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Convergence of sequence of measurable functions on fuzzy measure spaces[1]

Jun Li [a], Masami Yasuda [a], Qingshan Jiang [b,*], Hisakichi Suzuki [b], Zhenyuan Wang [c], George J. Klir [c]

[a] *Department of Mathematics, Faculty of Science, Chiba University, Chiba 263, Japan*
[b] *Department of Mathematics, Chiba Institute of Technology, Shibazono, Narashino 275, Japan*
[c] *Department of Systems Science and Industrial Engineering, Thomas J. Watson School of Engineering and Applied Science, State University of New York at Binghamton, Binghamton, NY 13902-6000, USA*

Abstract

The purpose of this paper is to investigate the convergence of sequence of measurable functions on fuzzy measure spaces. Several classical results on the convergence of measurable functions, such as Egoroff's theorem, Lebesgue's theorem and Riesz's theorem, are extended to fuzzy measure spaces by using the pseudometric generating property and order-continuity of fuzzy measures. © 1997 Elsevier Science B.V.

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1. Introduction

In classical measure theory, several types of convergence were introduced for sequence of measurable functions on a measure space, and basic relations among these types were established [1]. Egoroff's theorem, one of the most important of these results, states that almost everywhere convergence implies uniform convergence outside a negligibly small set. In the proof of the theorem [1], the following two properties of a measure, μ, are needed: subadditivity,

$$\mu\left(\bigcup_{n=1}^{+\infty} E_n\right) \leqslant \sum_{n=1}^{+\infty} \mu(E_n)$$

and continuity,

$$E_n \searrow \emptyset \Longrightarrow \lim_{n\to+\infty} \mu(E_n) = 0 .$$

Both these properties are direct consequences of σ-additivity of μ, which is called order-continuity in this paper.

Fuzzy measure theory is a generalization of classical measure theory. This generalization is obtained by replacing the additivity axiom of classical measures with weak axioms of monotonicity and continuity [9]. As elaborated in [6–9], some generalizations of Egoroff's theorem, Lebesgue's theorem, and Riesz's theorem for sequence of measurable functions on classical measure spaces remain valid for fuzzy measures with the finiteness and autocontinuity.

In this paper, we further investigate these convergent theorems under weaker characteristics, such as

* Corresponding author.

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the pseudometric generating property, property (s), and order-continuity, to generalize the classical results and obtain many fundamental results. Our main results are presented in Section 3. They are interesting not only as generalizations of their counterparts in classical measure theory, but also as an indication for the importance of many structural characteristics of set fuctions. In Section 4, we discuss some relations among these properties in the class of fuzzy measures.

2. Preliminary

Let X denote a non-empty set, $\mathscr{F}$ denote a σ-algebra of subsets of X, $N=\{1,2,\ldots,\}$, and $\overline{R_+}=[0,+\infty]$. Unless stated otherwise, all the sets are supposed to belong to $\mathscr{F}$ and we make the following conventions: $\sup\{i: i\in\emptyset\}=0, \infty-\infty=0$, and $0\cdot\infty=0$. The following terminology is used without any further reference.

A set function $\mu:\mathscr{F}\to\overline{R_+}$ is said to be *exhaustive* if $\lim_{n\to+\infty}\mu(E_n)=0$ for any infinite disjoint sequence $\{E_n\}_n$; *order-continuous* if $\lim_{n\to+\infty}\mu(E_n)=0$ whenever $E_n\searrow\emptyset$; *null-additive* if $\mu(E\cup F)=\mu(E)$ for any E whenever $\mu(F)=0$; and *autocontinuous* if $\lim_{n\to+\infty}\mu(F_n)=0$ implies $\lim_{n\to+\infty}\mu(E\cup F_n)=\mu(E)$ and $\lim_{n\to+\infty}\mu(E-F_n)=\mu(E)$ for any E.

