# Approximation of fuzzy neural networks by using Lusin's theorem

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September 23, 2006

# Abstract



In this note, we study an approximation property of regular fuzzy neural network(RFNN). It is shown that any fuzzy-valued measurable function can be approximated by the four-layer RFNN in the sense of fuzzy integral norm for the finite sub-additive fuzzy measure on $\mathbb{R}$.

*Keywords*:
Fuzzy measure; Lusin's theorem; Approximation; Regular fuzzy neural network

# Introduction

# Introduction 1/4

In neural network theory, the learning ability of a neural network is closely related to its approximating capabilities, so it is important and interesting to study the approximation properties of neural networks.

# Introduction 2/4



The studies on this matter were undertaken by many authors and a great number of important results were obtained.

- J.G Attali, G. Pages, Approximation of functions by a multilayer perceptron: a new approach, *Neural Networks* 10(1997) 1069-1081,
- R.M Burton, H.G. Dehling, Universal approximation in $p$-mean by neural networks, *Neural Networks* 11(1998) 661-667,
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# Introduction 3/4

The similar approximation problems in fuzzy environment were investigated by

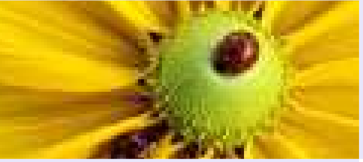
- J.J Buckley, Y. Hayashi, Fuzzy input-output controllers are universal approximators, *Fuzzy Sets and Systems* 58(1993) 273-278,
- J.J Buckley, Y. Hayashi, Can fuzzy neural nets approximate continuous fuzzy function, *Fuzzy Sets and Systems* 61(1994) 43-51,
- P. Liu, Analyses of regular fuzzy neural networks for approximation capabilities, *Fuzzy Sets and Systems* 114(2000) 329-338,
- P. Liu, Universal approximations of continuous fuzzy-valued functions by multi-layer regular fuzzy neural networks, *Fuzzy Sets and Systems* 119(2001) 313-320.

# Introduction 4/4



In P. Liu(2001) proved that continuous fuzzy-valued function can be closely approximated by a class of regular fuzzy neural networks (RFNNs) with real input and fuzzy-valued output.
In this note, by using Lusin's theorem on fuzzy measure space, we show that such RFNNs is pan-approximator for fuzzy-valued measurable function.
That is, any fuzzy-valued measurable function can be approximated by the four-layer RFNNs in the sense of fuzzy integral norm for the finite sub-additive measure on $\mathbb{R}$.

# Preliminaries



# Preliminaries 1/3



We suppose that $(X, \rho)$ is a metric space, and that $\mathcal{O}$ and $\mathcal{C}$ are the classes of all open and closed sets in $(X, \rho)$, respectively, and $\mathcal{B}$ is Borel $\sigma$-algebra on $X$, i.e., it is the smallest $\sigma$-algebra containing $\mathcal{O}$. A set function $\mu : \mathcal{B} \to [0, +\infty)$ is called *a fuzzy measure*(Narukawa/Murofushi(2004)), if it satisfies the following properties:

**(FM1)** $\mu(\emptyset) = 0$;
**(FM2)** $A \subset B$ implies $\mu(A) \leq \mu(B)$.

A fuzzy measure $\mu$ is called *null-additive* (Wang/Klir(1992)), if for any $E, F \in \mathcal{B}$ and $\mu(F) = 0$ imply $\mu(E \cup F) = \mu(E)$; *sub-additive* (Pap(1995)), if for any $E, F \in \mathcal{B}$ we have $\mu(E \cup F) \leq \mu(E) + \mu(F)$.

