A LIMIT THEOREM IN DYNAMIC FUZZY SYSTEMS WITH A MONOTONE PROPERTY

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Abstract : A dynamic fuzzy system defined on a compact state space is considered under a monotone property of the fuzzy relation. We study the limit of a sequence of fuzzy states and obtain a convergence theorem. The limit is characterized as the solution of a fuzzy relational equation. A numerical example is given to comprehend our idea in this paper.

Keyword : Sequence of fuzzy states; limit theorem; fuzzy relation; dynamic fuzzy system.

1. Introduction and notations

Kurano et al. [3] studied the limiting behavior of fuzzy states of dynamic fuzzy systems under a contractive condition for the fuzzy relation, and proved that the limiting fuzzy state is a unique solution of a fuzzy relational equation. The contractive case is a typical one of dynamic fuzzy systems. Another type of dynamic fuzzy systems is a monotone case, which was introduced by Yoshida [7] to study the recurrence behavior of dynamic fuzzy systems. In the monotone case, however, the limiting theorem has been not established yet. Here, we treat the monotone case and prove the limiting theorem under an additional condition concerning the directionality of fuzzy transition. Also, to comprehend our idea a numerical example is given.

Let E be a compact subset of some Banach space X with a norm $\|\cdot\|$. We put a metric d by $d(x,y) = \|x-y\|$ for $x, y \in X$. We denote by 2^E the collection of all closed subsets of E, and denote by $\mathcal{C}(E)$ the collection of all closed convex subsets of E. Let ρ be the Hausdorff metric on 2^E. Then it is well-known ([4]) that $(2^E, \rho)$ is a compact metric space. Let $\mathcal{F}(E)$ be the set of all fuzzy sets $\tilde{s} : E \to [0,1]$ which are upper semi-continuous and satisfy $\sup_{x\in E} \tilde{s}(x) = 1$. Let $\tilde{q} : E \times E \to [0,1]$ be a continuous fuzzy relation such that $\tilde{q}(x,\cdot) \in \mathcal{F}(E)$ for $x \in E$.

In this paper, we consider a sequence of fuzzy states $\{\tilde{p}_k\}_{k=0}^{\infty}$ defined by the following dynamic fuzzy system (1.1) (see [3]) :

$$\tilde{p}_{k+1}(y) := \sup_{x\in E} \{\tilde{p}_k(x) \wedge \tilde{q}(x,y)\}, \quad y \in E,\ k = 0,1,2,\cdots, \tag{1.1}$$

where $\tilde{p}_0 \in \mathcal{F}(E)$ and $a \wedge b = \min\{a,b\}$ for real numbers a and b.

For $\tilde{s} \in \mathcal{F}(E)$, the α-cut $\tilde{s}_\alpha$ is defined by

$$\tilde{s}_\alpha := \{x \in E \mid \tilde{s}(x) \geq \alpha\} \ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{s}_0 := \text{cl}\{x \in E \mid \tilde{s}(x) > 0\},$$

where cl denotes the closure of a set. We need the following convergence concept for a sequence of closed sets.

Definition 1. For $\{D_k\}_{k=1}^{\infty} \subset \mathcal{C}(E)$ and $D \in \mathcal{C}(E)$, $\lim_{k\to\infty} D_k = D$ means that

$$\overline{\lim}_{k\to\infty} D_k = \underline{\lim}_{k\to\infty} D_k = D,$$

where

$$\overline{\lim}_{k\to\infty} D_k := \{x \in E \mid \underline{\lim}_{k\to\infty} d(x, D_k) = 0\},$$
$$\underline{\lim}_{k\to\infty} D_k := \{x \in E \mid \overline{\lim}_{k\to\infty} d(x, D_k) = 0\}$$

and $d(x, D) := \inf_{y\in D} d(x, y), D \in \mathcal{C}(D)$.

The iterates $\tilde{q}^k, k \geq 1$, of the fuzzy relation $\tilde{q}$ are defined by setting $\tilde{q}^1 = \tilde{q}$, and iteratively,

$$\tilde{q}^{k+1}(x, y) = \sup_{z\in E} \left\{\tilde{q}^k(x, z) \wedge \tilde{q}^1(z, y)\right\} \quad (k \geq 1).$$

We define a map $\tilde{q}_\alpha : \mathcal{C}(E) \to 2^E$ $(\alpha \in [0, 1])$ by

$$\tilde{q}_\alpha(D) := \begin{cases} \{y \mid \tilde{q}(x, y) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha \neq 0,\ D \in \mathcal{C}(E), D \neq \emptyset, \\ \mathrm{cl}\{y \mid \tilde{q}(x, y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0,\ D \in \mathcal{C}(E), D \neq \emptyset, \\ E & \text{for } 0 \leq \alpha \leq 1,\ D = \emptyset. \end{cases} \tag{1.2}$$

Especially, we put $\tilde{q}_\alpha(x) := \tilde{q}_\alpha(\{x\})$ for $x \in E$. For $\alpha \in (0, 1]$ and $x \in E$, we also define a sequence $\{\tilde{q}_\alpha^k(x)\}_{k=1,2,\cdots}$ by

$$\tilde{q}_\alpha^1(x) := \tilde{q}_\alpha(x); \quad \text{and} \quad \tilde{q}_\alpha^{k+1}(x) := \tilde{q}_\alpha(\tilde{q}_\alpha^k(x)) \quad \text{for } k = 1, 2, \cdots.$$

First we consider a case when $\tilde{p}_0 = I_{\{x_0\}}$, where I_D is the indicator function for a set D. The α-cuts of the sequence $\{\tilde{p}_k\}_{k=0}^{\infty}$ defined in (1.1) are characterized in the following lemma, which is proved in [3].