A *fuzzy measure* is a set function $\mu:\mathscr{F}\to[0,+\infty]$ with the properties:

(FM1) $\mu(\emptyset)=0$;

(FM2) $A\subset B \Rightarrow \mu(A)\leqslant\mu(B)$;

(FM3) $A_1\subset A_2\subset\cdots \Rightarrow$

$$\mu\left(\bigcup_{n=1}^{\infty}A_n\right)=\lim_{n\to\infty}\mu(A_n);$$

(FM4) $A_1\supset A_2\supset\cdots$, and there exists n_0 with $\mu(A_{n_0})<+\infty$

$$\Rightarrow \mu\left(\bigcap_{n=1}^{\infty}A_n\right)=\lim_{n\to\infty}\mu(A_n).$$

Definition 1. μ is said to have the property (s) if for any $\{E_n\}_n$ with $\lim_{n\to+\infty}\mu(E_n)=0$, there exists a subsequence $\{E_{n(i)}\}_i$ of $\{E_n\}_n$ such that $\mu(\overline{\lim}_{i\to+\infty}E_{n(i)})=0$ [6].

Definition 2. μ is said to have the pseudometric generating property, abbreviated as p.g.p., if for any $\varepsilon>0$, $\exists\,\delta>0$ such that

$$\mu(E)\vee\mu(F)<\delta \quad\text{implies}\quad \mu(E\cup F)<\varepsilon.$$

Now we present some propositions concerning the introduced properties of fuzzy measures.

Proposition 1. *Let μ be a fuzzy measure, then μ is exhaustive if and only if μ is order-continuous* [2].

Proposition 2. *Any finite fuzzy measure is exhaustive* [2].

Proposition 3. *If a fuzzy measure is autocontinuous, then it has the pseudometric generating property* [3].

Proposition 4. *If fuzzy measure μ is null-additive and order-continuous, then $\lim_{n\to+\infty}\mu(A_n)=0$ whenever $A_n\searrow A$ and $\mu(A)=0$* [5].

Proposition 5. *If fuzzy measure μ has the pseudometric generating property, and $\lim_{n\to+\infty}\mu(E_n)=0$, then there exist a sequence $\{\delta_r\}_r$ of $\overline{R_+}$, and a subsequence $\{E_{n(i)}\}_i$ of $\{E_n\}_n$ such that $\delta_r\searrow 0$ and*

$$\mu\left(\bigcup_{i=r+1}^{+\infty}E_{n(i)}\right)\leqslant\delta_r, \forall r\geqslant 1 \text{ [5]}.$$

Proposition 6. *If a fuzzy measure has the pseudometric generating property, then it has the property (s)* [5].

Proposition 7. *Let fuzzy measure μ be exhaustive and null-additive. Then, μ has the pseudometric generating property if and only if it has the property (s)* [5].

3. Convergence of sequence of measurable functions

Let F be the class of all finite measurable functions on fuzzy measure space $(X,\mathscr{F},\mu)$, and let $f,g,f_n,g_n\in F$ $(n\in N)$. We denote that $\{f_n\}_n$ *everywhere converges to* f by $f_n\longrightarrow f$. We say that $\{f_n\}_n$ *almost everywhere converges to f on X* if there is subset $E\subset X$ such that $\mu(E)=0$ and $f_n\to f$ on $X-E$, and denote it by $f_n\xrightarrow{\text{a.e.}}f$; $\{f_n\}_n$ *almost uniformly converges to f on X* if for any $\varepsilon>0$ there is subset $E_\varepsilon\subset X$ such that $\mu(X-E_\varepsilon)<\varepsilon$ and f_n converges to f

uniformly on E_ε, and denote it by $f_n \xrightarrow{\text{a.u.}} f$; $\{f_n\}_n$ *converges to* f *in fuzzy measure* μ *on* X if for any $\varepsilon > 0$, $\lim_{n\to\infty}\mu(\{x : |f_n(x) - f(x)| \geqslant \varepsilon\}) = 0$, and denote it by $f_n \xrightarrow{\mu} f$; and $\{f_n\}_n$ is *fundamental in fuzzy measure* μ if $\lim_{n\wedge m\to+\infty}\mu(\{x : |f_n(x) - f_m(x)| \geqslant \varepsilon\}) = 0$ for any $\varepsilon > 0$.

Obviously, $f_n \xrightarrow{\text{a.u.}} f$ implies $f_n \xrightarrow{\mu} f$. Theorems 1, 3 and 5 below are generalizations of Egoroff's theorem, Lebesgue's theorem, and Riesz's theorem from classical measure spaces to fuzzy measure spaces.

