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A potential of fuzzy relations with a linear structure: The unbounded case

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Abstract

This paper is a sequel to Yoshida et al. (1993), in which the potential theory for linear fuzzy relations on the positive orthant R^n_+ is considered in the class of fuzzy sets with a compact support under the contractive assumption. In this paper, potential treatment for unbounded fuzzy sets is developed without the assumption of contraction and compactness. The objective of this paper is to give the existence and the characterization of potentials for linear fuzzy relations under some reasonable conditions.

Also, introducing a partial order in fuzzy sets, we prove Riesz decomposition theorem in the fuzzy potential theory. The proofs are shown by using only the linear structure and the monotonicity of fuzzy relations. In the one-dimensional case, the potential and its α-cuts are explicitly calculated. Numerical examples are given to comprehend further discussions.

Key words: Fuzzy potential; Superharmonic fuzzy set; Partial order; Linear structure; Fuzzy relation; Fuzzy relational equation

1. Introduction

A potential theory for linear fuzzy relations on the positive orthant R^n_+ of an n-dimensional Euclidean space is developed. Yoshida et al. [8] have introduced a linear structure for fuzzy relations and considered the potential theory in the class of fuzzy sets with a compact support. In this paper, the unbounded case is considered. We shall develop the relevant potential theory using only the linear structure and the monotonicity of fuzzy relations. Also, we introduce a partial order in fuzzy sets and prove Riesz-type decomposition theorem in the fuzzy potential theory. Moreover, we deal with the one-dimensional case, where numerical examples are also given to illustrate our approach.

We adopt the notations in [8]. Let R^n be an n-dimensional Euclidean space with a basis $\{e_1, e_2, \dots, e_n\}$. Let w_i be an orthogonal projection from R^n to the subspace $\{\lambda e_i \mid \lambda \in R^1\}$. Then, for $x \in R^n$, $x = \sum_{i=1}^n w_i(x)e_i$. We put a norm $\|\cdot\|$ and a metric d by $\|x\| = (\sum_{i=1}^n (w_i(x))^2)^{1/2}$ and $d(x, y) = \|x - y\|$ for $x, y \in R^n$. Let $R^n_+ := \{x \in R^n \mid w_i(x) \geqslant 0$ for all $i = 1, 2, \dots, n\}$ be a positive orthant of R^n. (R^n_+, d) is a complete separable metric

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space. We denote a fuzzy set on R^n_+ by its membership function $\tilde{s}: R^n_+ \mapsto [0, 1]$ (see Novák [6] and Zadeh [9]). For any fuzzy set $\tilde{s}$ on R^n_+ and $\alpha \in [0, 1]$, the α-cut is defined by $\tilde{s}_\alpha := \{x \in R^n_+ \mid \tilde{s}(x) \geqslant \alpha\}$ $(\alpha > 0)$ and $\tilde{s}_0 := \mathrm{cl}\{x \in R^n_+ \mid \tilde{s}(x) > 0\}$, where cl means the closure of a set. We call $\tilde{s}$ a convex fuzzy set if its α-cut $\tilde{s}_\alpha$ is a convex set for all $\alpha \in [0, 1]$. Let $\mathscr{F}(R^n_+)$ be the collection of all the convex fuzzy sets $\tilde{s}$ on R^n_+ which are upper semi-continuous.

For fuzzy sets $\tilde{s}, \tilde{r}$ and a scalar λ,

$$(\tilde{s} + \tilde{r})(x) := \sup_{y+z=x,\ y,z \in R^n_+} \{\tilde{s}(y) \wedge \tilde{r}(z)\}$$

and

$$(\lambda\tilde{s})(x) := \begin{cases} \tilde{s}(x/\lambda) & \text{if } \lambda > 0, \\ I_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases} \quad x \in R^n_+,$$

where $\lambda \wedge \mu := \min\{\lambda, \mu\}$ for scalars λ, μ, and $I_A(\cdot)$ is the classical characteristic function of $A \subset R^n_+$.

Let $\mathscr{C}(R^n_+)$ be the collection of all the closed convex subsets of R^n_+, and put $A + B := \{x + y \mid x \in A$ and $y \in B\}$ $(A, B \subset R^n_+)$, $\lambda A := \{\lambda x \mid x \in A\}$ $(A \subset R^n_+, \lambda \geqslant 0)$. Especially, put $\phi := \phi + A = A + \phi$ and $\phi := \lambda\phi$. It is known that $\tilde{s}_\alpha, \tilde{r}_\alpha \in \mathscr{C}(R^n_+)$ and $(\tilde{s} + \tilde{r})_\alpha = \tilde{s}_\alpha + \tilde{r}_\alpha$, $(\lambda\tilde{s})_\alpha = \lambda\tilde{s}_\alpha$ hold for fuzzy sets $\tilde{s}, \tilde{r} \in \mathscr{F}(R^n_+)$, $\lambda \in R^1_+$ and $\alpha \in [0, 1]$ (cf. Madan et al. [4]).

The following is easily shown.

Lemma 1.1. *Let $\tilde{s}, \tilde{r}, \tilde{p} \in \mathscr{F}(R^n_+)$ and $\lambda, \mu \in R^1_+$. Then*

(i) $\tilde{s} + \tilde{r} = \tilde{r} + \tilde{s}$.

(ii) $(\lambda(\mu\tilde{s})) = (\mu(\lambda\tilde{s}))$,

(iii) $(\tilde{s} + \tilde{r}) + \tilde{p} = \tilde{s} + (\tilde{r} + \tilde{p})$.

Let $\tilde{q}: R^n_+ \times R^n_+ \mapsto [0, 1]$ be a fuzzy relation on R^n_+. We assume that $\tilde{q}$ satisfies the following assumption.

Assumption A. The fuzzy relation $\tilde{q}$ satisfies the following conditions (A1)–(A5):

(A1) $\tilde{q}$ is continuous on $R^n_+ \times R^n_+ \setminus \{(0, 0)\}$,

(A2) $\tilde{q}(\cdot, y) \in \mathscr{F}(R^n_+)$ for all $y \in R^n_+$,

(A3) $\sup_{x \in R^n_+} \tilde{q}(x, y) = 1$ for all $y \in R^n_+$,

(A4) $\tilde{q}(\cdot, 0) = I_{\{0\}}$ and $\tilde{q}(0, \cdot) = I_{\{0\}}$,

(A5) $\tilde{q}(\cdot, \lambda y + \mu z) = \lambda\tilde{q}(\cdot, y) + \mu\tilde{q}(\cdot, z)$ for all $y, z \in R^n_+$ and $\lambda, \mu \in R^1_+$.

Assumption (A5) of the linear structure is firstly introduced in [8]. When a fuzzy set $\tilde{q}(\cdot, e_i) \in \mathscr{F}(R^n_+)$ is given for each e_i $(i = 1, 2, \ldots, n)$, we can construct a fuzzy relation $\tilde{q}$ on R^n_+ which satisfies Assumption A, by defining $\tilde{q}(\cdot, y) := \sum_{i=1}^n w_i(y)\tilde{q}(\cdot, e_i)$, $y \in R^n_+$ (see [8, Theorem 2.1]).

For any $\tilde{p} \in \mathscr{F}(R^n_+)$, let

$$\tilde{q}(\tilde{p})(x) := \sup_{y \in R^n_+} \{\tilde{q}(x, y) \wedge \tilde{p}(y)\} \quad (x \in R^n_+). \tag{1.1}$$

Inductively define the sequence of fuzzy states $\{\tilde{q}^k(\tilde{p})\}_{k=0}^\infty$ by $\tilde{q}^0(\tilde{p}) := \tilde{p}$ and $\tilde{q}^k(\tilde{p}) := \tilde{q}(\tilde{q}^{k-1}(\tilde{p}))$ $(k = 1, 2, \ldots)$. If a *formal* infinite sum

$$Q(\tilde{p}) := \sum_{k=0}^\infty \tilde{q}^k(\tilde{p}) \tag{1.2}$$

is well-defined, it is called a fuzzy potential or simply a potential given the fuzzy relation $\tilde{q}$.

