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A limit theorem in some dynamic fuzzy systems

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Abstract: In this paper, using a fuzzy relation we define a dynamic fuzzy system with a compact state space. By introducing a contraction property, we study the limit of a sequence of fuzzy states and obtain a theorem for the existence and uniqueness of the solution of a fuzzy relational equation. As an application, we deal with a dynamic fuzzy system with a terminal fuzzy gain and give some characterizations for the fuzzy expected gain. A numerical example is given to comprehend our idea in this paper.

Keywords: Fuzzy relational equation; sequence of fuzzy states; contraction property.

Introduction

Limit theorems for a sequence of fuzzy numbers are mathematically interesting and applicable to analyse multistage decision processes in a fuzzy environment. In fact, Bellman and Zadeh [1] considered a sequence of fuzzy numbers in a finite state space and solved the fuzzy relational equation, written in the matrix form, where a maximizing decision for fuzzy multistage decision processes with a defined terminal time was obtained.

In this paper, using a fuzzy relation, we formulate a dynamic fuzzy system with a compact state space. We introduce a contraction property for the transition of fuzzy states. By using this property, we consider a limit theorem for a sequence of fuzzy states defined by the dynamic fuzzy system and derive the existence and uniqueness of the solution for the fuzzy relational equation which represents the stability of this system.

Also, as an application of the results, we deal with a terminal fuzzy gain and obtain, as limit, a characterization for the fuzzy expected gain.

In Section 1, we specify the dynamic fuzzy system and define two related problems, which are analysed in Section 2. In Section 3, a numerical example is given.

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1. The dynamic fuzzy system

In this section, we shall formulate a dynamic fuzzy system and define the problems considered in the sequel.

Let X be a compact metric space. We denote by $\mathscr{C}(X)$ the collection of all the closed subsets of X. Let ρ be the Hausdorff metric on $\mathscr{C}(X)$. Then it is well-known [2, 4] that $(\mathscr{C}(X), \rho)$ is a compact metric space.

Throughout this paper, we define a fuzzy set on X by its membership function $\tilde{p}: X \to [0, 1]$. For the theory of fuzzy sets, we refer to Zadeh [10] and Novák [6].

Let $\mathscr{F}(X)$ be the set of all the fuzzy sets $\tilde{p}$ on X which are upper semi-continuous and satisfy $\sup_{x \in X} \tilde{p}(x) = 1$.

A discrete-time dynamic fuzzy system consists of four objects $(X, \tilde{q}, T, \tilde{p}_0)$ and satisfies the following conditions (i)–(iv):

(i) In general, the system is fuzzy, so that the state of the system is called a fuzzy state and denoted as an element of $\mathscr{F}(X)$. This representation includes also the case when the state of the system is denoted by a point x in X; the corresponding fuzzy state is $I_{\{x\}}(\cdot) \in \mathscr{F}(X)$, where for any ordinary subset G of X, $I_G(\cdot)$ is the classical characteristic function.

(ii) The law of motion of the system can be characterized by a time invariant fuzzy relation

$\tilde{q}$, which is supposed to be a continuous fuzzy relation $\tilde{q}: X \times X \to [0, 1]$. Then, when the system is in a state $x \in X$, we move to a fuzzy state $\tilde{q}(x, \cdot) \in \mathscr{F}(X)$ after unit time elapses.

(iii) The binary operation on the grades of membership in [0, 1] could be a triangular norm $T: [0, 1] \times [0, 1] \to [0, 1]$ (see [3, 8]). Throughout this paper for simplicity we assume that $T(x, y) = x \wedge y$, where $x \wedge y = \min\{x, y\}$.

(iv) The initial fuzzy state $\tilde{p}_0 \in \mathscr{F}(X)$ is arbitrary.

From the dynamic fuzzy system, we can define the sequence of fuzzy states $\{\tilde{p}_n\}_{n=0}^{\infty}$ by

$$\tilde{p}_{n+1}(y) = \sup_{x \in X}\{\tilde{p}_n(x) \wedge \tilde{q}(x, y)\}, \quad y \in X,\ n \geqslant 0. \tag{1.1}$$

The transition from $\tilde{p}_n$ to $\tilde{p}_{n+1}$ in (1.1) is called a fuzzy transition in the dynamic system.

Problem 1. Determine sufficient conditions under which the sequence $\{\tilde{p}_n\}_{n=0}^{\infty}$ converges (in the sense of Section 2) or there exists an invariant fuzzy state with respect to the fuzzy transition (1.1).

Before stating the other problem, we introduce several definitions referring to [6, 9, 10]. For any fuzzy state $\tilde{s} \in \mathscr{F}(X)$ and α $(0 \leqslant \alpha \leqslant 1)$, the α-cut is defined as follows:

$$\tilde{s}_\alpha := \{x \mid \tilde{s}(x) \geqslant \alpha\} \quad (\alpha \neq 0),$$

and

$$\tilde{s}_0 := \mathrm{cl}\{x \mid \tilde{s}(x) > 0\}$$

where cl means the closure of a set.