Theorem 1 (Egoroff's theorem). *Let* μ *be an order-continuous fuzzy measure with the pseudometric generating property, Then, for any sequence* $\{f_n\}_n$,

$$f_n \xrightarrow{\text{a.e.}} f \Longrightarrow f_n \xrightarrow{\text{a.u.}} f.$$

Proof. Since $f_n \xrightarrow{\text{a.e.}} f$, there is a subset $E \in \mathscr{F}$ with $\mu(E) = 0$ such that the sequence $\{f_n\}_n$ coverges to f everywhere on $X - E$. If we denote

$$E_n^{(m)} = \bigcap_{i=n}^{+\infty} \left\{x \in X - E : |f_i(x) - f(x)| < \frac{1}{m}\right\}$$

for any $m \geqslant 1$, then $E_n^{(m)}$ is increasing in n for each fixed m. Thus, we get

$$X - E = \bigcap_{m=1}^{+\infty}\bigcup_{n=1}^{+\infty} E_n^{(m)}$$

and $(X - E) - E_n^{(m)} \searrow \emptyset$ as $n \to +\infty$ for any fixed $m \geqslant 1$. Therefore, we have

$$\lim_{n\to+\infty} \mu((X - E) - E_n^{(m)}) = 0$$

for any $m \geqslant 1$ by using the order-continuity of μ. Thus, there exists a subsequence $\{(X - E) - E_{n(m)}^{(m)}\}_m$ of $\{(X - E) - E_n^{(m)} : n, m \geqslant 1\}$ satisfying

$$\mu((X - E) - E_{n(m)}^{(m)}) \leqslant \frac{1}{m}, \quad \forall m \geqslant 1$$

and, then,

$$\lim_{m\to+\infty} \mu((X - E) - E_{n(m)}^{(m)}) = 0.$$

On the other hand, from Propositions 5 and 7 there exists a sequence $\{\delta_r\}_r$ of real numbers, and a subsequence $\{(X - E) - E_{n(m_r)}^{(m_r)}\}_r$ of $\{(X - E) - E_{n(m)}^{(m)}\}_m$ such that $\delta_r \searrow 0$ and

$$\mu\left(\bigcup_{i=r+1}^{+\infty} ((X - E) - E_{n(m_i)}^{(m_i)})\right) \leqslant \delta_r, \quad \forall r \geqslant 1.$$

For any $\varepsilon > 0$, by using the p.g.p. of μ, we know that there exists $\delta > 0$ such that $\mu(E) \vee \mu(F) < \delta$ implies $\mu(E \cup F) < \varepsilon$. For above $\delta > 0$, there is $r_0 \geqslant 1$ such that

$$\mu\left(\bigcup_{i=r_0+1}^{+\infty} ((X - E) - E_{n(m_i)}^{(m_i)})\right) < \delta.$$

Puting $E_\varepsilon = \bigcap_{i=r_0+1}^{+\infty} E_{n(m_i)}^{(m_i)}$, we have $\mu((X - E) - E_\varepsilon) < \delta$. Thus,

$$\mu(X - E_\varepsilon) = \mu(((X - E) - E_\varepsilon) \cup ((X - E_\varepsilon) \cap E))) < \varepsilon$$

and $\{f_n\}_n$ converges to f uniformly on E_ε. This shows that $f_n \xrightarrow{\text{a.u.}} f$. □

By Theorem 1 and Propositions 1 and 7, we have the following theorem.

Theorem 2. *Let* μ *be a null-additive and order-continuous fuzzy measure with the property* (*s*). *Then, for any sequence* $\{f_n\}_n$,

$$f_n \xrightarrow{a.e.} f \Longrightarrow f_n \xrightarrow{a.u.} f.$$

Definition 3. Fuzzy measure μ is said to have the Egoroff's property, denoted by (*E.p.*), if $f_n \xrightarrow{\text{a.e.}} f$ implies $f_n \xrightarrow{\text{a.u.}} f$ for any sequence $\{f_n\}_n$.

From Theorem 2 and Proposition 1, we have the following corollary.

Corollary 1. *Let* μ *be a null-additve and exhaustive fuzzy measure with the property* (*s*). *Then,* μ *has the* (*E.p.*).

Remark 1. (i) If μ is a finite fuzzy measure, then the pseudometric generating property of Egoroff's theorem may be replaced by the null-additivity [8].

(ii) In general, the pseudometric generating property and order-continuity of Egoroff's theorem are inevitable as shown in the following two examples.

