**Proposition 0.3** Let $\mu$ be a finite, sub-additive and continuous fuzzy measure on $\mathbb{R}$. If $\tilde{F} \in \mathcal{L}(T)$, then for every $\epsilon > 0$, there exists $\tilde{\Phi}_\epsilon \in S(T)$ such that

$$\Delta_S(\tilde{F}, \tilde{\Phi}_\epsilon) < \epsilon.$$
Main Theorem
Define

\[ \mathcal{H}[^{\sigma}] \triangleq \left\{ \tilde{H} \mid \tilde{H}(x) = \sum_{i=1}^{n} \tilde{W}_i \, V_i[^{\sigma}] \right\} \]

where

\[ V_i[^{\sigma}] \triangleq \sum_{j=1}^{m} \tilde{V}_{ij} \cdot \sigma(x \cdot \tilde{U}_j + \tilde{\Theta}_j) \]

and \( \sigma \) is a given extended function of \( \sigma : \mathbb{R} \to \mathbb{R} \) (bounded, continuous and nonconstant), and \( x \in \mathbb{R}, \tilde{W}_i, \tilde{V}_{ij}, \tilde{U}_j, \tilde{\Theta}_j \in \mathcal{F}_0(\mathbb{R}) \).
Main Theorem 2/6

For any $\tilde{H} \in \mathcal{H}[\sigma]$, $\tilde{H}$ is a four-layer feedforward RFNN with activation function $\sigma$, threshold vector $(\tilde{\Theta}_1, \ldots, \tilde{\Theta}_m)$ in the first hidden layer (cf. Liu (2001)).
Main Theorem 3/6

Restricting fuzzy numbers $\tilde{V}_{ij}, \tilde{U}_j, \tilde{\Theta}_j \in \mathcal{F}_0(\mathbb{R})$, respectively, to be real numbers $v_{ij}, u_j, \theta_j \in \mathbb{R}$, we obtain the subset $\mathcal{H}_0[\sigma]$ of $\mathcal{H}[\sigma]$:

$$\mathcal{H}_0[\sigma] \triangleq \left\{ \tilde{H} \mid \tilde{H}(x) = \sum_{i=1}^{n} \tilde{W}_i v_i[\sigma] \right\}.$$  

where

$$v_i[\sigma] \triangleq \sum_{j=1}^{m} v_{ij} \cdot \sigma(x \cdot u_j + \theta_j).$$
Main Theorem 4/6

Let define two classes of pan-approximation which is fundamental to our results.

**Definition 0.2**

(1) \( \mathcal{H}_0[\sigma] \) is call the pan-approximator of \( S(T) \) in the sense of \( \triangle_S \), if for \( \forall \Phi \in S(T), \forall \epsilon > 0 \), there exists \( \tilde{H}_\epsilon \in \mathcal{H}_0[\sigma] \) such that \( \triangle_S(\Phi, \tilde{H}_\epsilon) < \epsilon \).

(2) For \( \tilde{F} \in \mathcal{L}(T) \), \( \mathcal{H}[\sigma] \) is call the pan-approximator for \( \tilde{F} \) in the sense of \( \triangle_S \), if \( \forall \epsilon > 0 \), there exists \( \tilde{H}_\epsilon \in \mathcal{H}[\sigma] \) such that \( \triangle_S(\tilde{F}, \tilde{H}_\epsilon) < \epsilon \).
Main Theorem 5/6

By using Lusin’s theorem (Theorem 0.1), Proposition 0.2 and 0.3 we can obtain the main result in this paper, which is stated in the following.
Main Theorem 6/6

**Theorem 0.2** Let \((T, \mathcal{B} \cap T, \mu)\) be fuzzy measure space and \(\mu\) be finite, sub-additive and continuous. Then,

1. \(\mathcal{H}_0[\sigma]\) is the pan-approximator of \(S(T)\) in the sense of \(\triangle_S\).
2. \(\mathcal{H}[\sigma]\) is the pan-approximator for \(\tilde{F}\) in the sense of \(\triangle_S\).
Pf of Main Theorem
Proof. By using the conclusion of (1) and Proposition 0.3 we can obtain (2). Now we only prove (1). Suppose that \( \tilde{\Phi}(x) \) is a fuzzy-valued simple function, i.e.,

\[
\tilde{\Phi}(x) = \sum_{k=1}^{m} \chi_{T_k}(x) \cdot \tilde{A}_k \quad (x \in T).
\]