Lemma 1.1 ([3, Lemma 1]). *Let $\{\tilde{p}_k\}_{k=0}^{\infty}$ be given in (1.1) with $\tilde{p}_0 = I_{\{x_0\}}$. Then, for any $\alpha \in [0, 1]$,*

(i) $\tilde{p}_{k+1,\alpha} = \tilde{q}_\alpha(\tilde{p}_{k,\alpha}) = \tilde{q}_\alpha^{k+1}(x_0)$;

(ii) $\tilde{q}^k(x_0, \cdot)_\alpha = \tilde{q}_\alpha^k(x_0) \quad (k \geq 1)$.

Here, several preliminary lemmas are given in order to use in the sequel.

Lemma 1.2 ([3, Lemma 2]). *Suppose a family of subsets $\{D_\alpha \mid \alpha \in [0, 1]\} \subset \mathcal{C}(E)$ satisfies the following conditions :*

(i) $D_\alpha \subset D_{\alpha'}$ *for* $\alpha' \leq \alpha$,

(ii) $\lim_{\alpha'\uparrow\alpha} D_{\alpha'} = D_\alpha$, *i.e.,* $\lim_{\alpha'\uparrow\alpha} \rho(D_{\alpha'}, D_\alpha) = 0$.

Then it holds that

$$\lim_{\alpha'\uparrow\alpha} \tilde{q}_{\alpha'}(D_{\alpha'}) = \tilde{q}_\alpha(D_\alpha). \tag{1.3}$$

Lemma 1.3 ([3, 6]). *We suppose that a family of subsets $\{D_\alpha \mid \alpha \in [0, 1]\} (\subset \mathcal{C}(E))$ satisfies the following conditions (i) and (ii) :*

(i) $D_\alpha \subset D_{\alpha'}$ *for* $0 \le \alpha' < \alpha \le 1$,

(ii) $\lim_{\alpha' \uparrow \alpha} D_{\alpha'} = D_\alpha$ *for* $\alpha \in (0,1]$.

Then $\tilde{s}(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge I_{D_\alpha}(x)\}$, $x \in E$, *satisfies* $\tilde{s} \in \mathcal{F}(E)$ *and* $\tilde{s}_\alpha = D_\alpha$ *for all* $\alpha \in [0,1]$.

Lemma 1.4 ([4]). *For a family of subsets* $\{D_k\}_{k=1}^\infty \subset \mathcal{C}(E)$ *the following (i) and (ii) hold:*

(i) *If* $D_{k+1} \supset D_k$ *for all* $k = 1, 2, \cdots$, *then* $\lim_{k\to\infty} D_k = \bigcap_{k=1}^\infty D_k \in \mathcal{C}(E)$.

(ii) *If* $D_{k+1} \subset D_k$ *for all* $k = 1, 2, \cdots$, *then* $\lim_{k\to\infty} D_k = cl\,(\bigcup_{k=1}^\infty D_k) \in \mathcal{C}(E)$.

Proof. It is trivial from the compactness of $(\mathcal{C}(E), \rho)$. □

In Section 2, we give some basic assumptions and the related results, by which the convergence theorem for the sequence of fuzzy states $\{\tilde{p}_k\}_{k=0}^\infty$ defined by (1.1) is established in Section 3. In Section 4, a numerical example is given to comprehend out idea in this paper.

2. Basic assumptions and related results

The continuity and unimodality of a fuzzy relation are defined as follows.

Definition 2. For $\alpha \in [0,1]$, we call the map $\tilde{q}_\alpha(\cdot) : E \mapsto 2^E$ is continuous if

$$\rho(\tilde{q}_\alpha(y), \tilde{q}_\alpha(x)) \to 0 \quad (y \to x) \quad \text{for all } x \in E.$$

Note that $\tilde{q}_\alpha(y) \to \tilde{q}_\alpha(x)$ $(y \to x)$ is equivalent to $\overline{\lim}_{y\to x}\tilde{q}_\alpha(y) = \underline{\lim}_{y\to x}\tilde{q}_\alpha(y)$ (See [4]).

Definition 3 ([7]). We call a fuzzy relation $\tilde{q}$ unimodal if $\tilde{q}_\alpha(x)$ is a closed convex subset of E for all $\alpha \in (0,1]$ and $x \in E$, i.e., $\tilde{q}_\alpha(x) \in \mathcal{C}(E)$ for all $x \in E$.

We also need some elementary notations in a finite dimensional Euclidean space: $x + y$ denotes the sum of $x, y \in E$ and γx denotes the product of a scalar γ and $x \in E$. We put $A + B := \{x + y \mid x \in A,\ y \in B\}$ for $A, B \in \mathcal{C}(E)$. Here we define a half line on E by

$$l(x,y) := \{\gamma(y - x) \mid \gamma \ge 0\} \quad \text{for } x, y \in E.$$

The following monotone property of a fuzzy relation was first introduced by Yoshida [7] to analyze the recurrence of the fuzzy transition in the dynamic system.

Definition 4 ([7]). We call a fuzzy relation $\tilde{q}$ monotone if

$$\tilde{q}_\alpha(y) \subset \tilde{q}_\alpha(x) + l(x,y) \quad \text{for all } \alpha \in (0,1],\ x, y \in E.$$

From now on, we suppose the following assumptions.

Assumption A. The following (A1) – (A3) hold:

(A1) The map $\tilde{q}_\alpha(\cdot) : E \mapsto \mathcal{C}(E)$ is continuous for $\alpha \in [0,1]$;

(A2) $\tilde{q}$ is unimodal;

(A3) $\tilde{q}$ is monotone.