For any fuzzy state $\tilde{p} \in \mathscr{F}(X)$, we denote by $\mu_{\tilde{p}}$ the fuzzy measure on $\mathscr{B}_X$ induced by $\tilde{p}$, which is defined by

$$\mu_{\tilde{p}}(D) := \sup_{x \in D} \tilde{p}(x) \quad (D \in \mathscr{B}_X)$$

where $\mathscr{B}_X$ is the set of all the Borel subsets of X.

Then, for any $\tilde{r}, \tilde{p} \in \mathscr{F}(X)$, the fuzzy expectation of $\tilde{r}$ with respect to the fuzzy measure $\mu_{\tilde{p}}$ is defined by the following fuzzy integral [7, 9]:

$$\int \tilde{r}\, \mathrm{d}\tilde{p} := \sup_{0 \leqslant \alpha \leqslant 1}\{\alpha \wedge \mu_{\tilde{p}}(\tilde{r}_\alpha)\} = \sup_{x \in X}\{\tilde{r}(x) \wedge \tilde{p}(x)\}.$$

Now, we consider a dynamic fuzzy system with terminal fuzzy gain $\tilde{r} \in \mathscr{F}(X)$. The fuzzy expected gain at time n is defined by

$$\psi_n^* := \int \tilde{r}\, \mathrm{d}\tilde{p}_n \quad (n \geqslant 0)$$

where $\{\tilde{p}_n\}_{n=0}^{\infty}$ is defined by (1.1). Let

$$\psi^* := \limsup_{n \to \infty} \psi_n^*.$$

Now, we can define the second problem.

Problem 2. Give a characterization for the limit fuzzy expected gain ψ^*.

2. Analysis

2.1. On Problem 1

Let us define the convergence of a sequence of fuzzy states.

Definition (see [5]). For $\tilde{s}_n, \tilde{s} \in \mathscr{F}(X)$,

$$\lim_{n \to \infty} \tilde{s}_n = \tilde{s} \tag{2.1}$$

means

$$\sup_{\alpha \in [0,1]} \rho(\tilde{s}_{n,\alpha}, \tilde{s}_\alpha) \to 0 \quad (n \to \infty)$$

provided that $\tilde{s}_{n,\alpha}$, $\tilde{s}_\alpha$ are α-cuts $(0 \leqslant \alpha \leqslant 1)$ for the fuzzy states $\tilde{s}_n$, $\tilde{s}$ respectively and ρ is the given Hausdorff metric.

In order to discuss the convergence of the sequence $\{\tilde{p}_n\}_{n=0}^{\infty}$, let us define the mapping $\tilde{q}_\alpha: \mathscr{C}(X) \to \mathscr{C}(X)$ $(0 \leqslant \alpha \leqslant 1)$ by

$$\tilde{q}_\alpha(D) := \begin{cases} \{y \mid \tilde{q}(x, y) \geqslant \alpha \text{ for some } x \in D\} \\ \quad \text{for } \alpha \neq 0,\ D \in \mathscr{C}(X),\ D \neq \emptyset, \\ \mathrm{cl}\{y \mid \tilde{q}(x, y) > 0 \text{ for some } x \in D\} \\ \quad \text{for } \alpha = 0,\ D \in \mathscr{C}(X),\ D \neq \emptyset, \\ X \quad \text{for } 0 \leqslant \alpha \leqslant 1,\ D = \emptyset. \end{cases} \tag{2.2}$$

Lemma 1. *The sequence* $\{\tilde{p}_n\}_{n=0}^{\infty}$ *defined by* (1.1) *satisfies the following*:

For any $\alpha \in [0, 1]$,

$$\tilde{p}_{n,\alpha} = \tilde{q}_\alpha(\tilde{p}_{n-1,\alpha}) = \tilde{q}_\alpha^{(n)}(\tilde{p}_{0,\alpha}) \tag{2.3}$$

where $\tilde{q}_\alpha^{(n)} = \tilde{q}_\alpha(\tilde{q}_\alpha^{(n-1)})$, $n \geqslant 1$.

Proof. Because of the upper-semicontinuity of $\tilde{p}_n$ and $\tilde{q}$, sup in (1.1) could be replaced with max. First we will show that $\tilde{p}_{n,\alpha} = \tilde{q}_\alpha(\tilde{p}_{n-1,\alpha})$.