In the previous paper [8], we have studied a potential under contractive conditions. Here a potential theory is developed in the unbounded case.

Define a map $\tilde{q}_\alpha$ on $\mathscr{C}(R^n_+)$ by

$$\tilde{q}_\alpha(D) := \begin{cases} \{x \in R^n_+ \mid \tilde{q}(x, y) \geqslant \alpha \text{ for some } y \in D\} & \text{for } \alpha > 0,\ D \in \mathscr{C}(R^n_+)\ (D \neq \phi), \\ \mathrm{cl}\{x \in R^n_+ \mid \tilde{q}(x, y) > 0 \text{ for some } y \in D\} & \text{for } \alpha = 0,\ D \in \mathscr{C}(R^n_+)\ (D \neq \phi), \\ \phi & \text{for } \alpha \in [0, 1],\ D = \phi. \end{cases}$$

Then, from Assumption (A5), it follows that

$$\tilde{q}_\alpha(\lambda y + \mu z) = \lambda \tilde{q}_\alpha(y) + \mu \tilde{q}_\alpha(z) \quad \text{for all } y, z \in R^n_+ \text{ and } \lambda, \mu \in R^1_+, \tag{1.3}$$

where $\tilde{q}_\alpha(y) = \tilde{q}_\alpha(\{y\})$. Note that $\tilde{q}_\alpha(D) = \bigcup_{y \in D} \tilde{q}_\alpha(y)$ holds for all $D \in \mathscr{C}(R^n_+)$.

For any $D \in \mathscr{C}(R^n_+)$, from the continuity of $\tilde{q}$ and (1.3), it follows that $\tilde{q}_\alpha(D) \in \mathscr{C}(R^n_+)$ and $\tilde{q}_\alpha : \mathscr{C}(R^n_+) \mapsto \mathscr{C}(R^n_+)$. Inductively we define maps $\tilde{q}^k_\alpha : \mathscr{C}(R^n_+) \mapsto \mathscr{C}(R^n_+)$ $(k = 0, 1, 2, \ldots)$ by $\tilde{q}^0_\alpha$ is an identity map and $\tilde{q}^k_\alpha := \tilde{q}_\alpha(\tilde{q}^{k-1}_\alpha)$ $(k = 1, 2, \ldots)$.

In Section 3 we shall need the following lemma regarding the map $\tilde{q}^k$.

Lemma 1.2 (see Kurano et al. [2]). *For any $\tilde{p} \in \mathscr{F}(R^n_+)$, it holds that*

$$(\tilde{q}^k(\tilde{p}))_\alpha = \tilde{q}^k_\alpha(\tilde{p}_\alpha), \quad k = 0, 1, 2, \ldots, \alpha \in [0, 1].$$

We shall introduce a partial order on $\mathscr{C}(R^n_+)$.

Definition 1.3. For $A, B \in \mathscr{C}(R^n_+)$,

$$A \succeq B$$

means that there exists $C \in \mathscr{C}(R^n_+)$ such that $A = B + C$.

We will introduce an order $\succeq$ on the class of convex fuzzy sets, $\mathscr{F}(R^n_+)$. The following Lemma 1.5 shows that $\succeq$ is a partial order on $\mathscr{F}(R^n_+)$.

Definition 1.4. For $\tilde{s}, \tilde{r} \in \mathscr{F}(R^n_+)$,

$$\tilde{s} \succeq \tilde{r}$$

means that there exists $\tilde{p} \in \mathscr{F}(R^n_+)$ such that $\tilde{s} = \tilde{r} + \tilde{p}$.

Lemma 1.5. *Let $\tilde{s}, \tilde{r}, \tilde{p} \in \mathscr{F}(R^n_+)$.*

(i) *If $\tilde{s} \succeq \tilde{r}$ and $\tilde{r} \succeq \tilde{s}$, then $\tilde{s} = \tilde{r}$.*

(ii) *If $\tilde{s} \succeq \tilde{r}$ and $\tilde{r} \succeq \tilde{p}$, then $\tilde{s} \succeq \tilde{p}$.*

The following lemma shows the monotonicity of the fuzzy relation $\tilde{q}$ with respect to the order.

Lemma 1.6. *For any $\tilde{s}, \tilde{r} \in \mathscr{F}(R^n_+)$, if $\tilde{s} \succeq \tilde{r}$ then*

(i) *$\tilde{s}_\alpha \succeq \tilde{r}_\alpha$ for all $\alpha \in [0, 1]$,*

(ii) *$\tilde{q}(\tilde{s}) \succeq \tilde{q}(\tilde{r})$.*

Proof. By Definition 1.4, there exists $\tilde{p} \in \mathscr{F}(R^n_+)$ such that $\tilde{s} = \tilde{r} + \tilde{p}$, which implies $\tilde{s}_\alpha = \tilde{r}_\alpha + \tilde{p}_\alpha$ for all $\alpha \in [0, 1]$. Namely $\tilde{s}_\alpha \succeq \tilde{r}_\alpha$ for all $\alpha \in [0, 1]$. (ii) is trivial from Definition 1.4, using the linearity of $\tilde{q}$ (see [8, Lemma 3.4]). □

In Section 2 we present the fundamental lemmas in order to develop a fuzzy potential theory. In Section 3 we prove the existence theorem of a fuzzy potential and give its characterization by means of a fuzzy relational equation. We also show a decomposition theorem of a superharmonic fuzzy set. In Section 4 a fuzzy potential is explicitly given in the one-dimensional case and some numerical examples are also given to comprehend the further discussions in this problem.

2. Preliminary lemmas

In this section we prepare several fundamental facts, which are used in Section 3.

Definition 2.1. For $\{A_k\}_{k=1}^{\infty} \subset \mathscr{C}(R^n_+)$ and $A \in \mathscr{C}(R^n_+)$, we denote

$$\lim_{k\to\infty} A_k = A \quad \text{if} \quad \overline{\lim}_{k\to\infty} A_k = \underline{\lim}_{k\to\infty} A_k = A,$$

where

$$\overline{\lim}_{k\to\infty} A_k := \{x \in R^n_+ \mid \underline{\lim}_{k\to\infty} d(x, A_k) = 0\},$$

$$\underline{\lim}_{k\to\infty} A_k := \{x \in R^n_+ \mid \overline{\lim}_{k\to\infty} d(x, A_k) = 0\}$$

and $d(x, D) := \inf_{y\in D} d(x, y)$, $D \in \mathscr{C}(R^n_+)$.

The following Lemmas 2.2 and 2.3 insist the monotone convergence regarding the partial order $\succeq$ on $\mathscr{C}(R^n_+)$.

Lemma 2.2 (Non-increasing case). *Let* $\{A_k\}_{k=1}^{\infty} \subset \mathscr{C}(R^n_+)$. *If* $A_k \succeq A_{k+1}$ $(k = 1, 2, \ldots)$, *then there exists* $A \in \mathscr{C}(R^n_+)$ *with* $\lim_{k\to\infty} A_k = A$.