For arbitrarily given \( \epsilon > 0 \), applying Theorem 0.1 (Lusin’s theorem) to each real measurable function \( \chi_{T_k}(x) \), for every fixed \( k \) \( (1 \leq k \leq m) \), there exists closed set \( F_k \in B \cap T \) such that

\[
F_k \subset L_k \quad \text{and} \quad \mu(L_k - F_k) < \frac{\epsilon}{2m}
\]

and \( \chi_{T_k}(x) \) is continuous on \( F_k \).
Pf of Main Thm 2/13

Therefore, for every $k$ there exist a Tauber-Wiener function $\sigma$ and $p_k \in \mathbb{N}, v_k^1, v_k^2, \ldots, v_k^{p_k}, \theta_k^1, \theta_k^2, \ldots, \theta_k^{p_k} \in \mathbb{R}$, and $w_k^1, w_k^2, \ldots, w_k^{p_k} \in \mathbb{R}^n$ such that

$$\left| \chi T_k(x) - \sum_{j=1}^{p_k} v_k^j \cdot \sigma(\langle w_k^j, x \rangle + \theta_k^j) \right| < \frac{\epsilon}{2 \sum_{k=1}^{m} |\tilde{A}_k|}$$

for $x \in L_k$. Note that we can assume $\sum_{k=1}^{m} |\tilde{A}_k| \neq 0$, without any loss of generality.
Denote \( L = \bigcap_{k=1}^{m} L_k \), then \( T = L \cup (T - L) \). By the subadditivity of \( \mu \), we have

\[
\mu(T - L) = \mu\left( \bigcup_{k=1}^{m} (T - L_k) \right)
\leq \sum_{k=1}^{m} \mu(T - L_k) < \frac{\varepsilon}{2}.
\]

We take \( \beta_1 = 0, \beta_k = \sum_{i=1}^{k-1} p_i, k = 2, \ldots, m \), and \( p = \sum_{k=1}^{m} p_k \). For \( k = 1, 2, \ldots, m, j = 1, 2, \ldots, p \), we denote

\[
v_{k,j} = \begin{cases} 
    v'_{k,(j-\beta_k)}, & \text{if } \beta_k < j \leq \beta_{k+1}, \\
    0, & \text{otherwise},
\end{cases}
\]
Pf of Main Thm 4/13

\[
\theta_{kj} = \begin{cases} 
\theta'_{k(j - \beta_k)}, & \text{if } \beta_k < j \leq \beta_{k+1}, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
w_{kj} = \begin{cases} 
w'_{k(j - \beta_k)}, & \text{if } \beta_k < j \leq \beta_{k+1}, \\
0, & \text{otherwise,}
\end{cases}
\]

then, for any $k \in \{1, 2, \cdots, m\}$, we have

\[
\sum_{j=1}^{p} v_{ij} \cdot \sigma(\langle w_{kj}, x \rangle + \theta_{kj})
= \sum_{j=1}^{p_k} v'_{ij} \cdot \sigma(\langle w'_{kj}, x \rangle + \theta'_{kj}).
\]
Pf of Main Thm 5/13

Now denote that

$$\tilde{H}(x) = \sum_{k=1}^{m} \tilde{A}_k \cdot \left( \sum_{j=1}^{p} u_{k,j} \cdot \sigma(\langle w_{k,j}, x \rangle + \theta_{k,j}) \right),$$

then $\tilde{H} \in \mathcal{H}_0[\sigma]$.

In the reminder part of this section we will prove $\Delta_S(\tilde{H}, \tilde{\Phi}) < \epsilon.$
Denote
\[ B_{kj} = v_{kj} \cdot \sigma(\langle w_{kj}, x \rangle + \theta_{kj}) \]
and
\[ B'_{kj} = v'_{kj} \cdot \sigma(\langle w'_{kj}, x \rangle + \theta'_{kj}). \]