# Preliminaries 2/3

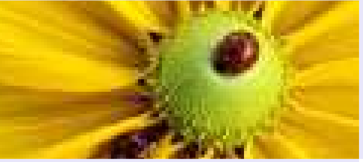


In this paper, we always assume that $\mu$ is a finite, sub-additive and continuous fuzzy measure on $\mathcal{B}$.
Consider a nonnegative real-valued measurable function $f$ on $A$ and the *fuzzy integral* of $f$ on $A$ with respect to $\mu$, which is denoted by

$$(S)\int_A f\,d\mu \triangleq \sup_{0\leq\alpha<+\infty}[\alpha\wedge\mu(\{x : f(x)\geq\alpha\}\cap A)]$$

# Preliminaries 3/3

**Theorem 0.1** (Lusin's theorem (Li/Yasuda(2004), Song/Li(2003)))
*Let $(X, \rho)$ be metric space and $\mu$ be null additive fuzzy measure on $\mathcal{B}$. If $f$ is a real-valued measurable function on $E \in \mathcal{B}$, then, for every $\epsilon > 0$, there exists a closed subset $F_\epsilon \in \mathcal{B}$ such that $f$ is continuous on $F_\epsilon$ and $\mu(E - F_\epsilon) < \epsilon$.*





# Approximation in fuzzy mean by RFNN

# Approximation 1/10

In this section, we study an approximation property of the four-layer RFNNs to fuzzy-valued measurable function in the sense of fuzzy integral norm for fuzzy measure on $\mathbb{R}$.

# Approximation 2/10



Let $\mathcal{F}_0(\mathbb{R})$ be the set of all bounded fuzzy numbers, i.e., for $\tilde{A} \in \mathcal{F}_0(\mathbb{R})$, the following conditions hold:

(i) $\forall\ \alpha \in (0,1], \tilde{A}_\alpha \triangleq \{x \in \mathbb{R} \mid \tilde{A}(x) \geq \alpha\}$ is the closed interval of $\mathbb{R}$;

(ii) The support Supp$(\tilde{A}) \triangleq$ cl$\{x \in \mathbb{R} \mid \tilde{A}(x) > 0\} \subset$ is a bounded set;

(iii) $\{x \in \mathbb{R} \mid \tilde{A}(x) = 1\} \neq \emptyset$.

For simplicity, supp$(\tilde{A})$ is also written as $\tilde{A}_0$. Obviously, $\tilde{A}_0$ is a bounded and closed interval of $\mathbb{R}$. For $\tilde{A} \in \mathcal{F}_0(\mathbb{R})$, let $\tilde{A}_\alpha = [a_\alpha^-, a_\alpha^+]$ for each $\alpha \in [0,1]$ and we denote

$$|\tilde{A}| \triangleq \bigvee_{\alpha \in [0,1]} (|a_\alpha^-| \vee |a_\alpha^+|).$$

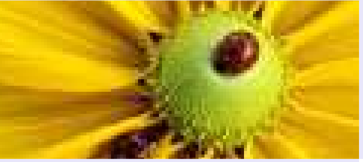
# Approximation 3/10



**Proposition 0.1** *Liu(2001) Assume* $\tilde{A}, \tilde{A}_1, \tilde{A}_2 \in \mathcal{F}_0(\mathbb{R})$, *and* $\tilde{W}_i, \tilde{V}_i \in \mathcal{F}_0(\mathbb{R})(i = 1, 2, \cdots, n)$.
*Then*

(1) $$d(\tilde{A} \cdot \tilde{A}_1, \tilde{A} \cdot \tilde{A}_2) \leq |\tilde{A}| \cdot d(\tilde{A}_1, \tilde{A}_2),$$

(2) $$d(\sum_{i=1}^{n} \tilde{W}_i, \sum_{i=1}^{n} \tilde{V}_i) \leq \sum_{i=1}^{n} d(\tilde{W}_i, \tilde{V}_i).$$