The case of $\alpha > 0$: Fix any $y \in \tilde{p}_{n,\alpha}$. Then the definition of α-cut implies $\tilde{p}_n(y) \geq \alpha$. So, because of (1.1), there exists $x \in X$ such that $\tilde{p}_{n-1}(x) \wedge \tilde{q}(x, y) \geq \alpha$. Since this means $\tilde{p}_{n-1,\alpha} \ni x$ and $\tilde{q}(x, y) \geq \alpha$, together with the definition of $\tilde{q}_\alpha$ in (2.2), we obtain $y \in \tilde{q}_\alpha(\tilde{p}_{n-1,\alpha})$. This means that $\tilde{p}_{n,\alpha} \subset \tilde{q}_\alpha(\tilde{p}_{n-1,\alpha})$. The reverse inclusion $\tilde{p}_{n,\alpha} \supset \tilde{q}_\alpha(\tilde{p}_{n-1,\alpha})$ is verified similarly.

The case of $\alpha = 0$: From the above result, it is immediate that

$$\bigcup_{\alpha>0} \tilde{p}_{n,\alpha} = \bigcup_{\alpha>0} \tilde{q}_\alpha(\tilde{p}_{n-1,\alpha}) \subset \bigcup_{\alpha>0} \tilde{q}_\alpha(\tilde{p}_{n-1,0}).$$

Taking closures, $\tilde{p}_{n,0} \subset \tilde{q}_0(\tilde{p}_{n-1,0})$ is obtained. To prove the reverse inclusion, let $y \in \tilde{q}_0(\tilde{p}_{n-1,0})$. It is clear that there is a sequence $y_k \to y$ such that

$$\{y_k\} \subset \{y \mid \tilde{q}(x, y) > 0 \text{ for some } x \in \tilde{p}_{n-1,0}\}$$

by the definition of $\tilde{q}_0$. The continuity of $\tilde{q}$ and the definition of $\tilde{p}_{n-1,0}$ imply that for each y_k there exists $x_k \in X$ which satisfies $\tilde{q}(x_k, y_k) > 0$ and $\tilde{p}_{n-1}(x_k) > 0$. Together with (1.1), we have $\tilde{p}_n(y_k) > 0$, that is, $y_k \in \tilde{p}_{n,0}$ $(k \geq 0)$. Since $\tilde{p}_{n,0}$ is closed, we obtain $y \in \tilde{p}_{n,0}$ tending as $k \to \infty$. Thus $\tilde{q}_0(\tilde{p}_{n-1,0}) \subset \tilde{p}_{n,0}$.

Therefore we have shown that $\tilde{p}_{n,\alpha} = \tilde{q}_\alpha(\tilde{p}_{n-1,\alpha})$, $0 \leq \alpha \leq 1$, and finally it is trivial that $\tilde{p}_{n,\alpha} = \tilde{q}_\alpha^{(n)}(\tilde{p}_{0,\alpha})$. □

From now on we shall assume the following contraction property concerning the fuzzy relation $\tilde{q}$.

Assumption (Contraction property). There exists a real number β $(0 < \beta < 1)$ satisfying the following condition:

$$\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \leq \beta \rho(A, B)$$

for all $A, B \in \mathscr{C}(X)$ and all α $(0 \leq \alpha \leq 1)$.

Here we need the following lemmas in order to prove the existence of a limit of the sequence $\{\tilde{p}_n\}_{n=0}^{\infty}$ and its independence of the initial fuzzy state.

Lemma 2. *Suppose a family of subsets* $\{D_\alpha \mid 0 \leq \alpha \leq 1\} \subset \mathscr{C}(X)$ *satisfies the following conditions*:

(i) $D_\alpha \subset D_{\alpha'}$ *for* $\alpha' \leq \alpha$.

(ii) $\lim_{\alpha' \uparrow \alpha} D_{\alpha'} = D_\alpha$, i.e.,

$$\lim_{\alpha' \uparrow \alpha} \rho(D_{\alpha'}, D_\alpha) = 0.$$

Then it holds that

$$\lim_{\alpha' \uparrow \alpha} \tilde{q}_{\alpha'}(D_{\alpha'}) = \tilde{q}_\alpha(D_\alpha). \tag{2.4}$$

Proof. From (i) and the definition of the map $\tilde{q}_\alpha$, it is clear that $\tilde{q}_{\alpha'}(D_{\alpha'}) \supset \tilde{q}_\alpha(D_\alpha)$ for $\alpha' \leq \alpha$. Let $\{\alpha_n\}$ be any sequence satisfying $\alpha_n \uparrow \alpha$, and let $\{y_n\} \subset X$ and $y \in X$ be any convergent sequence and its limit point satisfying $y_n \in \tilde{q}_{\alpha_n}(D_{\alpha_n})$. Owing to $y_n \in \tilde{q}_{\alpha_n}(D_{\alpha_n})$ and the definition of $\tilde{q}_{\alpha_n}$, there exist x_n satisfying $\tilde{q}(x_n, y_n) \geq \alpha_n$ and $x_n \in D_{\alpha_n}$. Hence from the compactness of X, we may take a convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and its limit point $x \in X$. Consequently, because of the continuity of $\tilde{q}$, we obtain $\tilde{q}(x, y) \geq \alpha$. Moreover (ii) implies $x \in D_\alpha$. Therefore we obtain $y \in \tilde{q}_\alpha(D_\alpha)$. This means $\lim_{\alpha' \uparrow \alpha} \tilde{q}_{\alpha'}(D_{\alpha'}) \subset \tilde{q}_\alpha(D_\alpha)$. Thus we completed the proof of this lemma. □

