Fuzzy Sets and Systems 60 (1993) 283–294
North-Holland

A potential of fuzzy relations with a linear structure: The contractive case

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Received June 1992
Revised May 1993

Abstract: In this paper we develop a potential theory of fuzzy relations on the positive orthant in a Euclidean space. By introducing a linear structure for fuzzy relations, the existence of a potential and its characterization by fuzzy relational equation are derived under the assumption of contraction and compactness. In the one-dimensional unimodal case, a potential is given explicitly. Also, a numerical example is shown to illustrate our approaches.

Keywords: Fuzzy potential; linear structure; fuzzy relation; fuzzy relational equation; contraction property.

1. Introduction and notation

The convergence theorems for a sequence of fuzzy sets defined successively by fuzzy relations are first found in Bellman and Zadeh [1]. They considered a sequence of fuzzy numbers in a finite space by solving a fuzzy linear equation written in matrix form. Kurano et al. [3], by introducing a contractive property, have studied a limit of a sequence of fuzzy sets defined by the dynamic fuzzy system with a compact state space. These works would present the basic tool for the limiting behavior of fuzzy sets and contribute to a fuzzy potential theory.

Our objective is to develop a potential theory of fuzzy relations on the positive orthant $\mathbb{R}^n_+$ of an n-dimensional Euclidean space. First we define an addition and scalar multiplication in fuzzy sets as additional internal fuzzy operations and consider the linear structure of fuzzy relations. In the following, under the assumption of contraction and compactness, we prove the existence theorem of a potential, which is characterized by a fuzzy relational equation. Moreover, we deal with the one-dimensional unimodal case, where a potential is given explicitly. A numerical example is shown to illustrate our approaches.

In the remainder of this section, we shall establish the notations that will be used throughout the paper and define the problems to be examined.

Let n be a positive integer. $\mathbb{R}^n$ denotes an n-dimensional Euclidean space with a basis $\{e_1, e_2, \ldots, e_n\}$. For $x, y \in \mathbb{R}^n$, the sum of x and y and the product of a scalar λ and x are written by $x + y$ and λx respectively. Let w_i be an orthogonal projection from $\mathbb{R}^n$ to the subspace $\{\lambda e_i \mid \lambda \in \mathbb{R}^n\}$:

$$x = \sum_{i=1}^{n} w_i(x) e_i \quad \text{for } x \in \mathbb{R}^n.$$

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We put a norm $\|\cdot\|$ and a metric d by $\|x\| = (\sum_{i=1}^{n} (w_i(x))^2)^{1/2}$ and $d(x, y) = \|x - y\|$ for $x, y \in \mathbb{R}^n$. A positive orthant of $\mathbb{R}^n$, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid w_i(x) \geq 0$ for all $i = 1, 2, \ldots, n\}$, is a closed convex cone and $(\mathbb{R}^n_+, d)$ is a complete separable metric space.

Throughout this paper, we denote a fuzzy set on $\mathbb{R}^n_+$ by its membership function $\tilde{s}: \mathbb{R}^n_+ \to [0, 1]$. For the details, refer to Novák [7] and Zadeh [8]. For any fuzzy set $\tilde{s}$ on $\mathbb{R}^n_+$ and $\alpha \in [0, 1]$, its α-cut is defined by

$$\tilde{s}_\alpha := \{x \in \mathbb{R}^n_+ \mid \tilde{s}(x) \geq \alpha\}\ (\alpha > 0) \quad \text{and} \quad \tilde{s}_0 := \mathrm{cl}\{x \in \mathbb{R}^n_+ \mid \tilde{s}(x) > 0\},$$

where cl means the closure of a set. We call $\tilde{s}_0$ a support of $\tilde{s}$.

Let $\mathcal{F}(\mathbb{R}^n_+)$ be the set of all fuzzy sets $\tilde{s}$ on $\mathbb{R}^n_+$, being upper semi-continuous, which has a compact support and satisfies $\sup_{x \in \mathbb{R}^n_+} \tilde{s}(x) = 1$. Let $\tilde{q}: \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to [0, 1]$ be a fuzzy relation on $\mathbb{R}^n_+$.

For any $\tilde{p} \in \mathcal{F}(\mathbb{R}^n_+)$, using the binary operation, max and min, on the grades of membership on $[0, 1]$, we can inductively define the sequence of fuzzy sets $\{\tilde{p}_k\}_{k=0}^{\infty}$ by

$$\tilde{p}_{k+1}(y) = \sup_{x \in \mathbb{R}^n_+} \{\tilde{p}_k(x) \wedge \tilde{q}(x, y)\}, \quad y \in \mathbb{R}^n_+,\ k \geq 1, \tag{1.1}$$

where $\tilde{p}_0 = \tilde{p}$ and $\lambda \wedge \mu = \min\{\lambda, \mu\}$ for $\mathbb{R}^1_+$.

In the previous paper [3], we have studied the limit of the sequence $\{\tilde{p}_k\}_{k=0}^{\infty}$ under some contractive conditions. Here a linear structure is introduced in the space of $\mathcal{F}(\mathbb{R}^n_+)$ and the infinite sum of the sequence is investigated provided that scalar multiplication and addition are additional internal operations.

For simplicitly, we define the map $\tilde{q}$ on $\mathcal{F}(\mathbb{R}^n_+)$. For any $\tilde{p} \in \mathcal{F}(\mathbb{R}^n_+)$,

$$\tilde{q}(\tilde{p})(x) = \sup_{y \in \mathbb{R}^n_+} \{\tilde{q}(x, y) \wedge \tilde{p}(y)\}, \quad x \in \mathbb{R}^n_+. \tag{1.2}$$

Then (1.1) is written by

$$\tilde{q}^0(\tilde{p}) = \tilde{p} \quad \text{and} \quad \tilde{q}^k(\tilde{p}) = \tilde{q}(\tilde{q}^{k-1}(\tilde{p})), \quad k = 1, 2, \ldots. \tag{1.3}$$

A linear structure in fuzzy sets is defined as follows: For fuzzy sets $\tilde{s}$, $\tilde{r}$ and a scalar λ,

$$\begin{aligned} &(\tilde{s} + \tilde{r})(x) := \sup_{y+z=x;\, y,z \in \mathbb{R}^n_+} \{\tilde{s}(y) \wedge \tilde{r}(z)\}, \\ &(\lambda\tilde{s})(x) := \begin{cases} \tilde{s}(x/\lambda) & \text{if } \lambda > 0, \\ I_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases} \quad x \in \mathbb{R}^n_+, \end{aligned} \tag{1.4}$$

where $I_A(\cdot)$ is the classical characteristic function for an ordinary subset A of $\mathbb{R}^n_+$.