# Approximation 4/10



For $\tilde{A}, \tilde{B} \in \mathcal{F}_0(\mathbb{R})$, define metric $d(\tilde{A}, \tilde{B})$ between $\tilde{A}$ and $\tilde{B}$ by

$$d(\tilde{A}, \tilde{B}) \triangleq \sup_{\alpha \in [0,1]} d_H(\tilde{A}_\alpha, \tilde{B}_\alpha)$$

where $d_H$ means Hausdorff metric: for $A, B \subset \mathbb{R}$,

$$\begin{aligned} & d_H(A, B) \\ \triangleq & \max\left\{ \sup_{x \in A} \inf_{y \in B} (|x-y|), \sup_{y \in B} \inf_{x \in A} (|x-y|) \right\}. \end{aligned}$$

# Approximation 5/10

It is known that $(\mathcal{F}_0(\mathbb{R}), d)$ is a completely separable metric space (Diamond/Kloeden(1994)).
Let $T$ be a measurable set in $\mathbb{R}^n$, $(T, \mathcal{B} \cap T, \mu)$ finite fuzzy measure space. Let $\mathcal{L}(T)$ denote the set of all fuzzy-valued measurable function

$$\tilde{F} : T \to \mathcal{F}_0(\mathbb{R}).$$

For any $\tilde{F}_1, \tilde{F}_2 \in \mathcal{L}(T)$, $d(\tilde{F}_1, \tilde{F}_2)$ is measurable function on $(T, \mathcal{B} \cap T)$, we will write a fuzzy integral norm as

$$\triangle_S(\tilde{F}_1, \tilde{F}_2) \triangleq (S) \int_T d(\tilde{F}_1, \tilde{F}_2) d\mu.$$

**Proposition 0.2** *Let $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3 \in \mathcal{L}(T)$, then*

$$\triangle_S(\tilde{F}_1, \tilde{F}_3) \leq 2(\triangle_S(\tilde{F}_1, \tilde{F}_2) + \triangle_S(\tilde{F}_2, \tilde{F}_3)).$$

# Approximation 7/10

**Proof.** From subadditivity of $\mu$, we have

$$\triangle_S(\tilde{F}_1, \tilde{F}_3) = (S)\int_T d(\tilde{F}_1, \tilde{F}_3)d\mu$$

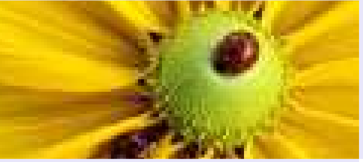
$$= \bigvee_{\alpha\in[0,\infty)} \{\alpha \wedge \mu(T \cap (d(\tilde{F}_1, \tilde{F}_3))_\alpha\}$$

$$\leq \bigvee_{\alpha\in[0,\infty)} \{\alpha \wedge \mu(T \cap$$

$$(d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}} \cup d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}}))\}$$

$$\leq \bigvee_{\alpha\in[0,\infty)} \{\alpha \wedge [\mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}})$$

$$+\mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})]\}.$$



# Approximation 8a/10



we have

$$
\begin{aligned}
\triangle_S(\tilde{F}_1, \tilde{F}_3) \quad &\leq \quad \bigvee_{\alpha \in [0,\infty)} \{\alpha \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}}) \\
&\quad +\alpha \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})\} \\
&\leq \quad \bigvee_{\alpha \in [0,\infty)} [\alpha \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}})] \\
&\quad + \bigvee_{\alpha \in [0,\infty)} [\alpha \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})]
\end{aligned}
$$

# Approximation 8b/10

$$
\begin{aligned}
\leq \quad & \bigvee_{\alpha\in[0,\infty)} \Big[\frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}}) \\
& + \frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}})\Big] \\
& + \bigvee_{\alpha\in[0,\infty)} \Big[\frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}}) \\
& + \frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})\Big]
\end{aligned}
$$