Example 1. Let $X = [0, +\infty)$, $\mathscr{F}$ be Borel σ-algebra of X, and μ_1 be Lebesgue's measure. Then μ_1 has the p.g.p., but μ_1 is not order-continuous. We denote the fuzzy measure space by $(X, \mathscr{F}, \mu_1)$ and put

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [0, n], \\ 0 & \text{if } x \in (n, +\infty). \end{cases}$$

Then, $f_n \longrightarrow 1$, but $f_n \xrightarrow{\mu_1} 1$ and $f_n \xrightarrow{\text{a.u.}} 1$ are not true in fuzzy measure space $(X, \mathscr{F}, \mu_1)$.

Example 2. Let $X = [0, +\infty)$, $\mathscr{F}$ be Borel σ-algebra of X, and m be Lebesgue's measure. Put

$$\mu_2(E) = \begin{cases} 0 & \text{if } 0 \notin E, \\ m(E) & \text{if } 0 \in E. \end{cases}$$

Then μ_2 is not null-additive and does not have the p.g.p., but μ_2 is exhaustive and has the property (s) [5]. On fuzzy measure space $(X, \mathscr{F}, \mu_2)$, if we put

$$f_n(x) = \begin{cases} 1 & \text{if } x \in (0, n], \\ 0 & \text{if } x = 0, \\ 0 & \text{if } x \in (n, +\infty), \end{cases}$$

then $f_n \xrightarrow{\text{a.e.}} 1$; but $f_n \xrightarrow{\mu_2} 1$ and $f_n \xrightarrow{\text{a.u.}} 1$ are not true in fuzzy measure space $(X, \mathscr{F}, \mu_2)$.

Theorem 3 (Lebesgue's theorem). *Let μ be a null-additive and order-continuous fuzzy measure. Then,*

$$f_n \xrightarrow{\text{a.e.}} f \Longrightarrow f_n \xrightarrow{\mu} f$$

for any sequence $\{f_n\}_n$.

Proof. Suppose that sequence $\{f_n\}_n$ coverges to f almost everywhere, and let D be the set of those points x at which $f_n(x)$ does not converge to $f(x)$. Then,

$$D = \bigcup_{m=1}^{+\infty} \bigcap_{n=1}^{+\infty} \bigcup_{k=n}^{+\infty} \left\{ x : |f_k(x) - f(x)| \geqslant \frac{1}{m} \right\}$$

and $\mu(D) = 0$; furthermore,

$$\mu\left(\bigcap_{n=1}^{+\infty} \bigcup_{k=n}^{+\infty} \left\{ x : |f_k(x) - f(x)| \geqslant \frac{1}{m} \right\} \right) = 0$$

for any $m \geqslant 1$. If we take $A_n^{(m)} = \bigcup_{k \geqslant n} \{x : |f_k(x) - f(x)| \geqslant \frac{1}{m}\}$ and

$$A^{(m)} = \bigcap_{n=1}^{+\infty} \bigcup_{k=n}^{+\infty} \left\{ x : |f_k(x) - f(x)| \geqslant \frac{1}{m} \right\}$$

for any $m \geqslant 1$, then $A_n^{(m)} \searrow A^{(m)}$ as $n \to +\infty$. By Proposition 4, we can obtain $\lim_{n\to+\infty} \mu(A_n^{(m)}) = 0$ for any $m \geqslant 1$ and, hence,

$$\lim_{n\to+\infty} \mu\left(\left\{ x : |f_n(x) - f(x)| \geqslant \frac{1}{m} \right\} \right) \leqslant \mu(A_n^{(m)}) = 0, \quad \forall m \geqslant 1.$$

This shows that $f_n \xrightarrow{\mu} f$. □

We can prove the following theorem in a similar way.

Theorem 4. *Let μ be an exhaustive fuzzy measure. Then,*

$$f_n \longrightarrow f \Longrightarrow f_n \xrightarrow{\mu} f$$

for any sequence $\{f_n\}_n$.

Definition 4. Fuzzy measure μ is said to have the *Lebesgue's property*, denoted by $(L.p.)$, if $f_n \xrightarrow{\text{a.e.}} f$ implies $f_n \xrightarrow{\mu} f$ for any sequence $\{f_n\}_n$.