Proof. Let $\{A_k\}_{k=1}^{\infty} \subset \mathscr{C}(R^n_+)$ satisfying $A_k \succeq A_{k+1}$ $(k = 1, 2, \ldots)$. Then there exists $\{C_k\}_{k=1}^{\infty} \subset \mathscr{C}(R^n_+)$ such that $A_k = C_k + A_{k+1}$ $(k = 1, 2, \ldots)$. Therefore

$$A_1 = C_1 + C_2 + \cdots + C_k + A_{k+1} \succeq C_1 + C_2 + \cdots + C_k \quad \text{for all } k = 1, 2, \ldots. \tag{2.1}$$

Clearly it holds that

$$\sum_{i=1}^{\infty} \delta(C_i) < \infty, \tag{2.2}$$

where $\delta(B) = \inf_{x\in B} \|x\|$ for $B \in \mathscr{C}(R^n_+)$. Thus, there exists a sequence $\{z_i\}_{i=1}^{\infty}$ such that $z_i \in C_i$ $(i = 1, 2, \ldots)$ and

$$\sum_{i=1}^{\infty} \|z_i\| < \infty. \tag{2.3}$$

To prove this lemma, it is sufficient to show that $\overline{\lim}_{k\to\infty} A_k \subset \underline{\lim}_{k\to\infty} A_k$. It is trivial when $\overline{\lim}_{k\to\infty} A_k = \phi$. If we assume that $\overline{\lim}_{k\to\infty} A_k \neq \phi$ and $x \in \overline{\lim}_{k\to\infty} A_k$, there exists a sub-sequence $\{x_{k_j}\}_{j=1}^{\infty}$ such that $x_{k_j} \in A_{k_j}$ $(j = 1, 2, \ldots)$ and $x_{k_j} \to x$ $(j \to \infty)$. Then we have $A_l = C_l + C_{l+1} + \cdots + C_{k_{j+1}-1} + A_{k_{j+1}}$ for $k_j < l < k_{j+1}$ and $j = 1, 2, \ldots$. Here we define a sequence $\{y_l\}_{l=1}^{\infty}$ by

$$y_l := \begin{cases} \sum_{i=l}^{k_{j+1}-1} z_i + x_{k_{j+1}} & \text{for } k_j < l < k_{j+1} \text{ and } j = 1, 2, \ldots \\ x_{k_j} & \text{for } l = k_j \text{ and } j = 1, 2, \ldots. \end{cases}$$

Then it is trivial that $y_l \in A_l$ $(l = 1, 2, \ldots)$. Further, we have.

$$\|y_l - x\| \leqslant \sum_{i=l}^{\infty} \|z_i\| + \|x_{k_{j+1}} - x\| \quad \text{for } k_j < l < k_{j+1} \text{ and } j = 1, 2, \ldots.$$

Letting $j, l \to \infty$, we obtain $y_l \to x$ from (2.3). Namely $x \in \underline{\lim}_{k\to\infty} A_k$. This completes the proof. □

Lemma 2.3 (Non-decreasing case). *Let* $\{A_k\}_{k=1}^{\infty} \subset \mathscr{C}(R^n_+)$. *If* $A_k \preceq A_{k+1}$ $(k = 1, 2, \ldots)$, *then there exists* $A \in \mathscr{C}(R^n_+)$ *with* $\lim_{k\to\infty} A_k = A$.

Proof. From Definition 1.3, there exists $\{D_k\}_{k=1}^{\infty} \subset \mathscr{C}(R^n_+)$ such that $A_{k+1} = D_k + A_k$ $(k = 1, 2, \ldots)$. Therefore

$$A_{k+1} = D_1 + D_2 + \cdots + D_k + A_1 \succeq D_1 + D_2 + \cdots + D_k \quad \text{for all } k = 1, 2, \ldots. \tag{2.4}$$

It is enough to show that $\overline{\lim}_{k\to\infty} A_k \subset \underline{\lim}_{k\to\infty} A_k$ holds. We may assume $\overline{\lim}_{k\to\infty} A_k \neq \phi$. Let $x \in \overline{\lim}_{k\to\infty} A_k$. There exists a sub-sequence $\{x_{k_j}\}_{j=1}^{\infty}$ such that $x_{k_j} \in A_{k_j}$, $x_{k_j} \to x (j \to \infty)$, and a real number $\varepsilon > 0$ such that $A_{k_j} \cap U_\varepsilon(x) \neq \phi$ for sufficiently large all j, where $U_\varepsilon(x) := \{z \in R^n_+ \mid \|z - x\| \leqslant \varepsilon\}$ denotes the ε-neighborhood of x. For sufficiently large j, there exists a constant K satisfying $\delta(A_{k_j}) < K$. This shows, by (2.4), that there exists a sequence $\{z_i\}_{i=1}^{\infty}$ such that $z_i \in D_i$ $(i = 1, 2, \ldots)$ and $\sum_{i=1}^{\infty} \|z_i\| < \infty$. We have $A_l = D_{l-1} + D_{l-2} + \cdots + D_{k_j} + A_{k_j}$ for $k_j < l < k_{j+1}$ and $j = 1, 2, \ldots$. Define a sequence $\{y_l\}_{l=1}^{\infty}$ by

$$y_l := \begin{cases} \sum_{i=k_j}^{l-1} z_i + x_{k_j} & \text{for } k_j < l < k_{j+1} \text{ and } j = 1, 2, \ldots \\ x_{k_j} & \text{for } l = k_j \text{ and } j = 1, 2, \ldots. \end{cases}$$

Therefore $y_l \in A_l$ $(l = 1, 2, \ldots)$ and

$$\|y_l - x\| \leqslant \sum_{i=k_j}^{\infty} \|z_i\| + \|x_{k_j} - x\| \quad \text{for } k_j < l < k_{j+1} \text{ and } j = 1, 2, \ldots.$$

Letting $j, l \to \infty$, we obtain $y_l \to x$. So, $x \in \underline{\lim}_{k\to\infty} A_k$. Thus the lemma is proved. □

The following lemma is easily proved from the linear structure of $\tilde{q}$.

Lemma 2.4. *It holds that*

$$\tilde{q}_\alpha(A + B) = \tilde{q}_\alpha(A) + \tilde{q}_\alpha(B) \quad \textit{for } \alpha \in [0, 1] \textit{ and } A, B \in \mathscr{C}(R^n_+).$$

Theorem 2.5. *Let* $\{A_k\}_{k=1}^{\infty} \subset \mathscr{C}(R^n_+)$ *and* $A \in \mathscr{C}(R^n_+)$ *such that* $\{A_k\}_{k=1}^{\infty}$ *is non-increasing (non-decreasing) with respect to the order* $\succeq$ *on* $\mathscr{C}(R^n_+)$ *and* $\lim_{k\to\infty} A_k = A$. *Then it holds that*

$$\lim_{k\to\infty} \tilde{q}_\alpha(A_k) = \tilde{q}_\alpha(A) \quad \textit{for } \alpha \in (0, 1].$$

Proof. Let $\{A_k\}_{k=1}^{\infty} \subset \mathscr{C}(R^n_+)$ and $A \in \mathscr{C}(R^n_+)$ such that $A_k \succeq A_{k+1}$ $(k = 1, 2, \ldots)$ and $\lim_{k\to\infty} A_k = A$. From Lemma 2.4 we have $\tilde{q}_\alpha(A_k) \succeq \tilde{q}_\alpha(A_{k+1})$ $(k = 1, 2, \ldots)$ for $\alpha > 0$. By Lemma 2.2, there exists $\lim_{k\to\infty} \tilde{q}_\alpha(A_k)$. On the other hand, in the case of $A_k \preceq A_{k+1}$ $(k = 1, 2, \ldots)$, using Lemma 2.3, we also obtain the same facts.