Then the corresponding α-cut representations are given as follows (see Madan et al. [5]):

$$(\tilde{s} + \tilde{r})_\alpha = \tilde{s}_\alpha + \tilde{r}_\alpha, \quad (\lambda\tilde{s})_\alpha = \lambda\tilde{s}_\alpha \qquad \text{for any } \alpha \in [0, 1] \tag{1.5}$$

where the addition and scalar multiplication in the right-hand-side of equations are ordinary set operations, that is, $A + B = \{x + y \mid x \in A$ and $y \in B\}$ $(A, B \subset \mathbb{R}^n_+)$ and $\lambda A = \{\lambda x \mid x \in A\}$ $(A \subset \mathbb{R}^n_+, \lambda \in \mathbb{R}^1_+)$. We also introduce the finite sum of fuzzy sets $\{\tilde{s}_i\}_{i=0}^{k}$ and the 'formal' infinite sum of fuzzy sets $\{\tilde{s}_i\}_{i=0}^{\infty}$ respectively by

$$\sum_{i=0}^{k} \tilde{s}_i := \tilde{s}_1 + \tilde{s}_2 + \cdots + \tilde{s}_k \quad \text{and} \quad \sum_{i=0}^{\infty} \tilde{s}_i := \tilde{s}_1 + \tilde{s}_2 + \cdots.$$

The finite sum is well-defined, however, the infinite sum is precisely defined later.

If the 'formal' infinite sum

$$Q(\tilde{p}) := \sum_{k=0}^{\infty} \tilde{q}^k(\tilde{p}) \tag{1.6}$$

converges (in the sense of the definition of Section 3), we call it a fuzzy potential or simply a potential.

In Section 2 fundamental assumptions are discussed to develop the potential theory in the class of fuzzy sets. The existence theorem of a potential and its characterization by a fuzzy relational equation are obtained in Section 3. In Section 4 a potential is given explicitly in the one-dimensional unimodal case and a numerical example is also given.

2. Assumptions and preliminary lemmas

In this section we introduce some assumptions for the fuzzy relation $\tilde{q}$ and give some lemmas in preparation for the theorem in the next section.

Throughout this paper, assume that

(i) $\tilde{q}$ is continuous on $\mathbb{R}_+^n \times \mathbb{R}_+^n \setminus \{(0, 0)\}$ and

(ii) $\tilde{q}(\cdot, y) \in \mathcal{F}(\mathbb{R}_+^n)$ for all $y \in \mathbb{R}_+^n$.

We call a fuzzy set $\tilde{s} \in \mathcal{F}(\mathbb{R}_+^n)$ to be convex if its α-cut $\tilde{s}_\alpha$ is a convex set for all $\alpha \in [0, 1]$. Note that $\tilde{q}$ has the discontinuity at $(0, 0)$. See the remark at Theorem 2.1. From now on, we impose the convexity and linearity concerning the fuzzy relation $\tilde{q}$ by the following Assumption A.

Assumption A (Convexity and linearity for the fuzzy relation).

(A1) $\tilde{q}(\cdot, y)$ is convex for all $y \in \mathbb{R}_+^n$,

(A2) $\tilde{q}(\cdot, \lambda y + \mu z) = \lambda \tilde{q}(\cdot, y) + \mu \tilde{q}(\cdot, z)$ for all $y, z \in \mathbb{R}_+^n$ and $\lambda, \mu \in \mathbb{R}_+^1$.

We note that Assumption A2 is equivalent to the following representation of α-cuts:

$$\tilde{q}_\alpha(\lambda y + \mu z) = \lambda \tilde{q}_\alpha(y) + \mu \tilde{q}_\alpha(z) \quad \text{for all } y, z \in \mathbb{R}_+^n,\ \lambda, \mu \in \mathbb{R}_+^1, \text{ and } \alpha \in [0, 1], \tag{2.1}$$

where

$$\tilde{q}_\alpha(y) := \{x \in \mathbb{R}_+^n \mid \tilde{q}(x, y) \geq \alpha\}, \quad \alpha \in [0, 1].$$

Here, we shall give a concrete construction of the fuzzy relation $\tilde{q}$ satisfying Assumption A by Theorem 2.1. The following lemma is trivial, but used extensively in what follows.

Lemma 2.1. *For a convex subset A of $\mathbb{R}_+^n$, it holds that*

$$\lambda A + \mu A = (\lambda + \mu)A \quad \textit{for all } \lambda, \mu \in \mathbb{R}_+^1. \tag{2.2}$$

Theorem 2.1. *Suppose that a convex fuzzy set $\tilde{q}(\cdot, e_i) \in \mathcal{F}(\mathbb{R}_+^n)$ is given for each basis e_i $(i = 1, 2, \ldots)$. Let the fuzzy relation $\tilde{q}$ on $\mathbb{R}_+^n$ be defined by*

$$\tilde{q}(\cdot, y) := \sum_{i=1}^{n} w_i(y) \tilde{q}(\cdot, e_i), \quad y \in \mathbb{R}_+^n. \tag{2.3}$$

Then $\tilde{q}$ of (2.3) satisfies Assumption A.

Proof. It is immediate that $\tilde{q}$ satisfies Assumption A1. For Assumption A2 it is sufficient to show (2.1). Let $\lambda, \mu \in \mathbb{R}^1_+$, $y, z \in \mathbb{R}^n_+$ and $\alpha \in [0, 1]$. Then, using (1.4), (1.5), (2.3) and Lemma 2.1, we obtain

$$\begin{aligned}\tilde{q}_\alpha(\lambda y + \mu z) &= \sum_{i=1}^{n} w_i(\lambda y + \mu z)\tilde{q}_\alpha(e_i) \\ &= \sum_{i=1}^{n} (\lambda w_i(y) + \mu w_i(z))\tilde{q}_\alpha(e_i) \\ &= \lambda \sum_{i=1}^{n} w_i(y)\tilde{q}_\alpha(e_i) + \mu \sum_{i=1}^{n} w_i(z)\tilde{q}_\alpha(e_i) \\ &= \lambda\tilde{q}_\alpha(y) + \mu\tilde{q}_\alpha(z),\end{aligned}$$

which is as required. □

We note by (2.3) that

$$\tilde{q}(\cdot, 0) = I_{\{0\}}, \tag{2.4}$$

which is a natural consequence of the linearity in Assumption A2 and that $\tilde{q}$ has a discontinuity at $(0, 0)$. See Figure 2 for example.