Lemma 3 ([6]). (i) *For any* $\tilde{p} \in \mathscr{F}(X)$, *the set of* α-*cuts* $\tilde{p}_\alpha \in \mathscr{C}(X)$ $(0 \leq \alpha \leq 1)$ *satisfies the following property*:

(a) $\tilde{p}_\alpha \subset \tilde{p}_{\alpha'}$ *for* $\alpha' \leq \alpha$,

(b) $\lim_{\alpha' \uparrow \alpha} \tilde{p}_{\alpha'} = \tilde{p}_\alpha$,

(c) $\tilde{p}(x) = \sup_{\alpha \in [0,1]}\{\alpha \wedge I_{\tilde{p}_\alpha}(x)\}$, $x \in X$.

(ii) *Further if* $A_\alpha \in \mathscr{C}(X)$ $(0 \leq \alpha \leq 1)$ *satisfies the above conditions* (a), (b), *then*

$$\tilde{q}(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge I_{A_\alpha}(x)\}, \quad x \in X,$$

satisfies

$$\tilde{q} \in \mathscr{F}(X) \quad \text{and} \quad \tilde{q}_\alpha = A_\alpha, \quad 0 \leq \alpha \leq 1.$$

(iii) *Let* $\tilde{p}, \tilde{s} \in \mathscr{F}(X)$. *If* $\tilde{p}_\alpha = \tilde{s}_\alpha$ *for all* $0 \leq \alpha \leq 1$, *then* $\tilde{p} = \tilde{s}$.

The following theorem holds for Problem 1.

Theorem 1. (i) *There exists a unique fuzzy state* $\tilde{p} \in \mathscr{F}(X)$ *satisfying*

$$\tilde{p}(y) = \max_{x \in X}\{\tilde{p}(x) \wedge \tilde{q}(x, y)\} \quad \text{for all } y \in X. \tag{2.5}$$

(ii) *The sequence* $\{\tilde{p}_n\}$ *defined by* (1.1) *converges to a unique solution* $\tilde{p} \in \mathscr{F}(X)$ *of* (2.5) *independently of the initial fuzzy state* $\tilde{p}_0$.

Namely,

$$\lim_{n\to\infty} \tilde{p}_n = \tilde{p}. \tag{2.6}$$

Proof. Since the metric space $(\mathscr{C}(X), \rho)$ is compact, from the contraction property and Banach's fixed point theorem it follows that there exists a family of closed subsets $\{A_\alpha \mid 0 \leq \alpha \leq 1\} \subset \mathscr{C}(X)$ of X such that

$$\tilde{q}_\alpha(A_\alpha) = A_\alpha, \quad 0 \leq \alpha \leq 1,$$

and $\lim_{n\to\infty} \tilde{q}_\alpha^{(n)}(D) = A_\alpha$ holds for any $D \in \mathscr{C}(X)$. For $\alpha' \leq \alpha$, it holds from the definition of $\tilde{q}_\alpha$ that $\tilde{q}_{\alpha'}(D) \supset \tilde{q}_\alpha(D)$. Further by induction we have that $\tilde{q}_{\alpha'}^{(n)}(D) \supset \tilde{q}_\alpha^{(n)}(D)$ for $n \geq 1$. Therefore by letting $n \to \infty$, we obtain

$$A_{\alpha'} \supset A_\alpha. \tag{2.7}$$

Also for $\alpha' \leq \alpha$, we can obtain that

$$\begin{aligned}\rho(A_\alpha, A_{\alpha'}) &= \rho(\tilde{q}_\alpha^{(n)}(A_\alpha), \tilde{q}_{\alpha'}^{(n)}(A_{\alpha'})) \\ &\leq \rho(\tilde{q}_{\alpha'}^{(n)}(A_\alpha), \tilde{q}_\alpha^{(n)}(A_\alpha)) \\ &\quad + \rho(\tilde{q}_{\alpha'}^{(n)}(A_{\alpha'}), \tilde{q}_{\alpha'}^{(n)}(A_\alpha)) \\ &\leq \rho(\tilde{q}_{\alpha'}^{(n)}(A_\alpha), \tilde{q}_\alpha^{(n)}(A_\alpha)) \\ &\quad + \beta^n \rho(A_{\alpha'}, A_\alpha)\end{aligned}$$