First we show $\tilde{q}_\alpha(A) \subset \lim_{k\to\infty} \tilde{q}_\alpha(A_k)$ for $\alpha > 0$. From the linearity of $\tilde{q}$, we have

$$\tilde{q}_\alpha(A) = \bigcup_{y\in A} \tilde{q}_\alpha(y) = \bigcup_{y\in A} \sum_{i=1}^{n} w_i(y)\, \tilde{q}_\alpha(e_i).$$

Therefore, for any $x \in \tilde{q}_\alpha(A)$, there exist $y \in A$ and $z_i \in \tilde{q}_\alpha(e_i)$ $(i = 1, 2, \dots, n)$ such that

$$x = \sum_{i=1}^{n} w_i(y) z_i.$$

Since $\lim_{k\to\infty} A_k = A$, we can take a sequence $\{y_k\}_{k=0}^{\infty}$ such that $y_k \in A_k$ $(k = 1, 2, \dots)$ and $\lim_{k\to\infty} y_k = y$. Then, by putting $x_k = \sum_{i=1}^{n} w_i(y_k) z_i$ $(k = 1, 2, \dots)$,

$$\lim_{k\to\infty} x_k = x \quad \text{and} \quad x_k = \sum_{i=1}^{n} w_i(y_k) z_i \in \bigcup_{y' \in A_k} \sum_{i=1}^{n} w_i(y') \tilde{q}_\alpha(e_i) = \tilde{q}_\alpha(A_k) \quad (k = 1, 2, \dots).$$

Therefore $x \in \lim_{k\to\infty} \tilde{q}_\alpha(A_k)$, which shows $\tilde{q}_\alpha(A) \subset \lim_{k\to\infty} \tilde{q}_\alpha(A_k)$.

Next we show the reverse inclusion. Let $x \in \lim_{k\to\infty} \tilde{q}_\alpha(A_k)$. Then there exists a sequence $\{(x_k, y_k)\}_{k=0}^{\infty} \subset R_+^n \times R_+^n$ such that

$$\lim_{k\to\infty} x_k = x, \quad y_k \in A_k \quad \text{and} \quad \tilde{q}(x_k, y_k) \geqslant \alpha \qquad (k = 1, 2, \dots). \tag{2.5}$$

To show $x \in \tilde{q}_\alpha(A)$, it is sufficient to prove that there exists a convergent subsequence of $\{y_k\}_{k=0}^{\infty}$. Because in this case there exists $y \in A$ satisfying $\tilde{q}(x, y) \geqslant \alpha$ from the continuity of $\tilde{q}$ and this means $x \in \tilde{q}_\alpha(A)$. Here we suppose that there do not exist any convergent subsequences of $\{y_k\}_{k=0}^{\infty}$. Then we have $\lim_{k\to\infty} \|y_k\| = \infty$. From Assumption (A5) and $\tilde{q}(\cdot, y') \in \mathscr{F}(R_+^n)$ $(y' \in A)$,

$$\tilde{q}(x_k, y_k) = \tilde{q}\left(x_k, \frac{y_k}{\|y_k\|}\|y_k\|\right) = \|y_k\| \, \tilde{q}\left(x_k, \frac{y_k}{\|y_k\|}\right) = \tilde{q}\left(\frac{x_k}{\|y_k\|}, \frac{y_k}{\|y_k\|}\right) \quad \text{for } k = 1, 2, \dots.$$

By taking a subsequence if necessary, we may assume that $y_k/\|y_k\|$ converges to some limit $z(\|z\| = 1)$. Therefore, from Assumption A1 and A3, we obtain

$$\underline{\lim}_{k\to\infty} \tilde{q}(x_k, y_k) = \underline{\lim}_{k\to\infty} \tilde{q}\left(\frac{x_k}{\|y_k\|}, \frac{y_k}{\|y_k\|}\right) = \tilde{q}(0, z) = 0.$$

This contradicts (2.5). Thus we obtain $\tilde{q}_\alpha(A) \supset \lim_{k\to\infty} \tilde{q}_\alpha(A_k)$. Therefore the proof is completed. □

To prove the main theorem in the next section, we need the following lemma.

Lemma 2.6. *Let $\{A_\alpha \mid \alpha \in [0, 1]\} \subset \mathscr{C}(R_+^n)$ such that $A_{\alpha'} \supset A_\alpha$ for $0 < \alpha' < \alpha \leqslant 1$. Then,*

$$\tilde{q}_\alpha\left(\lim_{\alpha' \uparrow \alpha} A_{\alpha'}\right) = \lim_{\alpha' \uparrow \alpha} \tilde{q}_{\alpha'}(A_{\alpha'}) \quad \textit{for } \alpha > 0.$$

Proof. Let $\{A_\alpha | \alpha \in [0, 1]\} \subset \mathscr{C}(R_+^n)$ such that $A_{\alpha'} \supset A_\alpha$ $(0 < \alpha' < \alpha \leqslant 1)$. Put $B_\alpha := \lim_{\alpha' \uparrow \alpha} A_{\alpha'} = \bigcap_{\alpha' < \alpha} A_{\alpha'}$ for $\alpha > 0$. Then we have $\tilde{q}_{\alpha'}(A_{\alpha'}) \supset \tilde{q}_{\alpha'}(A_\alpha) \supset \tilde{q}_\alpha(A_\alpha)$ $(0 < \alpha' < \alpha \leqslant 1)$. So,

$$\lim_{\alpha' \uparrow \alpha} \tilde{q}_{\alpha'}(A_{\alpha'}) = \bigcap_{\alpha' < \alpha} \tilde{q}_{\alpha'}(A_{\alpha'}) \quad \text{for } \alpha > 0. \tag{2.6}$$

We first show

$$\tilde{q}_\alpha(B_\alpha) \subset \lim_{\alpha' \uparrow \alpha} \tilde{q}_{\alpha'}(A_{\alpha'}) \quad \text{for } \alpha > 0. \tag{2.7}$$

Let $x \in \tilde{q}_\alpha(B_\alpha)$. Then there exists $y \in B_\alpha = \bigcap_{\alpha' < \alpha} A_{\alpha'}$ satisfying $\tilde{q}(x, y) \geqslant \alpha$, which implies $x \in \tilde{q}_\alpha(A_{\alpha'})$ for all $\alpha' < \alpha$. So we obtain $x \in \lim_{\alpha' \uparrow \alpha} \tilde{q}_{\alpha'}(A_{\alpha'})$. Therefore we get (2.7).

In order to show the reverse inclusion of (2.7), let $x \in \lim_{\alpha' \uparrow \alpha} \tilde{q}_{\alpha'}(A_{\alpha'}) = \bigcap_{\alpha' < \alpha} \tilde{q}_{\alpha'}(A_{\alpha'})$. Then we can take a sequence $\{\alpha_k\}_{k=1}^{\infty} (\subset (0, 1])$ satisfying that $x \in \tilde{q}_{\alpha_k}(A_{\alpha_k})$ for all $k = 1, 2, \ldots$ and $\alpha_k \uparrow \alpha (k \to \infty)$. Therefore there exists a sequence $\{y_k\}_{k=1}^{\infty}$ such that

$$y_k \in A_{\alpha_k} \quad \text{and} \quad \tilde{q}(x, y_k) \geqslant \alpha_k \quad \text{for } k = 1, 2, \ldots. \tag{2.8}$$

Here in the same arguments as the latter part of the proof of Theorem 2.5, we may assume that $\{y_k\}_{k=1}^{\infty}$ converges to some finite limit $y \in B_\alpha$. By letting $k \to \infty$ in (2.8), from the continuity of $\tilde{q}$, we obtain $y \in B_\alpha$ and $\tilde{q}(x, y) \geqslant \alpha$. Namely $x \in \tilde{q}_\alpha(B_\alpha)$. Therefore the proof is completed. □

The following lemma is referred to the construction of a fuzzy set from a family of subsets $\{A_\alpha\}$.