In this paper we deal with the contraction case in fuzzy sets with a compact support, so that we need the following assumption, which is assumed from now on.

Assumption B (Contraction). The fuzzy relation $\tilde{q}$ satisfies the condition

$$\tilde{q}_\alpha(e_i) \subset \left\{x \in \mathbb{R}^n_+ \mid \|x\| < \frac{1}{n}\right\} \quad \text{for all } i = 1, 2, \ldots, n \text{ and } \alpha \in [0, 1].$$

Later it is seen that Assumption B for $\tilde{q}$ corresponds to the contraction property introduced by Kurano et al. [3].

Let $\mathscr{C}(\mathbb{R}^n_+)$ be the collection of all the non-empty closed subsets of $\mathbb{R}^n_+$. For $\alpha \in [0, 1]$, we define the map $\tilde{q}_\alpha : \mathscr{C}(\mathbb{R}^n_+) \to \mathscr{C}(\mathbb{R}^n_+)$ by

$$\tilde{q}_\alpha(D) := \begin{cases}\{x \in \mathbb{R}^n_+ \mid \tilde{q}(x, y) \geq \alpha \text{ for some } y \in D\} & \text{for } \alpha > 0,\ D \in \mathscr{C}(\mathbb{R}^n_+), \\ \mathrm{cl}\{x \in \mathbb{R}^n_+ \mid \tilde{q}(x, y) > 0 \text{ for some } y \in D\} & \text{for } \alpha = 0,\ D \in \mathscr{C}(\mathbb{R}^n_+).\end{cases} \tag{2.5}$$

From the continuity of $\tilde{q}$, $\tilde{q}_\alpha$ maps any closed subset of $\mathbb{R}^n_+$ to a closed subset of $\mathbb{R}^n_+$. So, the definition of $\tilde{q}_\alpha$ is consistent. From (2.2) and (2.5) we note that $\tilde{q}_\alpha(y) = \tilde{q}_\alpha(\{y\})$ for $y \in \mathbb{R}^n_+$ and that

$$\tilde{q}_\alpha(D) = \bigcup_{y \in D} \tilde{q}_\alpha(y) \quad \text{for all } D \in \mathscr{C}(\mathbb{R}^n_+). \tag{2.6}$$

Here, using the map $\tilde{q}_\alpha$, for each $k = 0, 1, 2, \ldots$, we define the map $\tilde{q}^k_\alpha : \mathscr{C}(\mathbb{R}^n_+) \to \mathscr{C}(\mathbb{R}^n_+)$ by

$$\tilde{q}^0_\alpha \text{ is an identity map} \quad \text{and} \quad \tilde{q}^k_\alpha = \tilde{q}_\alpha(\tilde{q}^{k-1}_\alpha) \ (k = 1, 2, \ldots).$$

Then we obtain the following lemma (see Kurano et al. [3, Lemma 1]).

Lemma 2.2. *For $\tilde{p} \in \mathscr{F}(\mathbb{R}^n_+)$, it holds that*

$$(\tilde{q}^k(\tilde{p}))_\alpha = \tilde{q}^k_\alpha(\tilde{p}_\alpha) \quad \textit{for all } k = 0, 1, 2, \ldots, \textit{ and } \alpha \in [0, 1],$$

where $\tilde{q}^k(\tilde{p})$, $k = 0, 1, 2, \ldots$, are defined by (1.3).

Next we shall show the contraction property for the fuzzy relation $\tilde{q}$. For any positive number a, we define a rectangle $J(a)$ of $\mathbb{R}_+^n$ by

$$J(a):=\left\{x=\sum_{i=1}^{n} w_i(x)e_i \in \mathbb{R}_+^n \mid 0 \leqslant w_i(x) \leqslant a\right\}.$$

Then $(J(a), d)$ is a compact matric space. Further let $\mathscr{C}(J(a))$ be the collection of all the closed subsets of $J(a)$. Then $(\mathscr{C}(J(a)), \rho)$ becomes a compact metric space with a Hausdorff metric ρ (see [2, 4]). The following lemma holds for the map $\tilde{q}_\alpha$.

Lemma 2.3. *For $\alpha \in [0, 1]$ and real $a > 0$, the map $\tilde{q}_\alpha$ satisfies the following* (i)–(iii) *and there exists a real number β $(0<\beta<1)$ satisfying* (ii) *and* (iii), *which is independent of α and a:*

(i) $\tilde{q}_\alpha(D) \in \mathscr{C}(J(a))$ *for all* $D \in \mathscr{C}(J(a))$,

(ii) $\tilde{q}_\alpha(J(a)) \subset J(\beta a)$,

(iii) $\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \leqslant \beta\rho(A, B)$ *for all* $A, B \in \mathscr{C}(J(a))$.

Proof. (i) follows immediately from (ii). First we shall show (ii). Let $\alpha \in [0, 1]$ and $a > 0$. By (2.6) and (2.1), we have

$$\tilde{q}_\alpha(J(a)) = \bigcup_{y\in J(a)} \tilde{q}_\alpha(y) = \bigcup_{y\in J(a)} \sum_{i=1}^{n} w_i(y)\tilde{q}_\alpha(e_i). \tag{2.7}$$

From Assumption B, there exists a real number c $(0<c<1/n)$ such that

$$\tilde{q}_\alpha(e_i) \subset J(c) \quad \text{for all } i=1, 2, \ldots, n. \tag{2.8}$$

Together with (2.7), this implies

$$\tilde{q}_\alpha(J(a)) \subset \bigcup_{y\in J(a)} \sum_{i=1}^{n} w_i(y)J(c) \subset \sum_{i=1}^{n} J(ca) \subset J(nca).$$

By taking $\beta = nc$ $(0<\beta<1)$, we obtain (ii).