The continuity of $\tilde{q}_\alpha^k(\cdot)$ $(k \geq 1)$ is proved under Assumption A.

Proposition 2.1. *Let $k = 1, 2, \cdots$. The map $\tilde{q}_\alpha^k(\cdot) : E \mapsto 2^E$ is continuous for $\alpha \in [0,1]$.*

Proof. Let $\alpha \in [0,1]$. Let $\{x_n\}_{n=0}^\infty \subset E$ and $x \in E$ be a convergent sequence and its limit point. Then, by induction on k, it is sufficient to prove

$$\tilde{q}_\alpha^2(x) = \lim_{n\to\infty} \tilde{q}_\alpha^2(x_n). \tag{2.1}$$

Let $z \in \overline{\lim}_{n\to\infty}\tilde{q}_\alpha^2(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ and $\{z_{n_i}\}$ such that $z_{n_i} \in \tilde{q}_\alpha^2(x_{n_i})$ $(i = 0, 1, \cdots)$ and $z_{n_i} \to z$ $(i \to \infty)$. By the definition, there exists a sequence $\{y_{n_i}\}_{i=0}^\infty$ such that $y_{n_i} \in \tilde{q}_\alpha(x_{n_i})$ and $z_{n_i} \in \tilde{q}_\alpha(y_{n_i})$ $(i = 0, 1, \cdots)$. Since $\lim_{i\to\infty} \tilde{q}_\alpha(x_{n_i}) = \tilde{q}_\alpha(x)$ from Assumption A1, there exists a convergent subsequence $\{y_{n'_j}\}_{j=0}^\infty$ of $\{y_{n_i}\}_{i=0}^\infty$ and its limit point $y \in E$. It is seen that

$$y = \lim_{j\to\infty} y_{n'_j} \in \lim_{j\to\infty} \tilde{q}_\alpha(x_{n'_j}) = \tilde{q}_\alpha(x),$$

and

$$z = \lim_{j\to\infty} z_{n'_j} \in \lim_{j\to\infty} \tilde{q}_\alpha(y_{n'_j}) = \tilde{q}_\alpha(y) \subset \tilde{q}_\alpha^2(x).$$

Therefore we obtain $\overline{\lim}_{n\to\infty}\tilde{q}_\alpha^2(x_n) \subset \tilde{q}_\alpha^2(x)$.

Conversely let $z \in \tilde{q}_\alpha^2(x)$. Then there exists a point y such that $y \in \tilde{q}_\alpha(x)$ and $z \in \tilde{q}_\alpha(y)$. Since $\tilde{q}_\alpha(x) = \lim_{n\to\infty} \tilde{q}_\alpha(x_n)$, there exists a convergent sequence $\{y_n\}_{n=0}^\infty$ such that $y_n \in \tilde{q}_\alpha(x_n)$ $(n = 0, 1, \cdots)$ and $y_n \to y$ $(n \to \infty)$. From Assumption A1, we have

$$z \in \tilde{q}_\alpha(y) = \lim_{n\to\infty} \tilde{q}_\alpha(y_n).$$

Therefore there exists a convergent sequence $\{z_n\}_{n=0}^\infty$ such that $z_n \in \tilde{q}_\alpha(y_n)$ $(n = 0, 1, \cdots)$ and $z_n \to z$ $(n \to \infty)$. Then $z_n \in \tilde{q}_\alpha(y_n) \subset \tilde{q}_\alpha^2(x_n)$ $(n = 0, 1, \cdots)$. Thus we obtain $\tilde{q}_\alpha^2(x) \subset \underline{\lim}_{n\to\infty}\tilde{q}_\alpha^2(x_n)$. Therefore (2.1) holds and we get this proposition. $\square$

We need the following lemma for Proposition 2.2 below.

Lemma 2.1. *Let $\alpha \in [0,1]$ and $x_1, x_2, y_1 \in E$. Define*

$$C(\lambda) := \tilde{q}_\alpha(\lambda x_1 + (1-\lambda)x_2) \quad \text{for } \lambda \in [0,1],$$

and

$$l_1 := \{y_1 + \gamma(x_2 - x_1) \mid \gamma \geq 0\}.$$

Then the map

$$\lambda\ (\in [0,1]) \mapsto C(\lambda) \cap l_1\ (\in \mathcal{C}(E)) \tag{2.2}$$

is continuous.

Proof. From Assumptions A1 and A2, the map

$$C(\cdot) : [0,1] \mapsto \mathcal{C}(E) \tag{2.3}$$

is continuous. So Lemma is easily proved. $\square$

Proposition 2.2. *Let $k = 1, 2, \cdots$. A fuzzy relation $\tilde{q}_\alpha^k$ is unimodal, i.e., $\tilde{q}_\alpha^k(x) \in \mathcal{C}(E)$ for all $x \in E$.*

Proof. Since $\tilde{q}$ is unimodal, $\tilde{q}_\alpha(x)$ is closed convex for all $x \in E$. Since

$$\tilde{q}_\alpha^{k+1}(x) := \bigcup_{y \in \tilde{q}_\alpha^k(x)} \tilde{q}_\alpha(y), \quad x \in E, \ k \in \mathbf{N},$$

it is sufficient to check that

$$\bigcup_{x \in K} \tilde{q}_\alpha(x) \text{ is convex for any closed convex } K. \tag{2.4}$$