Lemma 2.7 (cf. Novak, [6]). *We suppose a family of subsets $\{A_\alpha \mid \alpha \in [0, 1]\} \subset \mathscr{C}(R^n_+)$ satisfies the following conditions* (i) *and* (ii):

(i) *$A_\alpha \subset A_{\alpha'}$ for $0 \leqslant \alpha' < \alpha \leqslant 1$,*
(ii) *$\lim_{\alpha' \uparrow \alpha} A_{\alpha'} = A_\alpha$ for $\alpha \in (0, 1]$.*

Then,

$$\tilde{s}(x) := \sup_{\alpha \in [0, 1]} \{\alpha \wedge I_{A_\alpha}(x)\}, \quad x \in R^n_+$$

satisfies $\tilde{s} \in \mathscr{F}(R^n_+)$ and $\tilde{s}_\alpha = A_\alpha$ for all $\alpha \in [0, 1]$.

3. General potential theorems

In this section we show the existence of potentials in $\mathscr{F}(R^n_+)$, which are defined formally in Section 1. Further, we develop a fuzzy potential theory to show the decomposition theorem of a superharmonic fuzzy set. First we define a convergence in $\mathscr{F}(R^n_+)$, which is weaker than the one given in [2, 5, 8].

Definition 3.1. For $\{\tilde{s}_k\}_{k=0}^{\infty} \subset \mathscr{F}(R^n_+)$ and $\tilde{r} \in \mathscr{F}(R^n_+)$,

$$\lim_{k \to \infty} \tilde{s}_k = \tilde{r}$$

means that $\tilde{s}_{k,\alpha} \to \tilde{r}_\alpha (k \to \infty)$ for all $\alpha \in [0, 1]$.

In order to show the existence of a potential, we consider the following reasonable assumption.

Assumption B. $\tilde{q}_1(e_i) = \{e_i\}$ for $i = 1, 2, \ldots, n$.

Under Assumption B, it holds that $\tilde{q}_1(A) = A$ for all $A \in \mathscr{C}(R^n_+)$, which simplifies our discussion. It is restrictive, however, we need not assume the contractivity of $\tilde{q}$ (cf. [8]). The fuzzy relation $\tilde{q}$ satisfying Assumption A is illustrated in Examples 4.2 and 4.4 in Section 4.

Lemma 3.2. *Suppose that Assumption* B *holds. For a non-empty $A \in \mathscr{C}(R^n_+)$, the following* (i)–(iii) *hold.*

(i) *If $0 \notin A$, then $\sum_{k=0}^{\infty} \tilde{q}_1^k(A) = \phi$.*
(ii) *If A satisfies $A \cap E_i \neq \phi$ for all $i = 1, 2, \ldots, n$, then $\lim_{k \to \infty} \tilde{q}_\alpha^k(A) = R^n_+$ for $0 < \alpha < 1$, where $E_i := \{\lambda e_i \mid 0 < \lambda < \infty\}$ for $i = 1, 2, \ldots, n$.*
(iii) *If A satisfies $A \cap E_i \neq \phi$ for all $i = 1, 2, \ldots, n$, then $\sum_{k=1}^{\infty} \tilde{q}_\alpha^k(A) \cap E_i \neq \phi$ for all $0 < \alpha < 1$ and $= 1, 2, \ldots, n$.*

Proof. (i) Since $A \in \mathscr{C}(R^n_+)$ and $0 \notin A$, we have $\delta(A) > 0$, where $\delta(B) = \inf_{x \in B} \|x\|$ for $B \in \mathscr{C}(R^n_+)$. From Assumption B, $\sum_{k=0}^{l} \tilde{q}_1^k(A) = \sum_{k=0}^{l} A \to \phi (l \to \infty)$.

(ii) From Assumption B and the continuity of $\tilde{q}$, there exists the ε-neighborhood of e_i: $\{\lambda e_i \mid \lambda \in [1-\varepsilon, 1+\varepsilon]\} \subset \tilde{q}_\alpha(e_i) \cap E_i$ for some $\varepsilon \in (0, 1)$. Repeating this procedure, it holds that

$$\{\lambda e_i \mid \lambda \in [(1-\varepsilon)^k, (1+\varepsilon)^k]\} \subset \tilde{q}_\alpha^k(e_i). \tag{3.1}$$

Since $A \cap E_i \neq \phi$ and A is closed and convex, letting $k \to \infty$ in (3.1) we obtain $\lim_{k\to\infty} \tilde{q}_\alpha^k(A) = \lim_{k\to\infty} \bigcup_{y \in A} \sum_{i=1}^{n} w_i(y) \tilde{q}_\alpha^k(e_i) = R^n_+$.

(iii) From (3.1), $\sum_{k=1}^{\infty} \tilde{q}_\alpha^k(A) \supset \sum_{k=1}^{\infty} w_i(y) \tilde{q}_\alpha^k(e_i) \supset w_i(y) \{\lambda e_i \mid \lambda \in [1/\varepsilon, \infty)\}$ for $y \in A \cap E_i$. Therefore $\sum_{k=1}^{\infty} \tilde{q}_\alpha^k(A) \cap E_i \neq \phi$. The proof is completed. □

Let $\mathscr{F}^*(R^n_+)$ be the set of all $\tilde{p} \in \mathscr{F}(R^n_+)$ such that $0 \notin \tilde{p}_1$ and $\tilde{p}_\alpha \cap E_i \neq \phi$ for all $\alpha \in (0, 1)$, where E_i is defined in the above. By the next theorem, we see that for any $\tilde{p} \in \mathscr{F}^*(R^n_+)$ its potential $Q(\tilde{p})$ is well-defined.

Theorem 3.3. *Suppose that Assumption* B *holds. For any $\tilde{p} \in \mathscr{F}^*(R^n_+)$ the potential $u := Q(\tilde{p})$ exists in $\mathscr{F}(R^n_+)$ and it satisfies the following fuzzy relational equation*:

$$\tilde{u} = \tilde{p} + \tilde{q}(\tilde{u}). \tag{3.2}$$

Proof. Let $\tilde{p} \in \mathscr{F}(R^n_+)$ such that $0 \notin \tilde{p}_1$ and $\tilde{p}_\alpha \cap E_i \neq \phi$ for all $\alpha \in (0, 1)$ and $i = 1, 2, \ldots, n$. We define $Q_l(\tilde{p}) := \sum_{k=0}^{l} \tilde{q}^k(\tilde{p})$ for $l = 0, 1, 2, \ldots$. Then we have $Q_{l+1}(\tilde{p}) \succeq Q_l(\tilde{p})$ $(l = 1, 2, \ldots)$. Therefore from Lemma 1.6(i) and 2.3, there exists

$$A_\alpha := \lim_{l\to\infty} Q_l(\tilde{p})_\alpha \quad \text{for } \alpha \in [0, 1]. \tag{3.3}$$

On the other hand, from the definition of Q_l, we also have $Q_l(\tilde{p}) = \tilde{p} + \tilde{q}(Q_{l-1}(\tilde{p}))$ $(l = 1, 2, \ldots)$. Applying Lemma 1.6, we have

$$Q_l(\tilde{p})_\alpha = \tilde{p}_\alpha + \tilde{q}_\alpha(Q_{l-1}(\tilde{p})_\alpha) \quad \text{for all } \alpha \in [0, 1] \text{ and } l = 1, 2, \ldots.$$

Letting $l \to \infty$, from Theorem 2.5, we obtain

$$A_\alpha = \tilde{p}_\alpha + \tilde{q}_\alpha(A_\alpha) \quad \text{for all } \alpha \in (0, 1]. \tag{3.4}$$

Put $B_\alpha := \lim_{\alpha' \uparrow \alpha} A_{\alpha'}$ for $\alpha \in (0, 1]$ and $B_0 := \mathrm{cl}(\bigcup_{\alpha > 0} B_\alpha)$. In order to show the existence of the potential $Q(\tilde{p})$, it is sufficient to prove that $B_\alpha = A_\alpha$ for all $\alpha \in [0, 1]$. In fact, $\{A_\alpha \mid \alpha \in [0, 1]\}$ satisfies Lemma 2.7(i) and (ii). Putting $Q(\tilde{p})(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge I_{A_\alpha}(x)\}$, $x \in R^n_+$, (3.3) implies that $Q(\tilde{p}) = \lim_{l\to\infty} Q_l(\tilde{p}) \in \mathscr{F}(R^n_+)$ and $Q(\tilde{p})_\alpha = A_\alpha$ for all $\alpha \in [0, 1]$.