Next we shall show (iii). Let $A, B \in \mathscr{C}(J(a))$. Let $x_i \in \tilde{q}_\alpha(e_i)$ $(i=1, 2, \ldots, n)$, $y \in A$ and $z \in B$. We put $y_x = \sum_{i=1}^n w_i(y)x_i$ and $z_x = \sum_{i=1}^n w_i(z)x_i$. Then from (2.8) we have

$$d(y_x, z_x) = \left\|\sum_{i=1}^{n} (w_i(y)-w_i(z))x_i\right\| \leqslant c\sum_{i=1}^{n} |(w_i(y)-w_i(z))| \leqslant nc\ \|y-z\| = \beta\ \|y-z\|. \tag{2.9}$$

On the other hand, replacing $J(a)$ with A and B in (2.7), we have

$$\tilde{q}_\alpha(A) = \bigcup_{y'\in A} \sum_{i=1}^{n} w_i(y')\tilde{q}_\alpha(e_i) \quad \text{and} \quad \tilde{q}_\alpha(B) = \bigcup_{z'\in B} \sum_{i=1}^{n} w_i(z')\tilde{q}_\alpha(e_i).$$

Therefore we obtain $y_x \in \tilde{q}_\alpha(A)$ and $z_x \in \tilde{q}_\alpha(B)$. Together with (2.9), this implies

$$d(y_x, \tilde{q}_\alpha(B)) = \min_{z'\in\tilde{q}_\alpha(B)} d(y_x, z') \leqslant d(y_x, z_x) \leqslant \beta\ \|y-z\|.$$

By moving z in B, it follows that $d(y_x, \tilde{q}_\alpha(B)) \leqslant \beta d(y, B)$. Further by moving y in A, we obtain

$$\max_{y'\in\tilde{q}_\alpha(A)} d(y', \tilde{q}_\alpha(B)) \leqslant \beta \max_{y\in A} d(y, B).$$

It also holds, by the symmetry,

$$\max_{z'\in\tilde{q}_\alpha(B)} d(\tilde{q}_\alpha(A), z') \leqslant \beta \max_{z\in B} d(A, z).$$

These two inequalities imply

$$\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \leq \max\left\{ \max_{y' \in \tilde{q}_\alpha(A)} d(y', \tilde{q}_\alpha(B)), \max_{z' \in \tilde{q}_\alpha(B)} d(\tilde{q}_\alpha(A), z') \right\} \leq \beta\rho(A, B).$$

This completes the proof. □

3. Main results

In this section we shall show the existence of potentials defined by (1.6). Further we shall give its characterization by a fuzzy relational equation.

First we shall define the convergence in $\mathcal{F}(\mathbb{R}^n_+)$.

Definition (see [3, 6]). For $\{\tilde{s}_k\}_{k=0}^\infty \subset \mathcal{F}(\mathbb{R}^n_+)$ and $\tilde{s} \in \mathcal{F}(\mathbb{R}^n_+)$,

$$\lim_{k\to\infty} \tilde{s}_k = \tilde{s}$$

means that there exists $a > 0$ satisfying $\tilde{s}_{k,0} \subset J(a)$ $(k = 0, 1, 2, \ldots)$ and $\tilde{s}_0 \subset J(a)$ and that

$$\sup_{\alpha \in [0,1]} \rho(\tilde{s}_{k,\alpha}, \tilde{s}_\alpha) \to 0 \quad (k \to \infty).$$

The following lemma, which can be easily checked (cf. [3, 7]), is needed to get results.

Lemma 3.1. *Let $\tilde{s}$ be a fuzzy set on $\mathbb{R}^n_+$. Then $\tilde{s} \in \mathcal{F}(\mathbb{R}^n_+)$ if and only if $\tilde{s}$ satisfies the following three conditions* (i)–(iii):

(i) *there exists a positive number a satisfying $\tilde{s}_\alpha \in \mathcal{C}(J(a))$ for all $\alpha \in [0, 1]$,*
(ii) $\lim_{\alpha' \uparrow \alpha} \tilde{s}_{\alpha'} = \tilde{s}_\alpha$,
(iii) $\tilde{s}_1 \neq \emptyset$.

The following theorem holds for the sequence of fuzzy sets $\{\tilde{q}^k(\tilde{p})\}_{k=0}^\infty$, which is defined by (1.3).

Theorem 3.1. *Let $\tilde{p} \in \mathcal{F}(\mathbb{R}^n_+)$. Then*
(i) $\tilde{q}^k(\tilde{p}) \in \mathcal{F}(\mathbb{R}^n_+)$ *for* $k = 0, 1, 2, \ldots$,
(ii) $\lim_{k\to\infty} \tilde{q}^k(\tilde{p}) = I_{\{0\}}$.

Proof. For (i), it is sufficient to prove that $\tilde{q}(\tilde{p})$ satisfies Lemma 3.1(i)–(iii). Since $\tilde{p} \in \mathcal{F}(\mathbb{R}^n_+)$, $\tilde{p}$ has a compact support. And there exists $a > 0$ such that $\tilde{p}_\alpha \subset J(a)$ for all $\alpha \in [0, 1]$. Together with Lemma 2.2 and 2.3(ii),

$$(\tilde{q}(\tilde{p}))_\alpha = \tilde{q}_\alpha(\tilde{p}_\alpha) \subset \tilde{q}_\alpha(J(a)) \subset J(\beta a) \subset J(a) \quad \text{for all } \alpha \in [0, 1]. \tag{3.1}$$

Then we obtain Lemma 3.1(i). Since $\tilde{p} \in \mathcal{F}(\mathbb{R}^n_+)$ and $\tilde{q}(\cdot, y) \in \mathcal{F}(\mathbb{R}^n_+)$ for all $y \in \mathbb{R}^n_+$, Lemma 3.1(iii) is proved easily. Further we can prove Lemma 3.1(ii) from the convergence theorem of compacts sets using Lemma 2.3(iii) (see Kurano et al. [3, Lemma 2]). Thus $\tilde{q}(\tilde{p})$ satisfies Lemma 3.1(i)–(iii), and so (i) holds.

Next we shall show (ii). From (i) and Lemma 3.1(i), we have

$$(\tilde{q}^k(\tilde{p}))_\alpha \subset J(a) \quad \text{for all } k = 1, 2, \ldots \text{ and } \alpha \in [0, 1].$$

From Lemma 2.3(iii), $(\tilde{q}(\tilde{p}))_\alpha : \mathcal{C}(J(a)) \to \mathcal{C}(J(a))$ has the contraction property. So, we see from Kurano et al. [3, Theorem 1] that $\lim_{k\to\infty} \tilde{q}^k(\tilde{p})$ exists and it is a unique solution $\tilde{u}$ of the following fuzzy relational equation:

$$\tilde{u}(x) = \sup_{y \in \mathbb{R}^n_+} \{\tilde{q}(x, y) \wedge \tilde{u}(y)\}, \quad x \in \mathbb{R}^n_+. \tag{3.2}$$

Moreover, from (2.4), we see that the unique solution of (3.2) is $I_{\{0\}}$. Thus we obtain (ii). The proof is completed. □

We need several lemmas to prove the existence theorem of potentials. The following lemma is easily checked regarding the Hausdorff metric ρ.