Let K be closed and convex in E, and $y_1, y_2 \in \bigcup_{x \in K} \tilde{q}_\alpha(x)$. Then there exists $x_1, x_2 \in K$ such that

$$y_1 \in \tilde{q}_\alpha(x_1), \quad y_2 \in \tilde{q}_\alpha(x_2).$$

Put

$$y := (y_1 + y_2)/2.$$

Then, to prove $\bigcup_{x \in K} \tilde{q}_\alpha(x)$ is convex, it is sufficient to show that there exists $\lambda \in [0,1]$ such that

$$y \in \tilde{q}_\alpha(\lambda x_1 + (1-\lambda)x_2).$$

Since $\tilde{q}$ is monotone, we have

$$y_1 \in \tilde{q}_\alpha(x_1) \subset \tilde{q}_\alpha(\lambda x_1 + (1-\lambda)x_2) + l(x_2, x_1) \quad \text{for all } \lambda \in [0,1], \tag{2.5}$$

$$y_2 \in \tilde{q}_\alpha(x_2) \subset \tilde{q}_\alpha(\lambda x_1 + (1-\lambda)x_2) + l(x_1, x_2) \quad \text{for all } \lambda \in [0,1]. \tag{2.6}$$

Define

$$l_1 := \{y_1 + \gamma(x_2 - x_1) \mid \gamma \geq 0\},$$

$$l_2 := \{y_2 + \gamma(x_1 - x_2) \mid \gamma \geq 0\}.$$

From (2.5), there exists $z_1 \in \tilde{q}_\alpha(\lambda x_1 + (1-\lambda)x_2)$ and $\gamma_1 \geq 0$ such that $y_1 = z_1 + \gamma_1(x_1 - x_2)$. Therefore

$$z_1 = y_1 + \gamma_1(x_2 - x_1) \in l_1.$$

Namely

$$\tilde{q}_\alpha(\lambda x_1 + (1-\lambda)x_2) \cap l_1 \neq \emptyset \quad \text{for all } \lambda \in [0,1].$$

Define

$$\gamma_1(\lambda) := \min\{\gamma \geq 0 \mid y_1 + \gamma(x_2 - x_1) \in \tilde{q}_\alpha(\lambda x_1 + (1-\lambda)x_2)\} \quad \text{for } \lambda \in [0,1].$$

Similarly define

$$\gamma_2(\lambda) := \min\{\gamma \geq 0 \mid y_2 + \gamma(x_1 - x_2) \in \tilde{q}_\alpha(\lambda x_1 + (1-\lambda)x_2)\} \quad \text{for } \lambda \in [0,1].$$

From Lemma 2.1, the map $\gamma_1(\cdot) : [0,1] \mapsto [0,\infty)$ is continuous. Similarly the map $\gamma_2(\cdot) : [0,1] \mapsto [0,\infty)$ is also continuous. Since $\gamma_1(0) \geq \gamma_1(1) = 0$ and $0 = \gamma_2(0) \leq \gamma_2(1)$, there exists a $\lambda^* \in [0,1]$ such that $\gamma_1(\lambda^*) = \gamma_2(\lambda^*)$. Then

$$\begin{aligned} y &= (y_1+y_2)/2 = \{y_1 + \gamma_1(\lambda^*)(x_2-x_1) + y_2 + \gamma_2(\lambda^*)(x_1-x_2)\}/2 \\ &\in \tilde{q}_\alpha(\lambda^* x_1 + (1-\lambda^*)x_2). \end{aligned}$$

Thus we get that $\bigcup_{x\in K} \tilde{q}_\alpha(x)$ is convex. The proof is completed. □

Proposition 2.3. *Let $k = 1, 2, \cdots$. The fuzzy relation $\tilde{q}^k$ is monotone.*

Proof. Let $\alpha \in (0,1]$. By induction, we show

$$\tilde{q}_\alpha^k(y) \subset \tilde{q}_\alpha^k(x) + l(x,y) \qquad \text{for } x, y \in E,\ k = 1, 2, \cdots. \tag{2.7}$$

It holds clearly for $k = 1$ from the definition. We assume (2.7) for some $k = 1, 2, \cdots$. Let $x, y \in E$ and $w \in \tilde{q}_\alpha^{k+1}(y)$. Then we have

$$\tilde{q}_\alpha^{k+1}(y) = \bigcup_{z \in \tilde{q}_\alpha^k(y)} \tilde{q}_\alpha(z) \subset \bigcup_{z \in \tilde{q}_\alpha^k(x) + l(x,y)} \tilde{q}_\alpha(z).$$

Therefore there exist $z' \in \tilde{q}_\alpha^k(x)$ and $\gamma \geq 0$ such that $w \in \tilde{q}_\alpha(z' + \gamma(y-x))$. Since $\tilde{q}$ is monotone, we have

$$\tilde{q}_\alpha(z' + \gamma(y-x)) \subset \tilde{q}_\alpha(z') + l(z', z' + \gamma(y-x)) \subset \tilde{q}_\alpha(z') + l(x,y).$$

Namely $z' \in \tilde{q}_\alpha^k(x)$ and $w \in \tilde{q}_\alpha(z') + l(x,y)$. Therefore we get $w \in \tilde{q}_\alpha^{k+1}(x) + l(x,y)$. Thus we get (2.7) for $k+1$. By induction on k, we obtain this result. □

3. A limit theorem

In this section, under Assumption A, we give the convergence theorem for a sequence of fuzzy sets $\{\tilde{p}_k\}_{k=0}^\infty$ defined in (1.1).