# Approximation 8c/10



$$
\begin{aligned}
\leq\ & \bigvee_{\frac{\alpha}{2}\in[0,\infty)} \left[\frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}})\right] \\
& + \bigvee_{\frac{\alpha}{2}\in[0,\infty)} \left[\frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_1, \tilde{F}_2)_{\frac{\alpha}{2}})\right] \\
& + \bigvee_{\frac{\alpha}{2}\in[0,\infty)} \left[\frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})\right] \\
& + \bigvee_{\frac{\alpha}{2}\in[0,\infty)} \left[\frac{\alpha}{2} \wedge \mu(T \cap d(\tilde{F}_2, \tilde{F}_3)_{\frac{\alpha}{2}})\right] \\
=\ & 2\left(\triangle_S(\tilde{F}_1, \tilde{F}_2) + \triangle_S(\tilde{F}_2, \tilde{F}_3)\right).
\end{aligned}
$$

# Approximation 9/10



**Definition 0.1** *(Liu(2001)) A fuzzy-valued function* $\tilde{\Phi} : T \to \mathcal{F}_0(\mathbb{R})$ *is called a fuzzy-valued simple function, if there exist* $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_m \in \mathcal{F}_0(\mathbb{R})$, *such that* $\forall\ x \in T$,

$$\tilde{\Phi}(x) = \sum_{k=1}^{m} \tilde{A}_k \cdot \chi_{T_k}(x)$$

*where* $T_k \in \mathcal{B} \cap T\ (k = 1, 2, \ldots, m),\ T_i \cap T_j = \emptyset\ (i \neq j)$ *and* $T = \bigcup_{k=1}^{m} T_k$.

# Approximation 10/10



Immediately, if $\mathcal{S}(T)$ denotes the set of all fuzzy-valued simple functions, then $\mathcal{S}(T) \subset \mathcal{L}(T)$.
Similar to the proof of Proposition 0.2 and by using subadditivity of $\mu$, we can obtain the following proposition.

**Proposition 0.3** *Let $\mu$ be a finite, sub-additive and continuous fuzzy measure on $\mathbb{R}$. If $\tilde{F} \in \mathcal{L}(T)$, then for every $\epsilon > 0$, there exists $\tilde{\Phi}_\epsilon \in \mathcal{S}(T)$ such that*

$$\triangle_S(\tilde{F}, \tilde{\Phi}_\epsilon) < \epsilon.$$

# Main Theorem

# Main Theorem 1/6



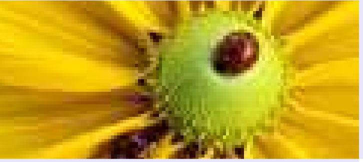
Define

$$\mathcal{H}[\sigma] \triangleq \left\{ \tilde{H} \;\middle|\; \tilde{H}(x) = \sum_{i=1}^{n} \tilde{W}_i \, V_i[\sigma] \right\}$$

where

$$V_i[\sigma] \triangleq \sum_{j=1}^{m} \tilde{V}_{ij} \cdot \sigma(x \cdot \tilde{U}_j + \tilde{\Theta}_j)$$

and $\sigma$ is a given extended function of $\sigma : \mathbb{R} \to \mathbb{R}$ (bounded, continuous and nonconstant), and $x \in \mathbb{R}, \tilde{W}_i, \tilde{V}_{ij}, \tilde{U}_j, \tilde{\Theta}_j \in \mathcal{F}_0(\mathbb{R})$.



# Main Theorem 2/6



For any $\tilde{H} \in \mathcal{H}[\sigma]$, $\tilde{H}$ is a four-layer feedforward RFNN with activation function $\sigma$, threshold vector $(\tilde{\Theta}_1, \ldots, \tilde{\Theta}_m)$ in the first hidden layer(cf. Liu(2001).

# Main Theorem 3/6

Restricting fuzzy numbers $\tilde{V}_{ij}, \tilde{U}_j, \tilde{\Theta}_j \in \mathcal{F}_0(\mathbb{R})$, respectively, to be real numbers $v_{ij}, u_j, \theta_j \in \mathbb{R}$, we obtain the subset $\mathcal{H}_0[\sigma]$ of $\mathcal{H}[\sigma]$:

$$\mathcal{H}_0[\sigma] \triangleq \left\{ \tilde{H} \;\middle|\; \tilde{H}(x) = \sum_{i=1}^{n} \tilde{W}_i \, v_i[\sigma] \right\}.$$

where

$$v_i[\sigma] \triangleq \sum_{j=1}^{m} v_{ij} \cdot \sigma(x \cdot u_j + \theta_j).$$

# Main Theorem 4/6

Let define two classes of pan-approximation which is fundamental to our results.