By using Proposition 0.1 and noting \( \mu(T - L) < \epsilon/2 \), we have
\[
\Delta_S(\tilde{H}, \tilde{\Phi})
= (S') \int_T d(\tilde{H}, \tilde{\Phi})d\mu
= \bigvee_{0 \leq \alpha < +\infty} \left[ \alpha \land \mu(T \cap (d(\tilde{H}, \tilde{\Phi}))_{\alpha}) \right].
\]
Since

\[
\mu(T \cap d(\tilde{H}, \Phi)_{\alpha})
\]

\[
= \mu \left( T \cap d \left( \sum_{k=1}^{m} \tilde{A}_k \cdot \sum_{j=1}^{p} B_{k,j}, \sum_{k=1}^{m} \chi_{T_k(x)} \cdot \tilde{A}_k \right)_{\alpha} \right)
\]

\[
\leq \mu \left( (L \cup (T - L)) \cap (C_{mp})_{\alpha} \right)
\]

\[
\leq \mu \left( L \cap (C_{mp})_{\alpha} \right) + \mu \left( (T - L) \cap (C_{mp})_{\alpha} \right)
\]
where the notation of set $C_{mp}$ is assigned as

$$C_{mp} = \sum_{k=1}^{m} |\tilde{A}_k| \cdot d \left( \sum_{k=1}^{p} B_{k,j}, \chi_{T_k}(x) \right).$$
Hence $\triangle_S(\tilde{H}, \tilde{\Phi})$ is dominated by

$$\bigvee_{0 \leq \alpha < +\infty} [\alpha \land \mu (L \cap (D_{mp}\alpha))]$$

$$+ \bigvee_{0 \leq \alpha < +\infty} [\alpha \land \mu ((T - L))]$$

$$\leq \bigvee_{0 \leq \alpha < +\infty} [\alpha \land \mu (L \cap (D_{mp}'\alpha))] + \frac{\epsilon}{2}$$
Here, for simplicity, we use the notations of

\[ D_{mp} = \sum_{k=1}^{m} |\tilde{A}_k| \cdot \left| \sum_{j=1}^{p} B_{kj} - \chi T_k(x) \right| \]

and

\[ D'_{mp} = \sum_{k=1}^{m} |\tilde{A}_k| \cdot d \left( \sum_{j=1}^{p} B'_{kj} - \chi T_k(x) \right). \]
Now we estimate the first part in the above formula. If \( x \in L \), then for every \( k = 1, 2, \ldots, m \), we have \( x \in L_k \), hence

\[
\left| \chi T_k (x) - \sum_{j=1}^{p_k} v'_{k,j} \cdot \sigma \left( \langle w'_{k,j}, x \rangle + \theta'_{k,j} \right) \right| < \frac{\epsilon}{m} \frac{1}{2 \sum_{k=1}^{m} |\tilde{A}_k|},
\]

for every \( k = 1, 2, \ldots, m \). That is, for \( x \in L \),

\[
D'_{mp} = \sum_{k=1}^{m} |\tilde{A}_k| \cdot d \left( \sum_{j=1}^{p_k} B'_{k,j} - \chi T_k (x) \right) < \frac{\epsilon}{2}.
\]
Therefore,

\[
\bigvee_{0 \leq \alpha < +\infty} \left[ \alpha \land \mu \left( L \cap (D'_{mp})_{\alpha} \right) \right]
\]

\[
= \bigvee_{\alpha \in [0, \frac{\epsilon}{2}]} \left[ \alpha \land \mu \left( L \cap (D'_{mp})_{\alpha} \right) \right]
\]

\[
+ \bigvee_{\alpha \in \left[\frac{\epsilon}{2}, \infty\right)} \left[ \alpha \land \mu \left( L \cap (D'_{mp})_{\alpha} \right) \right]
\]

\[
= \bigvee_{\alpha \in [0, \frac{\epsilon}{2}]} \left[ \alpha \land \mu \left( L \cap (D'_{mp})_{\alpha} \right) \right]
\]

\[
\leq \frac{\epsilon}{2}.
\]
Thus, combining with the previous evaluation, we obtain

\[
\begin{align*}
\triangle_S(\tilde{H}, \tilde{\Phi}) & \leq \bigvee_{0 \leq \alpha < +\infty} \left[ \alpha \wedge \mu \left( L \cap (D'_{mp})_\alpha \right) \right] + \frac{\epsilon}{2} \\
& < \epsilon.
\end{align*}
\]

The proof of (1) is now completed. \qed