Theorem 3.1. *Let $\alpha \in (0,1]$. For a sequence of the α-cuts $\{\tilde{q}_\alpha^k(x)\}_{k=0}^\infty$, it holds:*

(i) *There exists a limit*

$$A_\alpha(x) := \lim_{k\to\infty} \tilde{q}_\alpha^k(x) \quad \text{for } x \in E.$$

(ii) *It holds that*

$$\tilde{q}_\alpha(A_\alpha(x)) = A_\alpha(x) \quad \text{for } x \in E. \tag{3.1}$$

Proof. (i) Let $x \in E$. First, we consider a case of $x \in \tilde{q}_\alpha(x)$. Then

$$\tilde{q}_\alpha^k(x) \subset \bigcup_{y \in \tilde{q}_\alpha(x)} \tilde{q}_\alpha^k(y) = \tilde{q}_\alpha^{k+1}(x) \qquad \text{for } k = 1, 2, \cdots.$$

By Lemma 1.4, we obtain

$$\lim_{k \to \infty} \tilde{q}_\alpha^k(x) = \operatorname{cl}\left(\bigcup_{k=1,2,\cdots} \tilde{q}_\alpha^k(x) \right). \tag{3.2}$$

So, we obtain (i) in this case.

Next, we consider a case of $x \notin \tilde{q}_\alpha(x)$. Then from Proposition 2.3, we have

$$\tilde{q}_\alpha^{k+1}(x) = \bigcup_{y \in \tilde{q}_\alpha(x)} \tilde{q}_\alpha^k(y) \subset \bigcup_{y \in \tilde{q}_\alpha(x)} \{\tilde{q}_\alpha^k(x) + l(x,y)\} = \tilde{q}_\alpha^k(x) + \bigcup_{y \in \tilde{q}_\alpha(x)} l(x,y) \quad \text{for } k = 1, 2, \cdots.$$

We put $C_+ := \bigcup_{y \in \tilde{q}_\alpha(x)} l(x,y)$. Then we can easily check that C_+ is a closed convex cone since $\tilde{q}_\alpha(x)$ is closed convex and continuous in x. Since C_+ is a convex cone, inductively we obtain

$$\tilde{q}_\alpha^m(x) \subset \tilde{q}_\alpha^k(x) + C_+ \quad \text{for } k < m. \tag{3.3}$$

On the other hand, from Proposition 2.3, we have

$$\begin{aligned} \tilde{q}_\alpha^k(x) &\subset \tilde{q}_\alpha^k(y) + l(y,x) \\ &\subset \textstyle\bigcup_{y' \in \tilde{q}_\alpha(x)} \tilde{q}_\alpha^k(y') + l(y,x) \\ &= \tilde{q}_\alpha^{k+1}(x) + l(y,x) \quad \text{for } k = 1, 2, \cdots,\ y \in \tilde{q}_\alpha(x). \end{aligned}$$

Since $l(y,x)$ is a convex cone, inductively we also obtain

$$\tilde{q}_\alpha^k(x) \subset \tilde{q}_\alpha^m(x) + l(y,x) \quad \text{for } k < m,\ y \in \tilde{q}_\alpha(x). \tag{3.4}$$

To obtain (i), it is sufficient to prove $\overline{\lim}_{k\to\infty}\tilde{q}_\alpha^k(x) \subset \underline{\lim}_{k\to\infty}\tilde{q}_\alpha^k(x)$. Let $y \in \overline{\lim}_{k\to\infty}\tilde{q}_\alpha^k(x)$. Then there exists a sequence $\{y_{k_j}\}_{j=1}^\infty$ such that $y_{k_j} \in \tilde{q}_\alpha^{k_j}(x)$ $(j = 1, 2, \cdots)$ and $y_{k_j} \to y$ $(j \to \infty)$. Let $j = 1, 2, \cdots$ and $k_j < m < k_{j+1}$. From (3.3), we have

$$y_{k_{j+1}} \in \tilde{q}_\alpha^{k_{j+1}}(x) \subset \tilde{q}_\alpha^m(x) + C_+.$$

So there exists $y' \in \tilde{q}_\alpha(x)$ such that

$$y_{k_{j+1}} \in \tilde{q}_\alpha^m(x) + l(x, y').$$

So, there exist $u_1 \in \tilde{q}_\alpha^m(x)$ and $\lambda_1 \geq 0$ such that

$$y_{k_{j+1}} = u_1 + \lambda_1(y' - x). \tag{3.5}$$

On the other hand, from (3.4), we have

$$y_{k_j} \in \tilde{q}_\alpha^{k_j}(x) \subset \tilde{q}_\alpha^m(x) + l(y', x).$$

Therefore, there exist $u_2 \in \tilde{q}_\alpha^m(x)$ and $\lambda_2 \geq 0$ such that

$$y_{k_j} = u_2 + \lambda_2(x - y'). \tag{3.6}$$

We define $w_m := \lambda y_{k_j} + (1-\lambda) y_{k_{j+1}}$, where

$$\lambda := \begin{cases} 0 & \text{if } \lambda_1 = 0, \\ 1 & \text{if } \lambda_2 = 0, \\ \lambda_1/(\lambda_1+\lambda_2) & \text{otherwise.} \end{cases} \tag{3.7}$$

From (3.5), (3.6) and (3.7), we obtain

$$w_m = \lambda y_{k_j} + (1-\lambda) y_{k_{j+1}} = \lambda u_2 + (1-\lambda) u_1 \in \tilde{q}_\alpha^m(x) \tag{3.8}$$

since $\tilde{q}_\alpha^m(x)$ is convex from Proposition 2.2. Here we define a sequence $\{z_m\}_{m=1}^\infty$ by

$$z_m := \begin{cases} w_m & \text{for } k_j < m < k_{j+1} \text{ and } j = 1, 2, \cdots, \\ y_{k_j} & \text{for } m = k_j \text{ and } j = 1, 2, \cdots. \end{cases}$$

Then it is trivial that $z_m \in \tilde{q}_\alpha^m(x)$ $(m = 1, 2, \cdots)$. From (3.8), we obtain $z_m \to x$ $(j, m \to \infty)$. Namely $x \in \underline{\lim}_{k\to\infty} \tilde{q}_\alpha^k(x)$. Thus we get (i).