## Definition 0.2

(1) *$\mathcal{H}_0[\sigma]$ is call the pan-approximator of $\mathcal{S}(T)$ in the sense of $\triangle_S$, if for $\forall\ \tilde{\Phi} \in \mathcal{S}(T)$, $\forall\ \epsilon > 0$, there exists $\tilde{H}_\epsilon \in \mathcal{H}_0[\sigma]$ such that $\triangle_S(\tilde{\Phi}, \tilde{H}_\epsilon) < \epsilon$.*

(2) *For $\tilde{F} \in \mathcal{L}(T)$, $\mathcal{H}[\sigma]$ is call the pan-approximator for $\tilde{F}$ in the sense of $\triangle_S$, if $\forall\ \epsilon > 0$, there exists $\tilde{H}_\epsilon \in \mathcal{H}[\sigma]$ such that $\triangle_S(\tilde{F}, \tilde{H}_\epsilon) < \epsilon$.*

## Main Theorem 5/6



By using Lusin's theorem (Theorem 0.1), Proposition 0.2 and 0.3 we can obtain the main result in this paper, which is stated in the following.

# Main Theorem 6/6

**Theorem 0.2** *Let $(T, \mathcal{B} \cap T, \mu)$ be fuzzy measure space and $\mu$ be finite, sub-additive and continuous. Then,*

(1) $\mathcal{H}_0[\sigma]$ *is the pan-approximator of* $\mathcal{S}(T)$ *in the sense of* $\triangle_S$.

(2) $\mathcal{H}[\sigma]$ *is the pan-approximator for* $\tilde{F}$ *in the sense of* $\triangle_S$.

# Pf of Main Theorem

**Proof.** By using the conclusion of (1) and Proposition 0.3 we can obtain (2). Now we only prove (1). Suppose that $\tilde{\Phi}(x)$ is a fuzzy-valued simple function, i.e.,

$$\tilde{\Phi}(x) = \sum_{k=1}^{m} \chi_{T_k}(x) \cdot \tilde{A}_k \quad (x \in T).$$

For arbitrarily given $\epsilon > 0$, applying Theorem 0.1 (Lusin's theorem) to each real measurable function $\chi_{T_k}(x)$, for every fixed $k$ $(1 \leq k \leq m)$, there exists closed set $F_k \in \mathcal{B} \cap T$ such that

$$F_k \subset L_k \quad \text{and} \quad \mu(L_k - F_k) < \frac{\epsilon}{2m}$$

and $\chi_{T_k}(x)$ is continuous on $F_k$.

# Pf of Main Thm 2/13

Therefore, for every $k$ there exist a Tauber-Wiener function $\sigma$ and $p_k \in N, v'_{k1}, v'_{k2}, \cdots, v'_{kp_k}, \theta'_{k1}, \theta'_{k2}, \cdots, \theta'_{kp_k} \in \mathbb{R}$, and $\mathbf{w}'_{k1}, \mathbf{w}'_{k2}, \cdots, \mathbf{w}'_{kp_k} \in \mathbb{R}^n$ such that

$$\left| \chi_{T_k}(x) - \sum_{j=1}^{p_k} v'_{kj} \cdot \sigma(\langle \mathbf{w}'_{kj}, x\rangle + \theta'_{kj}) \right| < \frac{\epsilon}{2\sum_{k=1}^{m} |\tilde{A}_k|}$$

for $x \in L_k$. Note that we can assume $\sum_{k=1}^{m} |\tilde{A}_k| \neq 0$, without any loss of generality.