By replacing α for α' in (3.4) and letting $\alpha' \uparrow \alpha$, from $\tilde{p} \in \mathscr{F}(R^n_+)$ and Theorem 2.5, we have

$$B_\alpha = \tilde{p}_\alpha + \tilde{q}_\alpha(B_\alpha) \quad \text{for all } \alpha \in (0, 1]. \tag{3.5}$$

This follows $B_\alpha \succeq \tilde{q}_\alpha(B_\alpha)$ $(\alpha \in (0, 1])$. Let $\alpha \in (0, 1)$. Inductively we obtain $\tilde{q}_\alpha^l(B_\alpha) \succeq \tilde{q}_\alpha^{l+1}(B_\alpha)$ $(l = 1, 2, \ldots)$. From Lemma 2.2, there exists $C_\alpha := \lim_{l\to\infty} \tilde{q}_\alpha^l(B_\alpha)$. Since B_α is closed and convex from $\tilde{p} \in \mathscr{F}(R^n_+)$, applying Lemma 3.2(ii) and (iii), we have

$$C_\alpha = \lim_{l\to\infty} \tilde{q}_\alpha^l(B_\alpha) = R^n_+ \quad \text{for all } \alpha \in (0, 1). \tag{3.6}$$

Further applying Lemma 2.4 and 1.2 to (3.5), inductively we obtain $B_\alpha = Q_l(\tilde{p})_\alpha + \tilde{q}_\alpha^{l+1}(B_\alpha)$ $(\alpha \in (0, 1), l = 1, 2, \ldots)$. Hence, letting $l \to \infty$,

$$B_\alpha = A_\alpha + R^n_+ \quad \text{for all } \alpha \in (0, 1). \tag{3.7}$$

On the other hand, using (3.4) instead of (3.5), we obtain the same results for A_α as for B_α, that is,

$$A_\alpha = A_\alpha + R^n_+ \quad \text{for all } \alpha \in (0, 1). \tag{3.8}$$

Therefore, together with (3.7), $B_\alpha = A_\alpha$ for all $\alpha \in (0, 1)$. Further, from the definition of the 0-cut, $B_0 = A_0$. Moreover, regarding the 1-cut, since $0 \neq \tilde{p}_1$ and $Q_{l+1}(\tilde{p})_\alpha \succeq Q_l(\tilde{p})_\alpha$ ($\alpha \in [0, 1]$, $l = 1, 2, \ldots$), from (3.3), (3.8) and Lemma 3.2(i), it holds that $A_1 = \phi$ and

$$\begin{aligned} B_1 &= \bigcap_{\alpha<1} A_\alpha = \bigcap_{\alpha<1} (A_\alpha + R^n_+) \\ &= \bigcap_{\alpha<1} \lim_{l\to\infty} (Q_l(\tilde{p})_\alpha + R^n_+) = \bigcap_{\alpha<1} \bigcap_{l=1}^{\infty} (Q_l(\tilde{p})_\alpha + R^n_+) \\ &= \bigcap_{l=1}^{\infty} (Q_l(\tilde{p})_1 + R^n_+) = \lim_{l\to\infty} (Q_l(\tilde{p})_1 + R^n_+) \\ &= \phi. \end{aligned}$$

Consequently we get $B_\alpha = A_\alpha$ for all $\alpha \in [0, 1]$. Therefore the potential $Q(\tilde{p})$ exists.

Finally we show that $Q(\tilde{p})$ satisfies (3.2). Since $Q(\tilde{p})_\alpha = A_\alpha$ for all $\alpha \in [0, 1]$, it follows from (3.4) that

$$Q(\tilde{p})_\alpha = \tilde{p}_\alpha + \tilde{q}(Q(\tilde{p}))_\alpha \quad \text{for all } \alpha \in (0, 1]. \tag{3.9}$$

Taking the closure of the union for $\alpha \in (0, 1]$ in (3.9), we obtain $Q(\tilde{p})_0 = \tilde{p}_0 + \tilde{q}(Q(\tilde{p}))_0$. These results imply $Q(\tilde{p}) = \tilde{p} + \tilde{q}(Q(\tilde{p}))$. Therefore $\tilde{u} = Q(\tilde{p})$ satisfies (3.2). □

Definition 3.4. For $\tilde{s} \in \mathscr{F}(R^n_+)$, $\tilde{s}$ is called superharmonic (harmonic respectively) if

$$\tilde{s} \succeq \tilde{q}(\tilde{s}) \quad (\tilde{s} = \tilde{q}(\tilde{s})).$$

The next theorem shows Riesz decomposition of a superharmonic fuzzy set in its potential and harmonic parts.

Theorem 3.5. *Suppose Assumption* B *holds. Let $\tilde{s} \in \mathscr{F}(R^n_+)$ be a superharmonic fuzzy set satisfying $\tilde{s}_\alpha \cap E_i \neq \phi$ for all $\alpha \in (0, 1)$ and $i = 1, 2, \ldots, n$. Then*

(i) $\lim_{k\to\infty} \tilde{q}^k(\tilde{s})_\alpha = R^n_+$ *for all* $\alpha \in [0, 1)$,

(ii) *there exists a potential $\tilde{u}$ and a harmonic $\tilde{h}$ such that*

$$\tilde{s} = \tilde{u} + \tilde{h}. \tag{3.10}$$

(iii) *Further, if $\tilde{s} \succeq \tilde{p} + \tilde{q}(\tilde{s})$ for some $\tilde{p} \in \mathscr{F}^*(R^n_+)$, then $\tilde{s} \succeq Q(\tilde{p})$.*

Proof. (i) Let $\tilde{s} \in \mathscr{F}(R^n_+)$ be a superharmonic fuzzy set satisfying the hypothesis of the theorem. From Lemma 3.2(ii), we have $\lim_{k\to\infty} \tilde{q}^k(\tilde{s})_0 \supset \lim_{k\to\infty} \tilde{q}^k(\tilde{s})_\alpha = \lim_{k\to\infty} \tilde{q}^k_\alpha(\tilde{s})_\alpha = R^n_+$ for all $\alpha \in (0, 1)$. The result is obtained immediately.

(ii) Since $\tilde{s} \succeq \tilde{q}(\tilde{s})$, there exists $\tilde{p} \in \mathscr{F}(R^n_+)$ such that $\tilde{s} = \tilde{p} + \tilde{q}(\tilde{s})$. Therefore

$$\tilde{s}_\alpha = \tilde{p}_\alpha + \tilde{q}_\alpha(\tilde{s}_\alpha) \quad \text{for all } \alpha \in [0, 1]. \tag{3.11}$$

Using (3.11) inductively, from the linearity of $\tilde{q}$ (cf. [8, Lemma 3.4]), we obtain

$$\tilde{s}_\alpha = \sum_{k=0}^{l} \tilde{q}^k(\tilde{p})_\alpha + \tilde{q}^{l+1}(\tilde{s})_\alpha \quad \text{for all } \alpha \in [0, 1] \text{ and } l = 1, 2, \ldots. \tag{3.12}$$

We note that, by (i),

$$\lim_{l\to\infty} \tilde{q}^{l+1}(\tilde{s})_\alpha = R^n_+ \quad \text{for all } \alpha \in [0, 1).$$

So, letting $A_\alpha := \lim_{l\to\infty} \sum_{k=0}^{l} \tilde{q}^k(\tilde{p})_\alpha$ $(0 \leqslant \alpha \leqslant 1)$ implies

$$\tilde{s}_\alpha = A_\alpha + R^n_+ \quad \text{for all } \alpha \in [0, 1). \tag{3.13}$$

Here we must investigate the following two cases.