Corollary 2. *Let μ be a null- additive and exhaustive fuzzy measure. Then, μ has the $(L.p.)$.*

Proof. Trivial by Proposition 1 and Theorem 3. □

Remark 2. The null-additivity of Lebesgue's theorem is inevitable (see Example 2).

Theorem 5 (Riesz's theorem). *Let μ be a fuzzy measure with the property (s). If $f_n \xrightarrow{\mu} f$, then there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$.*

Proof. Let $f_n \xrightarrow{\mu} f$. Then,

$$\lim_{n\to+\infty} \mu\left(\left\{ x : |f_n(x) - f(x)| \geqslant \frac{1}{k} \right\} \right) = 0, \quad \forall k \geqslant 1.$$

If we take $E_n^{(k)} = \{x : |f_n(x) - f(x)| \geqslant 1/k\}$, then there exists a subsequence $\{n_k\}_k$ such that $\mu(E_{n_k}^{(k)}) \leqslant 1/k$ for any $k \geqslant 1$. Since μ has the property (s), there is a subsequence $\{E_{n_{k_i}}^{(k_i)}\}_i$ of $\{E_{n_k}^{(k)}\}_k$ such that $\mu\left(\overline{\lim}_{i\to+\infty} E_{n_{k_i}}^{(k_i)}\right) = 0$. This shows that $f_{n_{k_i}} \xrightarrow{\text{a.e.}} f$. □

Definition 5. Fuzzy measure μ is said to have the Riesz's property, denoted by $(R.p.)$, if $f_n \xrightarrow{\mu} f$ implies the existence of a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$.

By Proposition 6 and Theorem 5, we can prove the following corollary.

Corollary 3. *Let μ be a fuzzy measure with the pseudometric generating property. Then, μ has the $(R.p.)$.*

Next, we show that most of the results on convergence in measure still hold for any fuzzy measure that is exhaustive and has the pseudometric generating property.

Theorem 6. *Let μ be an order-continuous fuzzy measure with the pseudometric generating property. Then $f_n \xrightarrow{\mu} f$ if and only if, for any subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$, there exists a subsequence $\{f_{n_{k_i}}\}_i$ of $\{f_{n_k}\}_k$ with $f_{n_{k_i}} \xrightarrow{\text{a.e.}} f$.*

Proof. The "only if" part is trivial by Proposition 6 and Theorem 5. Now, we assume that the "if" part is not true. Then there exist $\varepsilon_0 > 0, \delta_0 > 0$ and a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ such that

$$\mu\left(\{x : |f_{n_k}(x) - f(x)| \geqslant \varepsilon_0\}\right) \geqslant \delta_0, \quad \forall k \geqslant 1.$$

For $\{f_{n_k}\}_k$, we have a subsequence $\{f_{n_{k_i}}\}_i$ of $\{f_{n_k}\}_k$ such that $f_{n_{k_i}} \xrightarrow{\text{a.e.}} f$ by the supposition. Thus, we have $f_{n_{k_i}} \xrightarrow{\text{a.u.}} f$ as $i \to +\infty$ and, hence, $f_{n_{k_i}} \xrightarrow{\mu} f$ as $i \to +\infty$. This is a contradiction to $\mu\left(\{x : |f_{n_{k_i}}(x) - f(x)| \geqslant \varepsilon_0\}\right) \geqslant \delta_0$. □

From Proposition 1, 7 and Theorem 6, we can directly obtain the following corollary.

Corollary 4. *Let μ be a null-additive and exhaustive fuzzy measure with the property (s). Then $f_n \xrightarrow{\mu} f$ if and only if for any subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$, there exists a subsequence $\{f_{n_{k_i}}\}_i$ of $\{f_{n_k}\}_k$ with $f_{n_{k_i}} \xrightarrow{a.e.} f$.*

Theorem 7. *Let μ be a fuzzy measure with the pseudometric generating property. We have the following statements:*

(1) *if $f_n \xrightarrow{\mu} f$, then $\{f_n\}_n$ is fundamental in fuzzy measure μ;*

(2) *if $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$, then $f \overset{a.e.}{=} g$;*

(3) *if $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$, then*

$$\alpha \cdot f_n + \beta \cdot g_n \xrightarrow{\mu} \alpha \cdot f + \beta \cdot g, \quad \forall \alpha, \beta \in R^1.$$

Furthermore, if μ is exhaustive, then

(4) *$f_n \xrightarrow{\mu} f$ implies $f_n \cdot g \xrightarrow{\mu} f \cdot g$ for any measurable function g;*