by using the induction on $\tilde{q}_\alpha$. When $\alpha' \uparrow \alpha$, because of (2.7), we may assume that $\rho(A_\alpha, A_{\alpha'})$ is bounded. So, without loss of generality, we put $\rho(A_\alpha, A_{\alpha'}) \leq 1$. Therefore the following relation is immediate:

$$\begin{aligned}\rho(A_{\alpha'}, A_\alpha) \leq \rho(\tilde{q}_{\alpha'}^{(n)}(A_\alpha), \tilde{q}_\alpha^{(n)}(A_\alpha)) \\ + \beta^n \quad (n \geq 1).\end{aligned} \tag{2.8}$$

By Lemma 2, we have $\lim_{\alpha' \uparrow \alpha} \tilde{q}_{\alpha'}(A_\alpha) = \tilde{q}_\alpha(A_\alpha)$. Again repeating these arguments,

$$\begin{aligned}\lim_{\alpha' \uparrow \alpha} \tilde{q}_{\alpha'}^{(2)}(A_\alpha) &= \lim_{\alpha' \uparrow \alpha} \tilde{q}_{\alpha'}(\tilde{q}_{\alpha'}(A_\alpha)) \\ &= \tilde{q}_\alpha(\tilde{q}_\alpha(A_\alpha)) = \tilde{q}_\alpha^{(2)}(A_\alpha).\end{aligned}$$

Similarly, by induction on n,

$$\lim_{\alpha' \uparrow \alpha} \tilde{q}_{\alpha'}^{(n)}(A_\alpha) = \tilde{q}_\alpha^{(n)}(A_\alpha).$$

This means that (see [2])

$$\lim_{\alpha' \uparrow \alpha} \rho(\tilde{q}_{\alpha'}^{(n)}(A_\alpha), \tilde{q}_\alpha^{(n)}(A_\alpha)) = 0. \tag{2.9}$$

Together with (2.8), it holds that

$$\lim_{\alpha' \uparrow \alpha} \rho(A_{\alpha'}, A_\alpha) \leq \beta^n \quad (n \geq 1).$$

By letting n tend to infinity, we obtain $\lim_{\alpha' \uparrow \alpha} A_{\alpha'} = A_\alpha$. Using a family of closed subsets $\{A_\alpha \mid 0 \leq \alpha \leq 1\}$, we define $\tilde{p}(x)$ by

$$\tilde{p}(x) := \sup_{0 \leq \alpha \leq 1} \{\alpha \wedge I_{A_\alpha}(x)\}.$$

Then from Lemma 3(ii), it holds that $\tilde{p} \in \mathscr{F}(X)$. Because A_α is the fixed point of $\tilde{q}_\alpha$,

$$\begin{aligned}\lim_{n\to\infty} \tilde{p}_{n,\alpha} &= \lim_{n\to\infty} \tilde{q}_\alpha^{(n)}(\tilde{p}_{0,\alpha}) \\ &= A_\alpha \quad (0 \leq \alpha \leq 1).\end{aligned}$$

Together with Lemma 3(iii), this implies

$$\lim_{n\to\infty} \tilde{p}_n = \tilde{p}.$$

Next we will show that $\tilde{p}$ is a solution of (2.5). Since $\tilde{p}_\alpha = A_\alpha$ by Lemma 3(ii), we note that

$$\tilde{q}_\alpha(\tilde{p}_\alpha) = \tilde{p}_\alpha, \quad 0 \leq \alpha \leq 1. \tag{2.10}$$

In the case of $0 < \alpha \leq 1$, it is proved that

$$\left\{y \in X \;\middle|\; \max_{x\in X}\{\tilde{p}(x) \wedge \tilde{q}(x, y)\} \geq \alpha\right\} = \tilde{q}_\alpha(\tilde{p}_\alpha).$$

Hence, by (2.10), we have

$$\left\{y \in X \;\middle|\; \max_{x\in X}\{\tilde{p}(x) \wedge \tilde{q}(x, y)\} \geq \alpha\right\} = \tilde{p}_\alpha. \tag{2.11}$$

For the case of $\alpha = 0$,

$$\mathrm{cl}\left\{y \in X \;\middle|\; \max_{x\in X}\{\tilde{p}(x) \wedge \tilde{q}(x, y)\} > 0\right\} = \tilde{q}_0(\tilde{p}_0)$$

can be proved in a similar way to the proof of Lemma 1. Further together with (2.10), we conclude that $\tilde{p}$ satisfies the relation (1.1) by using Lemma 3(iii).