Lemma 3.2. *For $a > 0$ and $A, B, C, D \in \mathscr{C}(J(a))$, the following* (i) *and* (ii) *hold:*

(i) $\rho(A, A + B) \leqslant \max_{x \in B} d(0, x)$,

(ii) $\rho(A + B, C + D) \leqslant \rho(A, C) + \rho(B, D)$.

Lemma 3.3. *For any $a > 0$ with $J(a) \supset \tilde{p}_0$, it holds that*

$$J(\beta^k a) \supset (\tilde{q}^k(\tilde{p}))_\alpha, \quad k = 0, 1, 2, \ldots \text{ and } \alpha \in [0, 1],$$

where β $(0 < \beta < 1)$ is the number given by Lemma 2.3.

Proof. It is easily proved from Lemma 2.3(i) by induction. □

Lemma 3.4. *For $\tilde{s}, \tilde{r} \in \mathscr{F}(R^n_+)$ and $\lambda, \mu \geqslant 0$, it holds that*

$$\tilde{q}^k(\lambda\tilde{s} + \mu\tilde{r}) = \lambda\tilde{q}^k(\tilde{s}) + \mu\tilde{q}^k(\tilde{r}), \quad k = 0, 1, 2, \ldots.$$

Proof. It is trivial when $k = 0$. It is sufficient to check the case of $k = 1$. Let $\tilde{s}, \tilde{r} \in \mathscr{F}(\mathbb{R}^n_+)$, $\lambda, \mu \geqslant 0$ and $\alpha \in [0, 1]$. From Lemma 2.2 and (2.1) we obtain

$$\begin{aligned}
(\tilde{q}(\lambda\tilde{s} + \mu\tilde{r}))_\alpha &= \tilde{q}_\alpha((\lambda\tilde{s} + \mu\tilde{r})_\alpha) \\
&= \tilde{q}_\alpha(\lambda\tilde{s}_\alpha + \mu\tilde{r}_\alpha) \\
&= \bigcup_{\lambda x + \mu y \in \lambda\tilde{s}_\alpha + \mu\tilde{r}_\alpha} \tilde{q}_\alpha(\lambda x + \mu y) \\
&= \bigcup_{\lambda x + \mu y \in \lambda\tilde{s}_\alpha + \mu\tilde{r}_\alpha} (\lambda\tilde{q}_\alpha(x) + \mu\tilde{q}_\alpha(y)) \\
&= \bigcup_{x \in \tilde{s}_\alpha} (\lambda\tilde{q}_\alpha(x)) + \bigcup_{y \in \tilde{r}_\alpha} (\mu\tilde{q}_\alpha(y)) \\
&= \lambda\tilde{q}_\alpha(\tilde{s}_\alpha) + \mu\tilde{q}_\alpha(\tilde{r}_\alpha) \\
&= \lambda(\tilde{q}(\tilde{s}))_\alpha + \mu(\tilde{q}(\tilde{r}))_\alpha.
\end{aligned}$$

Thus we obtain the lemma. □

The following two lemmas are easily checked from Lemma 2.3(iii).

Lemma 3.5. *Let $\{\tilde{p}_k\}_{k=0}^\infty \subset \mathscr{F}(\mathbb{R}^n_+)$ and $\tilde{p} \in \mathscr{F}(\mathbb{R}^n_+)$. If $\lim_{k\to\infty} \tilde{p}_k = \tilde{p}$, then $\lim_{k\to\infty} \tilde{q}(\tilde{p}_k) = \tilde{q}(\tilde{p})$.*

Lemma 3.6 (c.f. [6]). *For a real number $a > 0$ we suppose that a family of subsets $\{A_\alpha \mid \alpha \in [0, 1]\} \subset \mathscr{C}(J(a))$ satisfies the following conditions* (i)–(iii):

(i) $A_\alpha \subset A_{\alpha'}$ *for* $0 \leqslant \alpha' < \alpha \leqslant 1$,

(ii) $\lim_{\alpha' \uparrow \alpha} A_{\alpha'} = A_\alpha$ *for* $a \in (0, 1]$,

(iii) $A_1 \neq \emptyset$.

Then

$$\tilde{s}(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge I_{A_\alpha}(x)\}, \quad x \in \mathbb{R}^n_+$$

satisfies $\tilde{s} \in \mathcal{F}(\mathbb{R}^n_+)$ *and* $\tilde{s}_\alpha = A_\alpha$ *for all* $\alpha \in [0, 1]$.

Now, we can prove the existence of a potential and its linearity, which is one of our main results.

Theorem 3.2. *For any* $\tilde{p} \in \mathcal{F}(\mathbb{R}^n_+)$, *the potential*

$$Q(\tilde{p}) := \sum_{k=0}^{\infty} \tilde{q}^k(\tilde{p})$$

converges in $\mathcal{F}(\mathbb{R}^n_+)$ *and has the linearity*

$$Q(\lambda\tilde{s} + \mu\tilde{r}) = \lambda Q(\tilde{s}) + \mu Q(\tilde{r}) \quad \text{for all } \tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n_+) \text{ and } \lambda, \mu \geq 0. \tag{3.3}$$

Proof. Fix any $\tilde{p} \in \mathcal{F}(\mathbb{R}^n_+)$. We define $Q_l(\tilde{p}) := \sum_{k=0}^{l} \tilde{q}^k(\tilde{p})$ $(l = 0, 1, 2, \ldots)$. Then, considering its α-cuts, we have

$$(Q_l(\tilde{p}))_\alpha = \sum_{k=0}^{l} \tilde{q}^k_\alpha(\tilde{p}_\alpha) \quad \text{for all } l = 0, 1, 2, \ldots \text{ and } \alpha \in [0, 1]. \tag{3.4}$$

From $\tilde{p} \in \mathcal{F}(\mathbb{R}^n_+)$, $\tilde{p}$ has a compact support. So there exists $c > 0$ such that $\tilde{p}_\alpha \subset J(c)$ for all $\alpha \in [0, 1]$. Together with Lemma 3.3 we have

$$\tilde{q}^k_\alpha(\tilde{p}_\alpha) \subset J(\beta^k c) \quad \text{for all } k = 0, 1, 2, \ldots \text{ and } \alpha \in [0, 1]. \tag{3.5}$$