# Pf of Main Thm 3/13

Denote $L = \bigcap_{k=1}^{m} L_k$, then $T = L \cup (T - L)$. By the subadditivity of $\mu$, we have

$$\begin{aligned} \mu(T - L) &= \mu(\textstyle\bigcup_{k=1}^{m}(T - L_k)) \\ &\leq \textstyle\sum_{k=1}^{m} \mu(T - L_k) < \dfrac{\epsilon}{2}. \end{aligned}$$

We take $\beta_1 = 0$, $\beta_k = \sum_{i=1}^{k-1} p_i, k = 2, \cdots, m$, and $p = \sum_{k=1}^{m} p_k$. For $k = 1, 2, \cdots, m, j = 1, 2, \cdots, p$, we denote

$$v_{kj} = \begin{cases} v'_{k(j-\beta_k)}, & \text{if } \beta_k < j \leq \beta_{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$

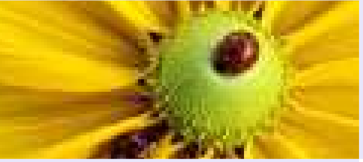
# Pf of Main Thm 4/13

$$\theta_{kj} = \begin{cases} \theta'_{k(j-\beta_k)}, & \text{if } \beta_k < j \le \beta_{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathbf{w}_{kj} = \begin{cases} \mathbf{w}'_{k(j-\beta_k)}, & \text{if } \beta_k < j \le \beta_{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$

then, for any $k \in \{1, 2, \cdots, m\}$, we have

$$\begin{aligned} &\textstyle\sum_{j=1}^{p} v_{ij} \cdot \sigma(\langle \mathbf{w}_{kj}, x\rangle + \theta_{kj}) \\ &= \textstyle\sum_{j=1}^{p_k} v'_{ij} \cdot \sigma(\langle \mathbf{w}'_{kj}, x\rangle + \theta'_{kj}). \end{aligned}$$



# Pf of Main Thm 5/13



Now denote that

$$\tilde{H}(x) = \sum_{k=1}^{m} \tilde{A}_k \cdot \left( \sum_{j=1}^{p} v_{kj} \cdot \sigma(\langle \mathbf{w}_{kj}, x \rangle + \theta_{kj}) \right),$$

then $\tilde{H} \in \mathcal{H}_0[\sigma]$.
In the reminder part of this section we will prove $\triangle_S(\tilde{H}, \tilde{\Phi}) < \epsilon$.

Denote

$$B_{kj} = v_{kj} \cdot \sigma(\langle \mathbf{w}_{kj}, x \rangle + \theta_{kj})$$

and

$$B'_{kj} = v'_{ij} \cdot \sigma(\langle \mathbf{w}'_{kj}, x \rangle + \theta'_{kj}).$$

By using Proposition 0.1 and noting $\mu(T - L) < \epsilon/2$, we have

$$\begin{aligned} & \triangle_S(\tilde{H}, \tilde{\Phi}) \\ = \;& (S)\int_T d(\tilde{H}, \tilde{\Phi}) d\mu \\ = \;& \bigvee_{0 \le \alpha < +\infty} \left[\alpha \wedge \mu(T \cap (d(\tilde{H}, \tilde{\Phi}))_\alpha)\right]. \end{aligned}$$

# Pf of Main Thm 7/13

Since

$$
\begin{aligned}
& \mu(T \cap d(\tilde{H}, \tilde{\Phi})_\alpha) \\
= \quad & \mu\left(T \cap d\left(\sum_{k=1}^{m} \tilde{A}_k \cdot \sum_{j=1}^{p} B_{kj},\right.\right. \\
& \left.\left.\sum_{k=1}^{m} \chi_{T_k}(x) \cdot \tilde{A}_k\right)_\alpha\right) \\
\leq \quad & \mu\left((L \cup (T - L)) \cap (C_{mp})_\alpha\right) \\
\leq \quad & \mu\left(L \cap (C_{mp})_\alpha\right) + \mu\left((T - L) \cap (C_{mp})_\alpha\right)
\end{aligned}
$$