(5) *$f_n \xrightarrow{\mu} f$ implies $f_n^2 \xrightarrow{\mu} f^2$; and*

(6) *$f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ implies $f_n \cdot g_n \xrightarrow{\mu} f \cdot g$.*

Proof. We only need to prove that (1) and (4) are valid. The others can be considered in similar ways. For any fixed $\varepsilon > 0$ there exists $\delta > 0$ satisfying

$$\mu(E) \vee \mu(F) < \delta \quad \text{implies} \quad \mu(E \cup F) < \varepsilon.$$

Let $f_n \xrightarrow{\mu} f$. Then, for any $\sigma > 0$, there is $n_0 \geqslant 1$ such that

$$\mu\left(\left\{x : |f_n(x) - f(x)| \geqslant \frac{\sigma}{2}\right\}\right) < \delta, \quad \forall n \geqslant n_0.$$

Hence, we have

$$\begin{aligned} &\mu(\{x : |f_n(x) - f_m(x)| \geqslant \sigma\}) \\ &\quad \leqslant \mu\left(\left\{x : |f_n(x) - f(x)| \geqslant \frac{\sigma}{2}\right\} \right. \\ &\qquad \left. \cup \left\{x : |f_m(x) - f(x)| \geqslant \frac{\sigma}{2}\right\}\right) < \varepsilon \end{aligned}$$

whenever $m \wedge n \geqslant n_0$. Therefore, $\{f_n\}_n$ is fundamental in fuzzy measure μ, so that (1) is true.

We assume that μ is exhaustive. Since $\{x : |g(x)| > m\} \searrow \emptyset$ as $m \to +\infty$, there is $m_0 \geqslant 1$ such that $\mu(\{x : |g(x)| > m_0\}) < \delta$ by Proposition 1. Let $f_n \xrightarrow{\mu} f$. Then, for any $\sigma > 0$ there exists $n_0 \geqslant 1$ such that

$$\mu\left(\left\{x : |f_n(x) - f(x)| \geqslant \frac{\sigma}{m_0}\right\}\right) < \delta, \quad \forall n \geqslant n_0.$$

Thus,

$$\begin{aligned} &\mu(\{x : |f_n(x) \cdot g(x) - f(x) \cdot g(x)| \geqslant \sigma\}) \\ &\quad \leqslant \mu\left(\left\{x : |f_n(x) - f(x)| \geqslant \frac{\sigma}{m_0}\right\} \right. \\ &\qquad \left. \cup \{x : |g(x)| > m_0\}\right) < \varepsilon \end{aligned}$$

whenever $n \geqslant n_0$. This shows that $f_n \cdot g \xrightarrow{\mu} f \cdot g$, so that (4) is true. □

Remark 3. We have known that if μ is a finite and autocontinuous fuzzy measure (or additive measure), then μ is null-additive, exhaustive, and has the pseudometric generating property, but the converse is not true. Thus, the above results are generalized forms of Lebesgue's theorem, Riesz's theorem and Egoroff's theorem as shown in [1,6,9], respectively.

Example 3. Let $X = N$, $\mathscr{F} = \mathscr{P}(X)$ and $m(E) = \sum_{n\in E} 1/2^n$,

$$\mu(E) = \begin{cases} m(E) & \text{if } 1 \notin E, \\ 1 & \text{if } E = \{1\}, \\ \mathrm{Card}(E - \{1\}) & \text{otherwise.} \end{cases}$$

Then, μ is an exhaustive and null-additive fuzzy measure with the pseudometric generating property. Although μ is autocontinuous from below, it is not autocontinuous from above [5].

4. Characteristics of fuzzy measures

In this section, we will study further relations among the pseudometric generating property, properties (s), (E.p.), (L.p.), (R.p.) and exhaustivity of fuzzy measures by using the convergence in fuzzy measure. We have the following theorems.