Finally we shall prove the uniqueness of solution of (1.1). Let us denote by $\tilde{p}' \in \mathscr{F}(X)$ another solution of (1.1). For $0 \leq \alpha \leq 1$, it is shown similarly that $\tilde{p}'_\alpha = \tilde{q}_\alpha(\tilde{p}'_\alpha)$. That is, $\tilde{p}'_\alpha$ is a fixed point of $\tilde{q}_\alpha : \mathscr{C}(X) \to \mathscr{C}(X)$. Further the uniqueness of the fixed point implies $\tilde{p}'_\alpha = \tilde{p}_\alpha$. So by Lemma 3(iii), $\tilde{p}' = \tilde{p}$. These arguments complete the proof. □

2.2. *On Problem 2*

When the terminal fuzzy gain $\tilde{r} \in \mathscr{F}(X)$ is given as in Problem 2, we will consider its expected gain ψ_n^*.

Lemma 4.

$$\psi_n^* = \sup_{0\le\alpha\le1} \{\alpha \mid \tilde{p}_{n,\alpha} \cap \tilde{r}_\alpha \ne \emptyset\} = \int \tilde{r} \, \mathrm{d}\tilde{p}_n. \qquad (2.12)$$

Proof. By the definition of the fuzzy integral, it is immediate that

$$\psi_n^* = \sup_{x\in X}\{\tilde{p}_n(x) \wedge \tilde{r}(x)\}$$

holds. For simplicity, let

$$a := \sup_{0\le\alpha\le1} \{\alpha \mid \tilde{p}_{n,\alpha} \cap \tilde{r}_\alpha \ne \emptyset\}.$$

Since $\tilde{p}_n(x) \wedge \tilde{r}(x)$ is upper-semicontinuous, there exists $x \in X$ such that $\psi_n^* = \tilde{p}_n(x) \wedge \tilde{r}(x)$. Therefore $\tilde{p}_n(x) \ge \psi_n^*$ and $\tilde{r}(x) \ge \psi_n^*$. So $\tilde{p}_{n,\alpha} \cap \tilde{r}_\alpha \ni x$ where $\alpha = \psi_n^*$, which implies $\psi_n^* \le a$. Conversely for any $\varepsilon > 0$, by putting $\alpha = a - \varepsilon$, there exists $x \in \tilde{p}_{n,\alpha} \cap \tilde{r}_\alpha$ since $\tilde{p}_{n,\alpha} \cap \tilde{r}_\alpha \ne \emptyset$. This state $x \in X$ satisfies that $\tilde{p}_n(x) \ge \alpha$ and $\tilde{r}(x) \ge \alpha$. Hence

$$\tilde{p}_n(x) \wedge \tilde{r}(x) \ge \alpha.$$

We can conclude that $\psi_n^* \ge \alpha = a - \varepsilon$. So we obtain $\psi_n^* \ge a$ by letting $\varepsilon \to 0$. □

Theorem 2. *Assume that $r \in \mathscr{F}(X)$ is continuous. Then*

$$\psi^* = \sup_{0\le\alpha\le1} \{\alpha \mid \tilde{p}_\alpha \cap \tilde{r}_\alpha \ne \emptyset\} = \int \tilde{r} \, \mathrm{d}\tilde{p} \qquad (2.13)$$

provided that $\tilde{p}_\alpha$ is the α-cut of the unique fuzzy state $\tilde{p} \in \mathscr{F}(X)$ which satisfies (2.5).

Proof. Let

$$a := \sup_{0\le\alpha\le1} \{\alpha \mid \tilde{p}_\alpha \cap \tilde{r}_\alpha \ne \emptyset\}.$$

By noting

$$\psi_n^* = \sup_{0\le\alpha\le1} \{\alpha \mid \tilde{p}_{n,\alpha} \cap \tilde{r}_\alpha \ne \emptyset\}$$

from Lemma 4, the definition of ψ^* implies the existence of a sequence $\{\alpha_n\}$ satisfying $\alpha_n \to \psi^*$ and $\tilde{p}_{n,\alpha_n} \cap \tilde{r}_{\alpha_n} \ne \emptyset$. It is sufficient to discuss the two cases $\alpha_n \downarrow \psi^*$ and $\alpha_n \uparrow \psi^*$.

Case $\alpha_n \downarrow \psi^*$: Since

$$\emptyset \ne \tilde{p}_{n,\alpha_n} \cap \tilde{r}_{\alpha_n} \subset \tilde{p}_{n,\psi^*} \cap \tilde{r}_{\psi^*} \quad \text{for all } n,$$

the compactness of X implies that there is a sequence $\{t_n : t_n \in \tilde{p}_{n,\psi^*} \cap \tilde{r}_{\psi^*}\}$, which tends to some limit t. It is clear that $t \in \tilde{r}_{\psi^*}$ from the closedness of $\tilde{r}_{\psi^*}$. Also the proof of Theorem 1 shows $\tilde{p}_{n,\psi^*} \to \tilde{p}_{\psi^*}$. Therefore $t \in \tilde{p}_{\psi^*} \cap \tilde{r}_{\psi^*}$ and $\psi^* \le a$.