(ii) Let $\alpha \in [0,1]$ and $x \in E$. First, we show $\tilde{q}_\alpha(A_\alpha(x)) \subset A_\alpha(x)$. If $z \in \tilde{q}_\alpha(A_\alpha(x)) = \bigcup_{y \in A_\alpha(x)} \tilde{q}_\alpha(y)$, then there exists $y \in A_\alpha(x)$ such that $z \in \tilde{q}_\alpha(y)$. From (i), there exists a sequence $\{y_k\}_{k=1}^\infty$ such that $y_k \in \tilde{q}_\alpha^k(x)$ $(k = 1, 2, \cdots)$ and $y_k \to y$ $(k \to \infty)$. From Assumption A1, $\tilde{q}_\alpha(y_k) \to \tilde{q}_\alpha(y)$ $(k \to \infty)$. Therefore there exists a sequence $\{z_k\}_{k=1}^\infty$ such that $z_k \in \tilde{q}_\alpha(y_k)$ $(k = 1, 2, \cdots)$ and $z_k \to z$ $(k \to \infty)$. Then

$$z_k \in \tilde{q}_\alpha(y_k) \subset \bigcup_{y \in \tilde{q}_\alpha^k(x)} \tilde{q}_\alpha(y) \subset \tilde{q}_\alpha^{k+1}(x), \quad k = 1, 2, \cdots.$$

From (i), we get $z \in A_\alpha(x)$. Therefore, we obtain $\tilde{q}_\alpha(A_\alpha(x)) \subset A_\alpha(x)$.

Conversely if $z \in A_\alpha(x)$, then there exists a sequence $\{z_k\}_{k=2}^\infty$ such that

$$z_{k+1} \in \tilde{q}_\alpha^{k+1}(x) = \bigcup_{w \in \tilde{q}_\alpha^k(x)} \tilde{q}_\alpha(w) \quad (k = 1, 2, \cdots) \quad \text{and} \quad z_{k+1} \to z \quad (k \to \infty).$$

There exists a sequence $\{w_k\}_{k=1}^\infty$ such that $w_k \in \tilde{q}_\alpha^k(x)$ and $z_{k+1} \in \tilde{q}_\alpha(w_k)$ $(k = 1, 2, \cdots)$. Since E is compact, there exist a convergent subsequence $\{w_{k_j}\}_{j=1}^\infty$ of $\{w_k\}_{k=1}^\infty$ and its limit point $w \in E$. From (i) and Assumption A1, we obtain $w \in \overline{\lim}_{k\to\infty} \tilde{q}_\alpha^k(x) = A_\alpha(x)$ and $z \in \tilde{q}_\alpha(w)$. Thus we get (ii). Therefore the proof is completed. □

Now, we introduce a notion of the convergence of fuzzy sets, which is weaker than that in [3] and [5]. The following convergence is well-defined since the fuzzy set are determined uniquely from the property of the α-cuts of a fuzzy set (c.f. Lemma 1.3).

Definition 5 (Convergence of fuzzy sets). For $\{\tilde{s}_k\}_{k=0}^\infty \subset \mathcal{F}(E)$ and $\tilde{r} \in \mathcal{F}(E)$, $\lim_{k\to\infty} \tilde{s}_k = \tilde{r}$ means that

$$\rho(\tilde{s}_{k,\alpha}, \tilde{r}_\alpha) \to 0 \ (k \to \infty) \quad \text{except for at most countable } \alpha \in [0,1].$$

In the rest of this section, we show that the sequence $\{\tilde{p}_k\}_{k=0}^\infty$, which is defined by (1.1), converges in the sense of this definition. We put the surface of a unit ball by $U := \{x \in X \mid \|x\| = 1\}$. For $D, D' \in \mathcal{C}(E)$ $(D' \supset D)$ and $u \in U$, we call D' u-directional to D if there exist a real number $\lambda > 0$, $y \in D$ and $z \in D'$ such that

(i) $d(z, y) = \rho(D', D)$,

(ii) $z - y = \lambda u$.

Then, we have $\rho(D', D) = \lambda$. For a subset $V \subset U$, we will introduce a binary relation $\supset_V$ as follows :

$$D' \supset_V D \iff \text{there exists } u \in V \text{ such that } D' \text{ is } u\text{-directional to } D.$$

In order to prove the convergence theorem we need the following assumption.

Assumption B. There exists a finite subset $V := \{u_1, u_2, \cdots, u_l\} \subset U$ such that for any $D, D' \in \mathcal{C}(E)$ $(D' \supset D)$, if $D' \supset_V D$, then $\tilde{q}_{\alpha'}(D') \supset_V \tilde{q}_\alpha(D)$ for all α', α $(0 \leq \alpha' < \alpha \leq 1)$.

The following lemma is trivial by induction.