From (3.4) and (3.5) we obtain

$$(Q_l(\tilde{p}))_\alpha \subset \sum_{k=0}^{l} J(\beta^k c) \subset J(a) \quad \text{for all } l = 0, 1, 2, \ldots \text{ and } \alpha \in [0, 1], \tag{3.6}$$

where $a = c(1-\beta)^{-1}$. On the other hand from Lemma 3.2(i) and (3.5), for $l = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$ $(l > m)$ we have

$$\begin{aligned}\rho((Q_l(\tilde{p}))_\alpha, (Q_m(\tilde{p}))_\alpha) &\leq \max_{x \in \sum_{k=m+1}^{l} \tilde{q}^k_\alpha(\tilde{p}_\alpha)} d(0, x) \\ &\leq \max_{x \in \sum_{k=m+1}^{l} J(\beta^k c)} d(0, x) \\ &= \sum_{k=m+1}^{l} \beta^k c \leq \beta^k c/(1-\beta).\end{aligned}$$

Therefore we obtain

$$\lim_{m,l \to \infty} \sup_{\alpha \in [0,1]} \rho((Q_l(\tilde{p}))_\alpha, (Q_m(\tilde{p}))_\alpha) = 0. \tag{3.7}$$

Together with (3.6), this implies that $\{(Q_l(\tilde{p}))_\alpha\}_{l=0}^{\infty}$ is a Cauchy sequence on the compact metric space $(\mathscr{C}(J(a)), \rho)$. Then for each $\alpha \in [0, 1]$ there exists

$$A_\alpha := \lim_{l \to \infty} (Q_l(\tilde{p}))_\alpha \in \mathscr{C}(J(a)). \tag{3.8}$$

First we shall show that $\{A_\alpha \mid \alpha \in [0, 1]\}$ of (3.8) satisfies Lemma 3.6(i)–(iii). From Theorem 3.1(i) we have $Q_l(\tilde{p}) \in \mathcal{F}(\mathbb{R}^n_+)$ and $(Q_l(\tilde{p}))_1 \neq \emptyset$ $(l = 0, 1, 2, \ldots)$. Clearly, $\{A_\alpha \mid \alpha \in [0, 1]\}$ satisfies Lemma 3.6(i), (iii). For $0 \leq \alpha' \leq \alpha \leq 1$ and $l = 0, 1, 2, \ldots$, we have

$$\rho(A_{\alpha'}, A_\alpha) \leq \rho((Q_l(\tilde{p}))_\alpha, A_\alpha)) + \rho(Q_l(\tilde{p}))_{\alpha'}, A_{\alpha'}) + \rho((Q_l(\tilde{p}))_\alpha, (Q_l(\tilde{p}))_{\alpha'}). \tag{3.9}$$

Noting that the convergence of (3.8) is uniform in $\alpha \in [0, 1]$ from (3.7), we have that for any $\epsilon > 0$ there exists l_0 such that

$$\rho((Q_l(\tilde{p}))_\alpha, A_\alpha) < \tfrac{1}{3}\epsilon \quad \text{for all } l \geq l_0 \text{ and } \alpha \in [0, 1].$$

Since $Q_l(\tilde{p}) \in \mathscr{F}(\mathbb{R}^n_+)$ $(l = 0, 1, 2, \ldots)$, from Lemma 3.1(ii), we have that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\rho((Q_l(\tilde{p}))_{\alpha'}, (Q_l(\tilde{p}))_\alpha) < \tfrac{1}{3}\epsilon \quad \text{if } \alpha - \alpha' < \delta.$$

From these facts, (3.9) implies that

$$\rho(A_{\alpha'}, A_\alpha) < \epsilon \quad \text{for } \alpha - \alpha' < \delta.$$

Then $\{A_\alpha \mid \alpha \in [0, 1]\}$ satisfies Lemma 3.6(ii). Therefore, putting (c.f. [7])

$$Q(\tilde{p})(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge I_{A_\alpha}(x)\} \quad \text{for } x \in \mathbb{R}^n_+,$$

we have $Q(\tilde{p}) \in \mathscr{F}(\mathbb{R}^n_+)$ and $Q(\tilde{p})_\alpha = A_\alpha$ for all $\alpha \in [0, 1]$ from Lemma 3.6. This, together with (3.8), implies

$$\lim_{l \to \infty} Q_l(\tilde{p}) = Q(\tilde{p}).$$

Finally we shall prove (3.3). Fix any $\tilde{s}, \tilde{r} \in \mathscr{F}(\mathbb{R}^n_+)$ and $\lambda, \mu \geq 0$. From Lemma 3.4,

$$Q(\lambda\tilde{s} + \mu\tilde{r}) = \lim_{l \to \infty} Q_l(\lambda\tilde{s} + \mu\tilde{r}) = \lim_{l \to \infty} (\lambda Q_l(\tilde{s}) + \mu Q_l(\tilde{r})) = \lambda Q(\tilde{s}) + \mu Q(\tilde{r}).$$

Thus we complete the proof of this theorem. □

The following theorem shows that a potential is given by the unique solution of related fuzzy relational equations.

Theorem 3.3. *For any $\tilde{p} \in \mathscr{F}(\mathbb{R}^n_+)$, the fuzzy relational equation*

$$\tilde{u} = \tilde{p} + \tilde{q}(\tilde{u}) \tag{3.10}$$

has a unique solution $\tilde{u} = Q(\tilde{p}) \in (\mathscr{F}(\mathbb{R}^n_+))$, which is the potential of $\tilde{p}$.

Proof. We use the notations of the proof of Theorem 3.2. From the definition of Q_l we have

$$Q_l(\tilde{p}) = \tilde{p} + \tilde{q}(Q_{l-1}(\tilde{p})) \quad \text{for all } l = 1, 2, \ldots.$$

By letting $l \to \infty$, from Theorem 3.2 and Lemma 3.5, we obtain

$$Q(\tilde{p}) = \tilde{p} + \tilde{q}(Q(\tilde{p})).$$

Therefore $Q(\tilde{p})$ is a solution of (3.10).