First if "$0 \in \tilde{p}_1$", then $0 \in \tilde{p}_1 \subset \tilde{p}_\alpha$ and $0 \in \tilde{q}^k(\tilde{p})_\alpha$ for all $\alpha \in (0, 1)$, $k \geqslant 1$. By (3.13), $\tilde{s}_\alpha = R^n_+$ for all $\alpha \in (0, 1)$. From $\tilde{s} \in \mathscr{F}(R^n_+)$, we have $\tilde{s}(x) = 1$ for all $x \in R^n_+$. Therefore (3.10) holds, by taking $\tilde{u} = Q(I_{\{0\}}) = I_{\{0\}}$ and $\tilde{h}(x) = 1$ for $x \in R^n_+$.

Next if "$0 \notin \tilde{p}_1$", then $\alpha \in (0, 1)$. Since $\tilde{s}_\alpha$ is closed, there exists a real number $\lambda_0 := \lambda_0(\alpha, i) \geqslant 0$ such that

$$\tilde{s}_\alpha \cap \{\lambda e_i \mid 0 \leqslant \lambda < \infty\} = \{\lambda e_i \mid \lambda_0 \leqslant \lambda < \infty\}. \tag{3.14}$$

Note that $\lambda_0(\alpha, i)$ is non-decreasing in α for each i. For any $i = 1, 2, \ldots, n$, put $\alpha_i := \sup\{\alpha \mid \lambda_0(\alpha, i) = 0\}$. We must consider the following two sub-cases.

If "$0 \leqslant \alpha \leqslant \alpha_i$", we have $\lambda_0(\alpha, i) = 0$. So, from (3.14), $0 \in \tilde{s}_\alpha$ and $\lambda_0(\alpha, j) = 0$ for all $j = 1, 2, \ldots, n$. Therefore α_i is independent of i. Further, since $\tilde{s}_\alpha$ is closed and convex and $\tilde{s}_\alpha \cap \{\lambda e_j \mid 0 \leqslant \lambda < \infty\} = \{\lambda e_j \mid 0 \leqslant \lambda < \infty\}$ $(j = 1, 2, \ldots, n)$, we obtain that $\tilde{s}_\alpha = R^n_+$ and $\lambda_0(\alpha_i, i) = 0$.

If "$\alpha_i < \alpha \leqslant 1$", from the continuity of $\tilde{q}$ and Assumption B, there exists an ε-neighborhood $\{\lambda e_i \mid \max\{\lambda_0 - \varepsilon, 0\} \leqslant \lambda < \infty\} \subset \tilde{q}_\alpha(\tilde{s}_\alpha)$, so that we get $\min\{\lambda_0, \varepsilon\} e_i \in \tilde{p}_\alpha \cap E_i$. Therefore $\tilde{p}_\alpha \cap E_i \neq \phi$. By observing the proof of Theorem 3.3, $\lim_{\alpha' \uparrow \alpha} A_{\alpha'} = A_\alpha$ for $\alpha > \alpha_i$.

From these facts and Lemma 2.7, we can define, $\tilde{r}, \tilde{u} \in \mathscr{F}(R^n_+)$ by

$$\tilde{r}_\alpha = \begin{cases} \tilde{p}_\alpha & \text{if } \alpha \in (\alpha_1, 1] \\ R^n_+ & \text{if } \alpha \in [0, \alpha_1] \end{cases} \quad \text{and} \quad \tilde{u}_\alpha = \begin{cases} A_\alpha & \text{if } \alpha \in (\alpha_1, 1] \\ R^n_+ & \text{if } \alpha \in [0, \alpha_1]. \end{cases} \tag{3.15}$$

Clearly $\tilde{u}$ is a potential and $\tilde{u} = Q(\tilde{r})$. Also, by (3.13)

$$\tilde{s}_\alpha = \tilde{u}_\alpha + R^n_+ \quad \text{for all } \alpha \in [0, 1).$$

Taking $\tilde{h} \in \mathscr{F}(R^n_+)$ by $\tilde{h}(x) = 1$ for all $x \in R^n_+$, we see that

$$\tilde{s}_\alpha = \tilde{u}_\alpha + \tilde{h}_\alpha \quad \text{for all } \alpha \in [0, 1). \tag{3.16}$$

By Assumption B, $\tilde{q}_1(\tilde{s}_1) = \tilde{s}_1$, so that from (3.11) $\tilde{s}_1 = \tilde{p}_1 + \tilde{s}_1$, which implies $\tilde{s}_1 = \phi$. Also, by Lemma 3.2(i) and (3.15), $\tilde{u}_1 = A_1 = \phi$. Therefore (3.16) hold for $\alpha = 1$. Combining these facts, we proved (3.10).

(iii) For $\tilde{p} \in \mathscr{F}^*(R^n_+)$ satisfying $\tilde{s} \succeq \tilde{p} + \tilde{q}(\tilde{s})$, there exists $\tilde{t} \in \mathscr{F}(R^n_+)$ such that $\tilde{s} = \tilde{p} + \tilde{t} + \tilde{q}(\tilde{s})$. From (ii) and (3.15), we see that there exists $\tilde{r}, \tilde{h} \in \mathscr{F}(R^n_+)$ satisfying $\tilde{s} = Q(\tilde{r}) + \tilde{h}$ and $\tilde{r} \succeq \tilde{p} + \tilde{t} \succeq \tilde{p}$. Thus, applying Lemma 1.6(ii), we get $\tilde{s} \succeq Q(\tilde{r}) \succeq Q(\tilde{p})$. □

4. One-dimensional case

In this section we consider a fuzzy potential of fuzzy number on $R_+ := R^1_+$. Using the results in Section 3 and [8], we could deal with the contractive and non-expansive examples simultaneously. Since any convex set of R_+ is an interval, $\tilde{s} \in \mathscr{F}(R_+)$ means that its α-cuts $\tilde{s}_\alpha$ are closed intervals of R_+ for $\alpha \in [0, 1]$. For $\tilde{p} \in \mathscr{F}(R_+)$, it clearly holds that $\tilde{q}(\tilde{p}) \in \mathscr{F}(R_+)$. Also a fuzzy number $\tilde{q}(\cdot, y) \in \mathscr{F}(R_+)$ $(y \in R_+)$ satisfies that $\sup_{x \in R_+} \tilde{q}(x, 1) = 1$, $\tilde{q}(0, 1) = 0$ and $\tilde{q}(\cdot, y) = y\tilde{q}(\cdot, 1)$ in the one-dimensional case. Applying Theorem 3.3 to the one-dimensional case, we obtain the following.

Theorem 4.1. *Suppose that $\tilde{q}(1,1) = 1$. Then, for any $\tilde{p} \in \mathscr{F}(R_+)$ with $0 \notin \tilde{p}_1$, the potential $\tilde{u} = Q(\tilde{p})$ exists. Its α-cut $\tilde{u}_\alpha$ is given by*

$$\tilde{u}_\alpha = \begin{cases} [\min \tilde{p}_\alpha/(1 - \min \tilde{q}_\alpha(1)), \infty) & \text{if } \min \tilde{q}_\alpha(1) < 1, \\ R & \text{if } \min \tilde{q}_\alpha(1) = 1 \text{ and } \min \tilde{p}_\alpha = 0, \\ \phi & \text{if } \min \tilde{q}_\alpha(1) = 1 \text{ and } \min \tilde{p}_\alpha > 0. \end{cases} \tag{4.1}$$

Proof. The existence of $Q(\tilde{p})$ follows from Theorem 3.3. The α-cut representation of (3.2) becomes

$$[\min \tilde{u}_\alpha, \max \tilde{u}_\alpha] = [\min \tilde{p}_\alpha, \max \tilde{p}_\alpha] + [\min \tilde{u}_\alpha \times \min \tilde{q}_\alpha(1), \max \tilde{u}_\alpha \times \max \tilde{q}_\alpha(1)].$$

So (4.1) is immediately obtained. □

We will calculate several examples which is related to the existence of the potential. Let us denote a finite sum by $\tilde{u}_l := \sum_{k=0}^{l} \tilde{q}^k(\tilde{p}) = Q_l(\tilde{p})$ $(l = 0, 1, 2, \dots)$.