Lemma 3.1. *Suppose that Assumption B holds. Let $D, D' \in \mathcal{C}(E)$ $(D' \supset D)$. Then, for V in Assumption B, $D' \supset_V D$ implies $\tilde{q}^k_{\alpha'}(D') \supset_V \tilde{q}^k_\alpha(D)$ for all α', α $(0 \leq \alpha' < \alpha \leq 1)$ and $k \geq 1$.*

For $A_\alpha(x)$ in Theorem 3.1, let

$$A^+_\alpha(x) := \bigcap_{\alpha' < \alpha} A_{\alpha'}(x) \quad (\alpha \in (0, 1]), \quad \text{and} \quad A^+_0(x) := A_0(x).$$

Then, we define a distance $\lambda(\alpha) := \rho(A^+_\alpha(x), A_\alpha(x))$ and the set of discontinuous levels $\Lambda := \{\alpha \in [0, 1] \mid \lambda(\alpha) > 0\}$.

Lemma 3.2. *Under Assumptions A and B, the set Λ is at most countable.*

Proof. First we prove

$$A^+_\alpha(x) \supset_V A_\alpha(x) \quad \text{for } \alpha \in (0, 1]. \tag{3.9}$$

Let α', α $(0 \leq \alpha' < \alpha \leq 1)$. From the continuity of $\rho(\cdot, \cdot)$ and Theorem 3.1, we have

$$\lim_{k \to \infty} \rho(\tilde{q}^k_{\alpha'}(x), \tilde{q}^k_\alpha(x)) = \rho(A_{\alpha'}(x), A_\alpha(x)). \tag{3.10}$$

On the other hand, since $\tilde{q}^k_{\alpha'}(x) \supset_V \tilde{q}^k_\alpha(x)$ from Lemma 3.1, there exist $x^k_\alpha \in \tilde{q}^k_\alpha(x)$, $x^k_{\alpha'} \in \tilde{q}^k_{\alpha'}(x)$, $\lambda_k > 0$, $u^k \in V$ such that

(a) $\rho(\tilde{q}^k_{\alpha'}(x), \tilde{q}^k_\alpha(x)) = d(x^k_{\alpha'}, x^k_\alpha)$ and

(b) $x^k_{\alpha'} - x^k_\alpha = \lambda_k u^k$.

Since E is compact and V is finite, by taking a subsequence if we need, we may assume that there exist $x_{\alpha'} \in A_{\alpha'}(x), x_\alpha \in A_\alpha(x), \lambda > 0$ and $u \in V$ such that

$$x^k_{\alpha'} \to x_{\alpha'}, \quad x^k_\alpha \to x_\alpha, \quad \lambda_k \to \lambda, \quad u_k = u \quad \text{as } k \to \infty.$$

From (3.10), (a) and (b), we have $\rho(A_{\alpha'}(x), A_\alpha(x)) = d(x_{\alpha'}, x_\alpha)$ and $x_{\alpha'} - x_\alpha = \lambda u$. This yields

$$A^+_{\alpha'}(x) \supset_V A_\alpha(x) \quad \text{for } \alpha', \alpha \ (0 \leq \alpha' < \alpha \leq 1)$$

Therefore we obtain (3.9).

Next we put

$$\Lambda(n) := \{\alpha \in \Lambda \mid \lambda(\alpha) \geq 1/n\} \quad \text{for } n = 1, 2, \cdots.$$

Then, we have $\rho(A^+_\alpha(x), A_\alpha(x)) \geq 1/n$ for all $\alpha \in \Lambda(n)$. From (3.9), the direction whose distance is greater than $1/n$, are contained in V. However, from Assumption B, V contains only finite vectors. Therefore, we obtain that $\Lambda(n)$ is a finite set since E is compact. Thus, $\Lambda = \bigcup_{n=1}^{\infty} \Lambda(n)$ is at most countable and the proof is completed. □

Here, we can prove the main theorem, which asserts the convergence of $\{\tilde{p}_k\}_{k=0}^{\infty}$ under Assumptions A and B.

Theorem 3.2. *Under Assumptions A and B, the sequence $\{\tilde{p}_k\}_{k=0}^{\infty}$, which is defined by (1.1), converges to $\tilde{p} \in \mathcal{F}(E)$ and $\tilde{p}$ satisfies*

$$\tilde{p}(y) = \sup_{x \in E} \{\tilde{p}(x) \wedge \tilde{q}(x, y)\}, \quad y \in E, \tag{3.11}$$

where $\tilde{p}_0 = I_{\{x\}}$ for some $x \in E$.

Proof. From Lemma 3.2, there exists a subset $F(\subset [0, 1])$ which is at most countable and satisfies

$$A^+_\alpha(x) = A_\alpha(x) \quad \text{for all } \alpha \in F^c, \tag{3.12}$$

where $F^c := [0, 1] \setminus F$. Since $\{A^+_\alpha(x), \alpha \in [0, 1]\}$ satisfies the conditions (i) and (ii) of Lemma 1.3, we can construct a fuzzy set $\tilde{p} \in \mathcal{F}(E)$ by

$$\tilde{p}(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge I_{A^+_\alpha(x)}(x)\}, \quad x \in E. \tag{3.13}$$

From (3.12) and Theorem 3.1, we have

$$\tilde{p}_{k,\alpha} = \tilde{q}^k_\alpha(x) \to \tilde{p}_\alpha \quad \text{as } k \to \infty \quad \text{for all } \alpha \in F^c.$$

From the definition of the convergence of fuzzy sets, we obtain

$$\tilde{p}_k \to \tilde{p} \quad \text{as } k \to \infty.$$

Then, from (3.12) and Theorem 3.1, we also have

$$\tilde{p}_\alpha = \tilde{q}_\alpha(\tilde{p}_\alpha) \quad \text{for all } \alpha \in F^c.$$

By the denseness of F^c in [0,1] and Lemma 1.2, we obtain

$$\tilde{p}_\alpha = \tilde{q}_\alpha(\tilde{p}_\alpha) \quad \text{for all } \alpha \in [0, 1].$$

This implies (3.11) and the proof is completed. □

As a remark,, the uniqueness of a solution (3.11) has been shown in the contractive case of the previous paper [3]. However the uniqueness in the monotone case does not holds generally.