Finally we shall show the uniqueness of the solution of (3.10). Let $\tilde{u}' \in \mathscr{F}(X)$ be another solution of (3.10). Then we have

$$\tilde{u}_\alpha = \tilde{p}_\alpha + \tilde{q}_\alpha(\tilde{u}_\alpha) \quad \text{and} \quad \tilde{u}'_\alpha = \tilde{p}_\alpha + \tilde{q}_\alpha(\tilde{u}'_\alpha) \qquad \text{for } \alpha \in [0, 1].$$

From Lemma 3.2(ii) and 2.3(iii) we obtain

$$\begin{aligned} \rho(\tilde{u}_\alpha, \tilde{u}'_\alpha) &\leq \rho(\tilde{p}_\alpha, \tilde{p}_\alpha) + \rho(\tilde{q}_\alpha(\tilde{u}_\alpha), \tilde{q}_\alpha(\tilde{u}'_\alpha)) \\ &\leq \rho(\tilde{q}_\alpha(\tilde{u}_\alpha), \tilde{q}_\alpha(\tilde{u}'_\alpha)) \\ &\leq \beta\rho(\tilde{u}_\alpha, \tilde{u}'_\alpha). \end{aligned}$$

Since $0<\beta<1$, it follows that $\rho(\tilde{u}_\alpha, \tilde{u}'_\alpha)=0$. That is, $\tilde{u}_\alpha=\tilde{u}'_\alpha$. Since $\alpha \in [0, 1]$ is arbitrary, we obtain $\tilde{u}=\tilde{u}'$. We complete the proof. □

4. One-dimensional unimodal case

In this section we investigate fuzzy potentials of unimodal fuzzy numbers on $\mathbb{R}_+ := \mathbb{R}^1_+$ by applying the results of Section 3.

Definition. For a fuzzy number $\tilde{s} \in \mathcal{F}(\mathbb{R}_+)$, $\tilde{s}$ is called unimodal if its α-cuts $\tilde{s}_\alpha$ are bounded closed intervals, say $[\min \tilde{s}_\alpha, \max \tilde{s}_\alpha] \subset \mathbb{R}_+$ for all $\alpha \in [0, 1]$.

Let $\mathcal{F}_u(\mathbb{R}_+)$ be the set of all the unimodal fuzzy numbers on $\mathbb{R}_+$. In the one-dimensional unimodal case, Assumptions A and B for the fuzzy relation $\tilde{q}$ are reduced to the following two conditions (C1) and (C2):

(C1) $\tilde{q}(\cdot, 1) \in \mathcal{F}_u(\mathbb{R}_+)$,

(C2) $\tilde{q}_\alpha(1) \subset [0, 1)$ for all $\alpha \in [0, 1]$.

From Condition C1, $\tilde{q}(\cdot, 1)$ is a bounded closed interval of $\mathbb{R}_+$ ($\alpha \in [0, 1]$). We write $\tilde{q}_\alpha(1) = [\min \tilde{q}_\alpha(1), \max \tilde{q}_\alpha(1)]$ ($\alpha \in [0, 1]$) for convenience. Further from Condition C2, we obtain $0 \leq \min \tilde{q}_\alpha(1) \leq \max \tilde{q}_\alpha(1) < 1$ for all $\alpha \in [0, 1]$. Then we have the following lemma.

Lemma 4.1. *For $\tilde{p} \in \mathcal{F}_u(\mathbb{R}_+)$, it holds that $\tilde{q}(\tilde{p}) \in \mathcal{F}_u(\mathbb{R}_+)$.*

Proof. From the above remark, it is sufficient to check its α-cuts $(\tilde{q}(\tilde{p}))_\alpha$ are bounded closed intervals on $[0, 1]$. Fix any $\alpha \in [0, 1]$. From Lemma 2.2 and (2.6), we have

$$(\tilde{q}(\tilde{p}))_\alpha = \tilde{q}_\alpha(\tilde{p}_\alpha) = \bigcup_{y \in \tilde{p}_\alpha} \tilde{q}_\alpha(y) = \bigcup_{y \in \tilde{p}_\alpha} y\tilde{q}_\alpha(1)$$

$$= \bigcup_{y \in [\min \tilde{p}_\alpha, \max \tilde{p}_\alpha]} y[\min \tilde{q}_\alpha(1), \max \tilde{q}_\alpha(1)]$$

$$= [\min \tilde{p}_\alpha \times \min \tilde{q}_\alpha(1), \max \tilde{p}_\alpha \times \max \tilde{q}_\alpha(1)]. \quad \square$$

We obtain the following result, by applying Theorems 3.2 and 3.3 to the one-dimensional unimodal case, which is useful to determine a potential concretely.

Corollary 4.1. *For $\tilde{p} \in \mathcal{F}_u(\mathbb{R}_+)$,* (i) *and* (ii) *hold:*

(i) *The potential $\tilde{u} := Q(\tilde{p}) \in \mathcal{F}_u(\mathbb{R}_+)$,*

(ii) *its α-cut $\tilde{u}_\alpha = [\min \tilde{u}_\alpha, \max \tilde{u}_\alpha]$ is given by*

$$\min \tilde{u}_\alpha = \frac{\min \tilde{p}_\alpha}{1 - \min \tilde{q}_\alpha(1)} \quad \textit{and} \quad \max \tilde{u}_\alpha = \frac{\max \tilde{p}_\alpha}{1 - \max \tilde{q}_\alpha(1)} \quad \textit{for } \alpha \in [0, 1]. \tag{4.1}$$

Proof. We can prove (i) in similar way to the proof of Theorem 3.2, using Lemma 4.1. We shall show only (ii). Fix any $\alpha \in [0, 1]$. Since $\tilde{u} \in \mathcal{F}_u(\mathbb{R}_+)$ satisfies (3.10), using the proof of Lemma 4.1, we obtain the α-cut representation of (3.10) as

$$[\min \tilde{u}_\alpha, \max \tilde{u}_\alpha] = [\min \tilde{p}_\alpha, \max \tilde{p}_\alpha] + [\min \tilde{u}_\alpha \times \min \tilde{q}_\alpha(1), \max \tilde{u}_\alpha \times \max \tilde{q}_\alpha(1)].$$

Noting the remark of Condition C2, we get (4.1). □

Finally we shall give a one-dimensional unimodal numerical example. Let the fuzzy set $\tilde{q}(\cdot, 1)$ be given by

$$\tilde{q}(x, 1) = \begin{cases} 1 - 2\,|3x - 1| & \frac{1}{6} \leq x \leq \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