Theorem 8. *Let μ be a fuzzy measure; we have the following statements:*

(1) *if μ has the (R.p.), then μ has the property (s);*

(2) *if μ has the (L.p.), then μ is exhaustive;*

(3) *if μ has the (E.p.), then μ has the (L.p.) and, hence, it is exhaustive.*

Proof. We only prove that (2) is valid. For any decreasing set sequence $\{E_n\}_n$ with $E_n \searrow \emptyset$, if we take

$$f_n(x) = \begin{cases} 0 & \text{if } x \notin E_n, \\ 1 & \text{if } x \in E_n \end{cases}$$

for any $n \geqslant 1$, then we have $f_n \to 0$. Since μ has the (L.p.), we can get

$$\lim_{n\to+\infty} \mu(E_n) = \lim_{n\to+\infty} \mu(\{x : f_n(x) \geqslant \tfrac{1}{2}\}) = 0,$$

so that μ is order-continuous. Thus, μ is exhaustive by Proposition 1. □

Theorem 9. *If μ is a fuzzy measure satisfying $f_n + g_n \xrightarrow{\mu} 0$ whenever $f_n \xrightarrow{\mu} 0$ and $g_n \xrightarrow{\mu} 0$, then μ has the pseudometric generating property.*

Proof. Suppose that μ does not have the pseudometric generating property. Then there exists $\varepsilon_0 \in (0,1)$ and two sequences $\{E_n\}_n$ and $\{F_n\}_n$ such that

$$\lim_{n\to+\infty} \mu(E_n) \vee \mu(F_n) = 0 \text{ while}$$
$$\mu(E_n \cup F_n) \geqslant \varepsilon_0, \quad \forall n \geqslant 1.$$

There is no loss of generality in assuming that $E_n \cap F_n = \emptyset$ for any $n \geqslant 1$. If we take that

$$f_n(x) = \begin{cases} 0 & \text{if } x \notin E_n, \\ 1 & \text{if } x \in E_n \end{cases}$$

and

$$g_n(x) = \begin{cases} 0 & \text{if } x \notin F_n, \\ 1 & \text{if } x \in F_n, \end{cases}$$

then $f_n \xrightarrow{\mu} 0$ and $g_n \xrightarrow{\mu} 0$. Thus,

$$f_n(x) + g_n(x) = \begin{cases} 0 & \text{if } x \notin E_n \cup F_n, \\ 1 & \text{if } x \in E_n \cup F_n, \end{cases}$$

satisfying $f_n + g_n \xrightarrow{\mu} 0$ and

$$\mu(E_n \cup F_n) = \mu(\{x : f_n(x) + g_n(x) \geqslant \varepsilon_0\}) \to 0 \quad \text{as } n \to +\infty.$$

This is a contradiction with the fact that $\mu(E_n \cup F_n) \geqslant \varepsilon_0$ for any $n \geqslant 1$.

Theorem 10. *Let μ be a fuzzy measure. Then, the following statements are equivalent:*

(1) *μ has the pseudometric generating property;*

(2) *$f_n + g_n \xrightarrow{\mu} 0$ whenever $f_n \xrightarrow{\mu} 0$ and $g_n \xrightarrow{\mu} 0$;*

(3) *$f_n \vee g_n \xrightarrow{\mu} 0$ whenever $f_n \xrightarrow{\mu} 0$ and $g_n \xrightarrow{\mu} 0$.*

Proof. By Theorems 7 and 9, we know that (1) $\Longleftrightarrow$ (2).

Let μ have the pseudometric generating property. Since

$$f_n(x) \vee g_n(x) = \tfrac{1}{2}\left((f_n(x) + g_n(x)) - |f_n(x) - g_n(x)|\right),$$

we have $f_n \vee g_n \xrightarrow{\mu} 0$ whenever $f_n \xrightarrow{\mu} 0$ and $g_n \xrightarrow{\mu} 0$. This completes the proof of $(1) \Rightarrow (3)$. In a similar way to the proof of Theorem 9, we can prove that $(3) \Rightarrow (1)$. □

Remark 4. For convenience, all the established relations among structural characteristics of fuzzy measures are summarized in the following.

(a) the autocontinuity $\Longrightarrow$ pseudometric generating property $\Longrightarrow$ property (s) $\Longleftrightarrow$ the (R.p.);

(b) the (E.p.) $\Longrightarrow$ the (L.p.) $\Longrightarrow$ exhaustivity $\Longleftrightarrow$ order-continuity; and

(c) the (E.p.) $\overset{\text{p.g.p.}}{\Longleftrightarrow}$ exhaustivity $\overset{\text{0-add.}}{\Longleftrightarrow}$ the (L.p.), where 0-add. denotes the null-additivity.

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