Case $\alpha_n \uparrow \psi^*$: For any $\alpha < \psi^*$, we have

$$\tilde{p}_{n,\psi^*} \subset \tilde{p}_{n,\alpha_n} \subset \tilde{p}_{n,\alpha}$$

for sufficiently large n. Hence, letting $n \to \infty$,

$$\tilde{p}_{\psi^*} \subset \liminf_{n\to\infty} \tilde{p}_{n,\alpha_n} \subset \limsup_{n\to\infty} \tilde{p}_{n,\alpha_n} \subset \tilde{p}_\alpha.$$

By using Lemma 3(i) and letting $\alpha \uparrow \psi^*$, it is immediate that

$$\liminf_{n\to\infty} \tilde{p}_{n,\alpha_n} = \limsup_{n\to\infty} \tilde{p}_{n,\alpha_n} = \tilde{p}_{\psi^*}.$$

So

$$\lim_{n\to\infty} \tilde{p}_{n,\alpha_n} \cap \tilde{r}_{\alpha_n} = \tilde{p}_{\psi^*} \cap \tilde{r}_{\psi^*}. \qquad (2.14)$$

Since $\tilde{p}_{n,\alpha_n} \cap \tilde{r}_{\alpha_n} \ne \emptyset$ for each n and X is compact, there exists a convergent sequence $\{t_n : t_n \in \tilde{p}_{n,\alpha_n} \cap \tilde{r}_{\alpha_n}\}$. Let its limit point be t. Then by (2.14), $t \in \tilde{p}_{\psi^*} \cap \tilde{r}_{\psi^*}$. This means that $\psi^* \le a$.

To show the reverse inequality, let $\alpha := a - \varepsilon$ for any $\varepsilon > 0$. We have $\tilde{p}_\alpha \cap \tilde{r}_\alpha \ne \emptyset$ from the notation of a. Hence there exists $t \in \tilde{p}_\alpha \cap \tilde{r}_\alpha$. Since $\tilde{p}_{n,\alpha} \to \tilde{p}_\alpha$, there exists $\{t_n : t_n \in \tilde{p}_{n,\alpha}\}$ such that $t_n \to t$. Because of the continuity of $\tilde{r}$, $\tilde{r}(t_n) \to \tilde{r}(t) \ge \alpha$, so that $t_n \in \tilde{r}_{\alpha-\varepsilon'}$ for any $\varepsilon' > 0$ and sufficiently large n. Then $t_n \in \tilde{p}_{n,\alpha-\varepsilon'} \cap \tilde{r}_{\alpha-\varepsilon'}$. Using Lemma 4, we have $\alpha - \varepsilon' \le \psi_n^*$ for sufficiently large n. Therefore

$$\alpha - \varepsilon' \le \limsup_{n\to\infty} \psi_n^* = \psi^*,$$

i.e., $a - \varepsilon - \varepsilon' \le \psi^*$. Letting $\varepsilon, \varepsilon' \downarrow 0$, we have shown the reverse inequality $a \le \psi^*$. These conclude that $a = \psi^*$. □

Now we shall show two corollaries. Let $0 \le \alpha \le 1$ and

$$D_\alpha := \{x \mid \tilde{r}_\alpha \cap \tilde{q}_\alpha(x) \ne \emptyset\},$$

where $\tilde{q}_\alpha(x) := \tilde{q}_\alpha(\{x\})$.

Corollary 1. *If $\tilde{p}_\alpha(D_\alpha) \subset D_\alpha$, then $\psi^* \ge \alpha$.*

Proof. We have $\tilde{p}_\alpha = \lim \tilde{q}_\alpha^{(n)}(D_\alpha) \subset D_\alpha$ and $\tilde{p}_\alpha = \tilde{q}_\alpha(\tilde{p}_\alpha)$. Therefore

$$\tilde{r}_\alpha \cap \tilde{p}_\alpha = \tilde{r}_\alpha \cap \tilde{q}_\alpha(\tilde{p}_\alpha) \supset \tilde{r}_\alpha \cap \tilde{q}_\alpha(x)$$