4. Numerical example

We consider a one-dimensional numerical example to illustrate our results in the preceding section. Let $X = (-\infty, \infty)$ and $E = [-2, 2]$. We give a fuzzy relation, which is monotone (see [7, Example 6.1]), by

$$\tilde{q}(x, y) = (1 - |y - x^3|) \vee 0, \quad x, y \in [-2, 2]. \tag{4.1}$$

Then, Assumptions A and B hold by taking $U = \{-1, 1\}$. For $x = 3/4$, we have

$$A_\alpha(3/4) = \begin{cases} [x_\alpha^-, x_\alpha^+] & \text{if} \quad 43/64 < \alpha \leq 1, \\ [x_\alpha^-, 3/4] & \text{if} \quad \alpha = 43/64, \\ [x_\alpha^-, 2] & \text{if} \quad 1 - 2/(3\sqrt{3}) < \alpha < 43/64, \\ [-1/\sqrt{3}, 2] & \text{if} \quad \alpha = 1 - 2/(3\sqrt{3}), \\ [-2, 2] & \text{if} \quad 0 \leq \alpha < 1 - (2/3\sqrt{3}), \end{cases} \tag{4.2}$$

where, for α satisfying the conditions of (4.2), x_α^- and x_α^+ denote the second least of three real solutions of the equation $x^3 - x - 1 + \alpha = 0$ and $x^3 - x + 1 - \alpha = 0$ respectively, and so

$$A_\alpha^+(3/4) = \begin{cases} [x_\alpha^-, x_\alpha^+] & \text{if} \quad 43/64 < \alpha \leq 1, \\ [x_\alpha^-, 2] & \text{if} \quad 1 - 2/(3\sqrt{3}) < \alpha \leq 43/64, \\ [-2, 2] & \text{if} \quad 0 \leq \alpha \leq 1 - 2/(3\sqrt{3}). \end{cases} \tag{4.3}$$

Therefore, $A_\alpha(3/4)$ is discontinuous at $\alpha = 43/64, (-3 + \sqrt{37})/8$.

Take an initial fuzzy state by $\tilde{p}_0 = I_{\{3/4\}}$. From (4.2) and (3.13), the sequence $\{\tilde{p}_k\}_{k=0}^\infty$ converges to the limiting fuzzy state $\tilde{p}$:

$$\tilde{p}(y) = \begin{cases} 1 - 2/(3\sqrt{3}) & \text{if} \quad -2 \leq y \leq -1/\sqrt{3}, \\ -y^3 + y + 1 & \text{if} \quad -1/\sqrt{3} < y \leq 0, \\ y^3 - y + 1 & \text{if} \quad 0 < y \leq (-3 + \sqrt{37})/8, \\ 43/64 & \text{if} \quad (-3 + \sqrt{37})/8 < y \leq 2. \end{cases}$$

Fig. 1 shows the convergence.

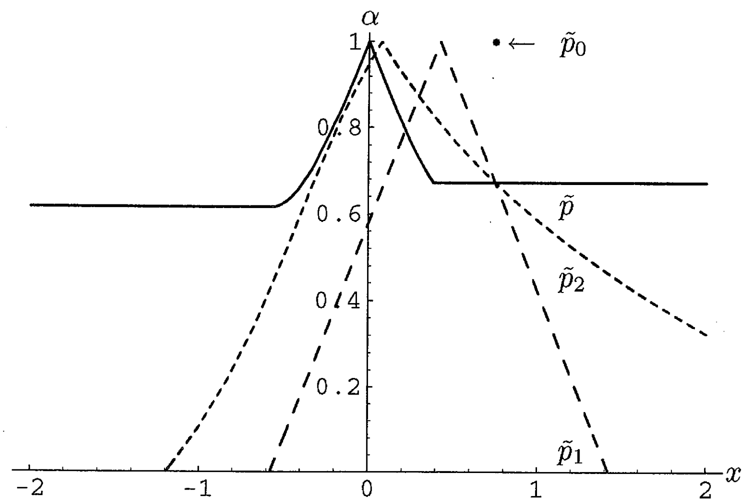

Fig. 1 : The convergence of the sequence of fuzzy states $\{\tilde{p}_k\}_{k=0}^\infty$.

References

[1] R.E.Bellman and L.A.Zadeh, Decision-making in a fuzzy environment, *Management Sci. Ser B.* **17** (1970) 141-164.

[2] G.J.Klir and T.A.Folger, *Fuzzy Sets, Uncertainty, and Information* (Prentice-Hall, London, 1988).

[3] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51** (1992) 83-88.

[4] K.Kuratowski, *Topology I* (Academic Press, New York, 1966).

[5] S.Nanda, On sequences of fuzzy numbers, *Fuzzy Sets and Systems* **33** (1989) 123-126.

[6] V.Novák, *Fuzzy Sets and Their Applications* (Adam Hilger, Bristol-Boston, 1989).

[7] Y.Yoshida, The recurrence of dynamic fuzzy systems, submitted.

[8] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure : The contractive case, *Fuzzy Sets and Systems* **60** (1993) 283-294.

[9] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure : The unbounded case, *Fuzzy Sets and Systems* **66** (1994) 83-95.