Example 4.2. Let a fuzzy set $\tilde{q}(\cdot, 1)$ be

$$\tilde{q}(x,1) = \begin{cases} 1 - 2|x-1|, & 1/2 \leqslant x \leqslant 3/2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Then using the linear structure of $\tilde{q}$, $\tilde{q}$ on R_+ is given by

$$\tilde{q}(x,y) = \begin{cases} \tilde{q}(x/y, 1), & x \geqslant 0 \text{ and } y > 0, \\ I_{\{0\}}(x), & x \geqslant 0 \text{ and } y = 0. \end{cases} \tag{4.3}$$

This fuzzy relation satisfies Assumption A and B. The graph of $\tilde{q}(x,y)$ is shown in Fig. 1. Here we take a fuzzy set $\tilde{p}$ defined by

$$\tilde{p}(x) = \begin{cases} 1 - |5/2x - 1|, & 0 \leqslant x \leqslant 4/5, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

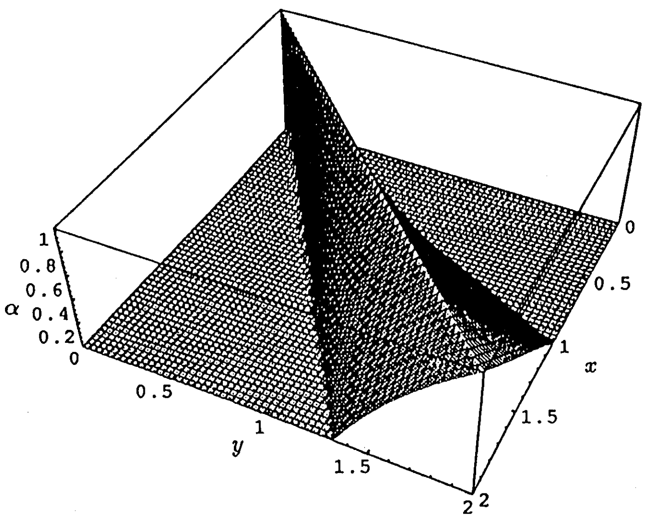

Fig. 1. The fuzzy relation $\tilde{q}(x,y)$ of (4.2), (4.3).

Fig. 2 shows the pointwise convergence of the sequence $\{\tilde{u}_l\}_{l=0}^{\infty}$ to the fuzzy potential $\tilde{u} = Q(\tilde{p})$ given by

$$\tilde{u}(x) = 5x/(4 + 5x), \quad x \geqslant 0.$$

Example 4.3. In this example we consider the case where $\tilde{q}_\alpha$ is contractive for $\alpha > 1/2$ (see [8]) and $\tilde{q}_\alpha$ is not contractive for $\alpha < 1/2$. Let the fuzzy set $\tilde{q}(\cdot, 1)$ be

$$\tilde{q}(x,1) = \begin{cases} 2x, & 0 \leqslant x \leqslant 1/2, \\ 3/2 - x, & 1/2 < x \leqslant 3/2, \\ 0, & \text{otherwise.} \end{cases}$$

In a similar way, we can define the fuzzy relation $\tilde{q}(x, y)$ by (4.3) and a fuzzy set $\tilde{p}$ by (4.4). Note that the fuzzy relation $\tilde{q}$ does not satisfy Assumption B. Fig. 3 shows the convergence of the sequences $\{\tilde{u}_l\}_{l=0}^{\infty}$ to the fuzzy potential $\tilde{u}$:

$$\tilde{u}(x) = \begin{cases} 10x/(4 + 5x), & 0 \leqslant x \leqslant 4/5, \\ (8 + 5x)/(4 + 10x), & 4/5 < x. \end{cases}$$

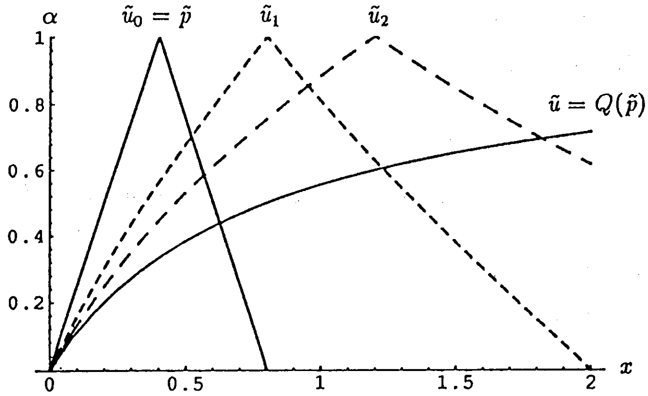

Fig. 2. The potential $\tilde{u}$ for Example 4.2.

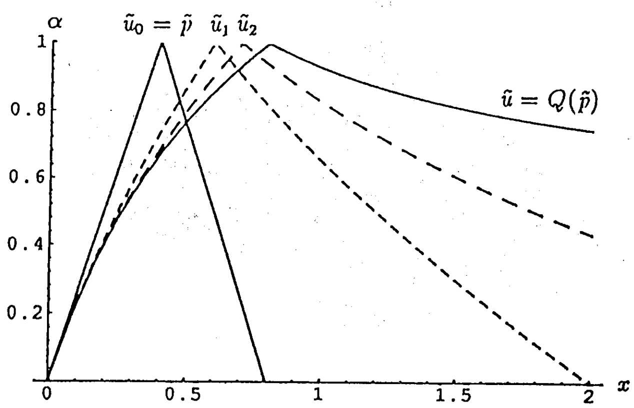

Fig. 3. The potential $\tilde{u}$ for Example 4.3.

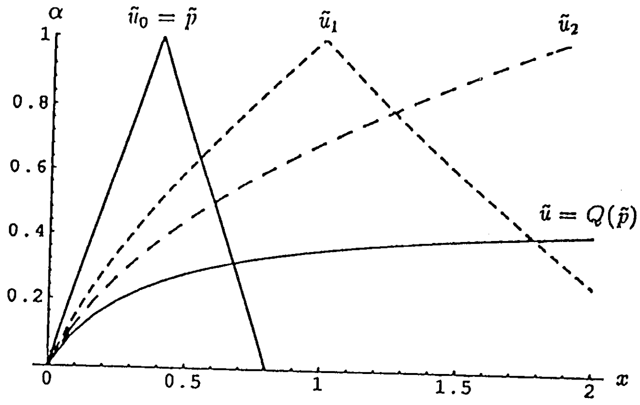

Fig. 4. The potential $\tilde{u}$ for Example 4.4.

Example 4.4. Finally we consider an example which is not contractive. If a fuzzy set $\tilde{q}(\cdot, 1)$ is given by

$$\tilde{q}(x,1) = \begin{cases} x - 1/2, & 1/2 \leqslant x \leqslant 3/2, \\ 4 - 2x, & 3/2 < x \leqslant 2, \\ 0, & \text{otherwise}. \end{cases}$$

Similarly determine the fuzzy relation $\tilde{q}(x, y)$ by (4.3) and a fuzzy set $\tilde{p}$ by (4.4). Then Fig. 4 shows the convergence of the sequences $\{\tilde{u}_l\}_{l=0}^{\infty}$ to the fuzzy potential $\tilde{u}$:

$$\tilde{u}(x) = 5x/(4 + 10x), \quad x \geqslant 0.$$

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