# Pf of Main Thm 8/13



where the notation of set $C_{mp}$ is assigned as

$$C_{mp} = \sum_{k=1}^{m} |\tilde{A}_k| \cdot d\left(\sum_{k=1}^{p} B_{kj}, \chi_{T_k}(x)\right).$$

# Pf of Main Thm 9/13

Hence $\triangle_S(\tilde{H}, \tilde{\Phi})$ is dominated by

$$\begin{aligned} & \bigvee_{0\le\alpha<+\infty} \left[\alpha \wedge \mu\left(L \cap (D_{mp})_\alpha\right)\right] \\ & + \bigvee_{0\le\alpha<+\infty} \left[\alpha \wedge \mu\left((T - L)\right)\right] \\ \le \quad & \bigvee_{0\le\alpha<+\infty} \left[\alpha \wedge \mu\left(L \cap \left(D'_{mp}\right)_\alpha\right)\right] + \frac{\epsilon}{2} \end{aligned}$$

# Pf of Main Thm 10/13

Here, for simplicity, we use the notations of

$$D_{mp} = \sum_{k=1}^{m} |\tilde{A}_k| \cdot \left| \sum_{j=1}^{p} B_{kj} - \chi_{T_k}(x) \right|$$

and

$$D'_{mp} = \sum_{k=1}^{m} |\tilde{A}_k| \cdot d\left( \sum_{j=1}^{p_k} B'_{kj} - \chi_{T_k}(x) \right).$$

# Pf of Main Thm 11/13

Now we estimate the first part in the above formula. If $x \in L$, then for every $k = 1, 2, \cdots, m$, we have $x \in L_k$, hence

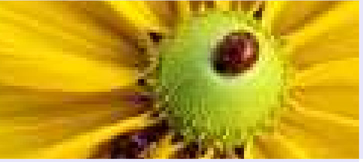
$$\left| \chi_{T_k}(x) - \sum_{j=1}^{p_k} v_{kj}^{'} \cdot \sigma(\langle \mathbf{w}_{kj}^{'}, x \rangle + \theta_{kj}^{'}) \right| < \frac{\epsilon}{2 \sum_{k=1}^{m} |\tilde{A}_k|},$$

for every $k = 1, 2, \cdots, m$. That is, for $x \in L$,

$$D'_{mp} = \sum_{k=1}^{m} |\tilde{A}_k| \cdot d\left( \sum_{j=1}^{p_k} B_{kj}^{'} - \chi_{T_k}(x) \right) < \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned}
& \bigvee_{0\leq\alpha<+\infty}\left[\alpha\wedge\mu\left(L\cap\left(D'_{mp}\right)_\alpha\right)\right] \\
= & \bigvee_{\alpha\in[0,\frac{\epsilon}{2}]}\left[\alpha\wedge\mu\left(L\cap\left(D'_{mp}\right)_\alpha\right)\right] \\
& + \bigvee_{\alpha\in[\frac{\epsilon}{2},\infty)}\left[\alpha\wedge\mu\left(L\cap\left(D'_{mp}\right)_\alpha\right)\right] \\
= & \bigvee_{\alpha\in[0,\frac{\epsilon}{2}]}\left[\alpha\wedge\mu\left(L\cap\left(D'_{mp}\right)_\alpha\right)\right] \\
\leq & \frac{\epsilon}{2}.
\end{aligned}$$





# Pf of Main Thm 13/13

Thus, combining with the previous evaluation, we obtain

$$\begin{aligned} & \triangle_S(\tilde{H}, \tilde{\Phi}) \\ \leq \; & \bigvee_{0\leq\alpha<+\infty} \left[\alpha \wedge \mu\left(L \cap \left(D'_{mp}\right)_\alpha\right)\right] + \frac{\epsilon}{2} \\ < \; & \epsilon. \end{aligned}$$

The proof of (1) is now completed. □