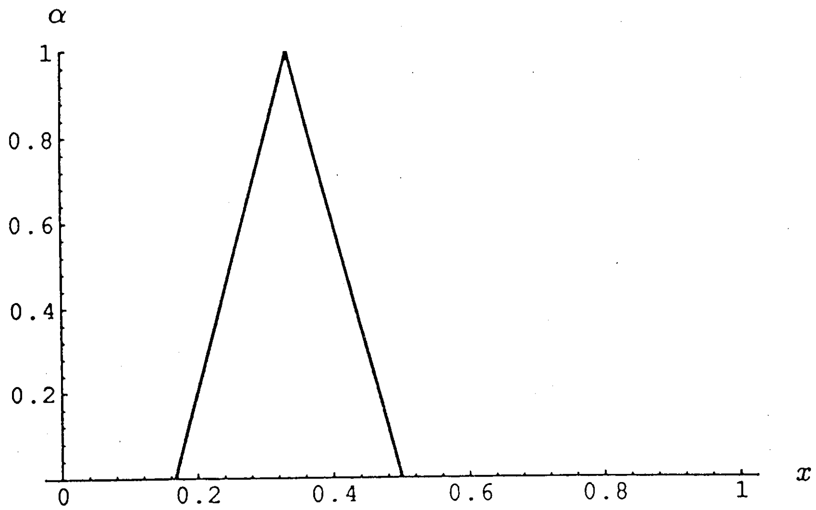

Fig. 1. The fuzzy relation $\tilde{q}(x, 1)$ at $y = 1$.

Then from Theorem 2.1, the fuzzy relation $\tilde{q}$ on $\mathbb{R}_+$, which satisfies Assumptions A and B, is given by

$$\tilde{q}(x, y) = \begin{cases} \tilde{q}(x/y, 1), & x \geqslant 0 \text{ and } y > 0, \\ I_{\{0\}}(x), & x \geqslant 0 \text{ and } y = 0. \end{cases}$$

The fuzzy set $\tilde{q}(x, 1)$ and the fuzzy relation $\tilde{q}(x, y)$ are shown respectively in Figures 1 and 2.

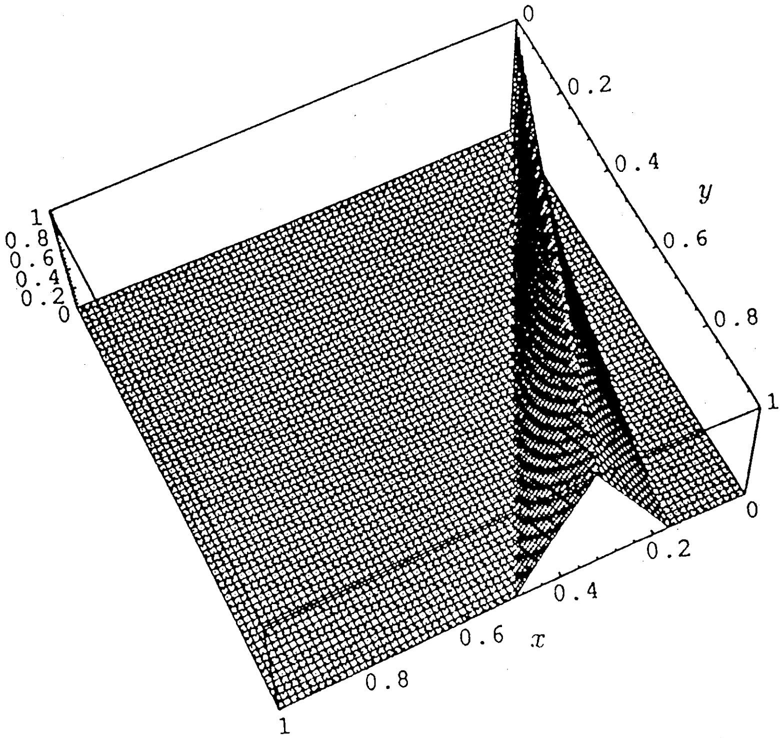

Fig. 2. The fuzzy relation $\tilde{q}(x, y)$.

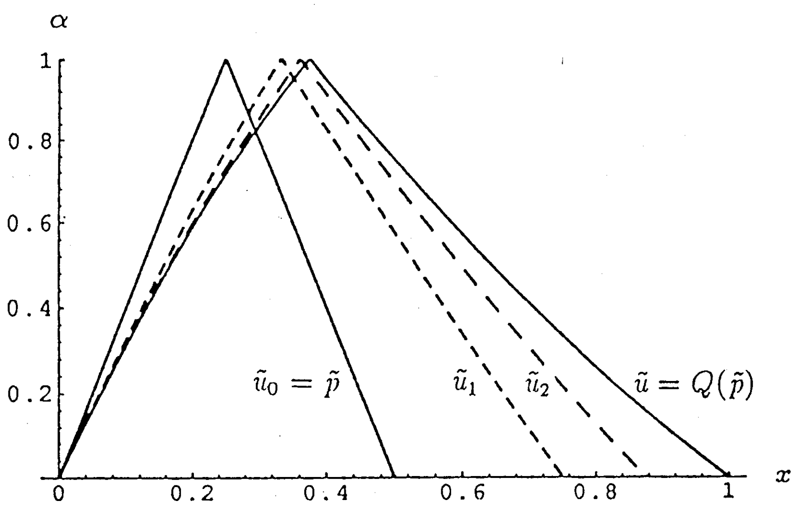

Fig. 3. The sequence $\{\tilde{u}_l\}_{l=0}^{\infty}$ and the potential $\tilde{u}$.

Here we put a fuzzy set $\tilde{p}$ by

$$\tilde{p}(x) = \begin{cases} 1 - |4x - 1|, & 0 \leq x \leq \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Calculating the fuzzy potential $\tilde{u} = Q(\tilde{p})$ of $\tilde{p}$ by (4.1), we obtain

$$\tilde{u}(x) = \begin{cases} \min\{10x/(3+2x), -6(1+x)/(3+2x)\}, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

We put $\tilde{u}_l := \sum_{k=0}^{l} \tilde{q}_k(\tilde{p}) = Q_l(\tilde{p})$ $(l = 0, 1, 2, \ldots)$. Figure 3 shows the convergence of the sequence of fuzzy states $\{\tilde{u}_l\}_{l=0}^{\infty}$ to the fuzzy potential $\tilde{u}$, which is the unique solution of (3.10).

Acknowledgement

The authors would like to thank the referee for valuable suggestions.

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