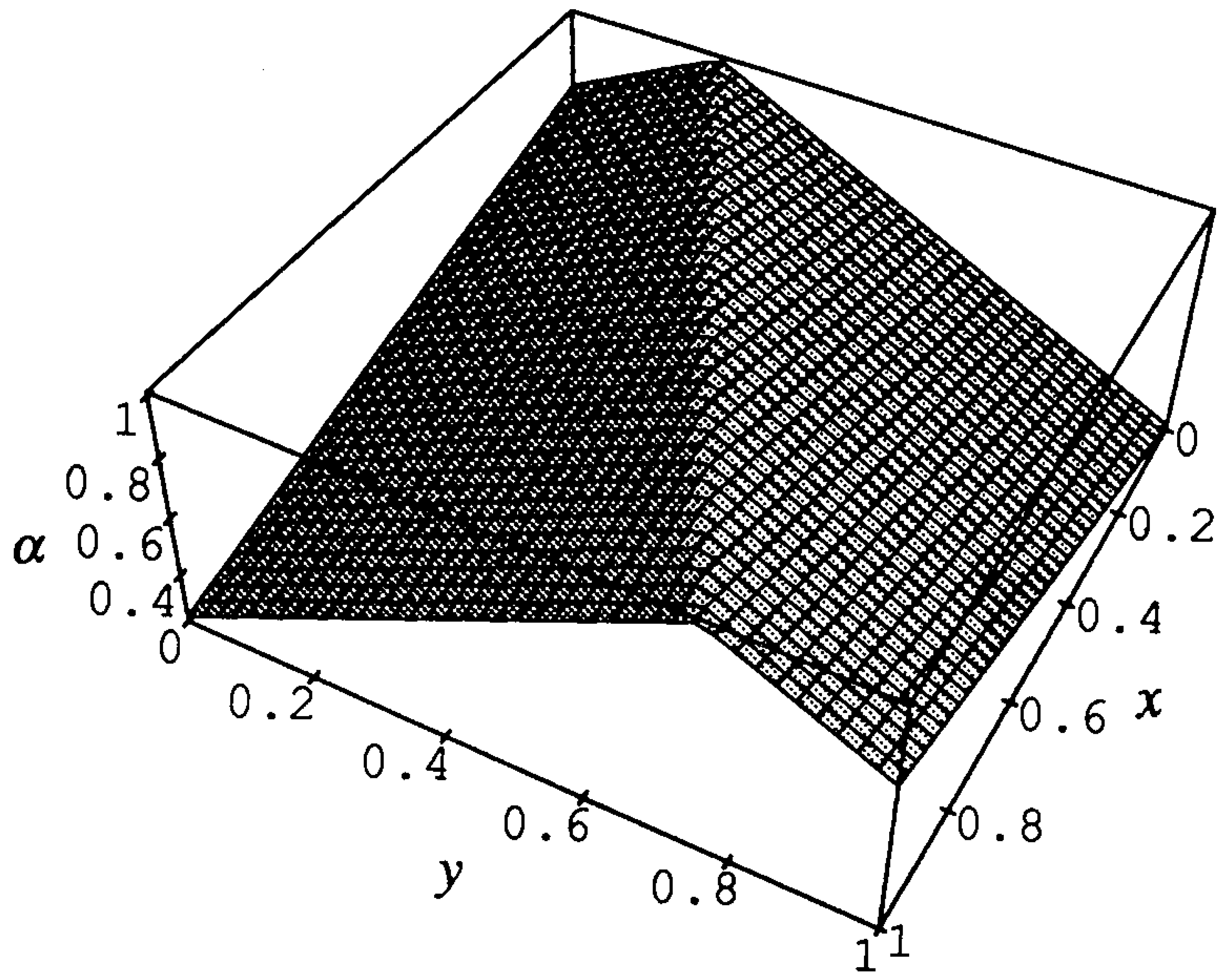

Fig. 1. The fuzzy relation $\tilde{q}(x, y)$.

for some $x \in D_\alpha$. The definition of D_α implies $\tilde{r}_\alpha \cap \tilde{q}_\alpha(x) \neq \emptyset$, so that $\tilde{r}_\alpha \cap \tilde{p}_\alpha \neq \emptyset$. By applying Theorem 2, it is shown that $\psi^* \geq \alpha$. □

Corollary 2. *If there exists* $x \in \tilde{r}_\alpha$ *which satisfies* $x \in \tilde{q}_\alpha(x)$, *then* $\psi^* \geq \alpha$ *holds.*

Proof. By the assumption, $\tilde{q}_\alpha^{(n)}(x) \ni x$ for all $n \geq 1$. Since $\lim_{n\to\infty} \tilde{q}_\alpha^{(n)}(x) = \tilde{p}_\alpha$, it must be $x \in \tilde{p}_\alpha$. Therefore $\tilde{r}_\alpha \cap \tilde{p}_\alpha \neq \emptyset$. It is clear from Theorem 2 that $\psi^* \geq \alpha$. □

3. Numerical example

Let $X = [0, 1]$ be a space of states, and consider a fuzzy relation

$$\tilde{q}(x, y) = 1 - |y - (\tfrac{1}{2}x + \tfrac{1}{4})|, \quad x, y \in X, \tag{3.1}$$

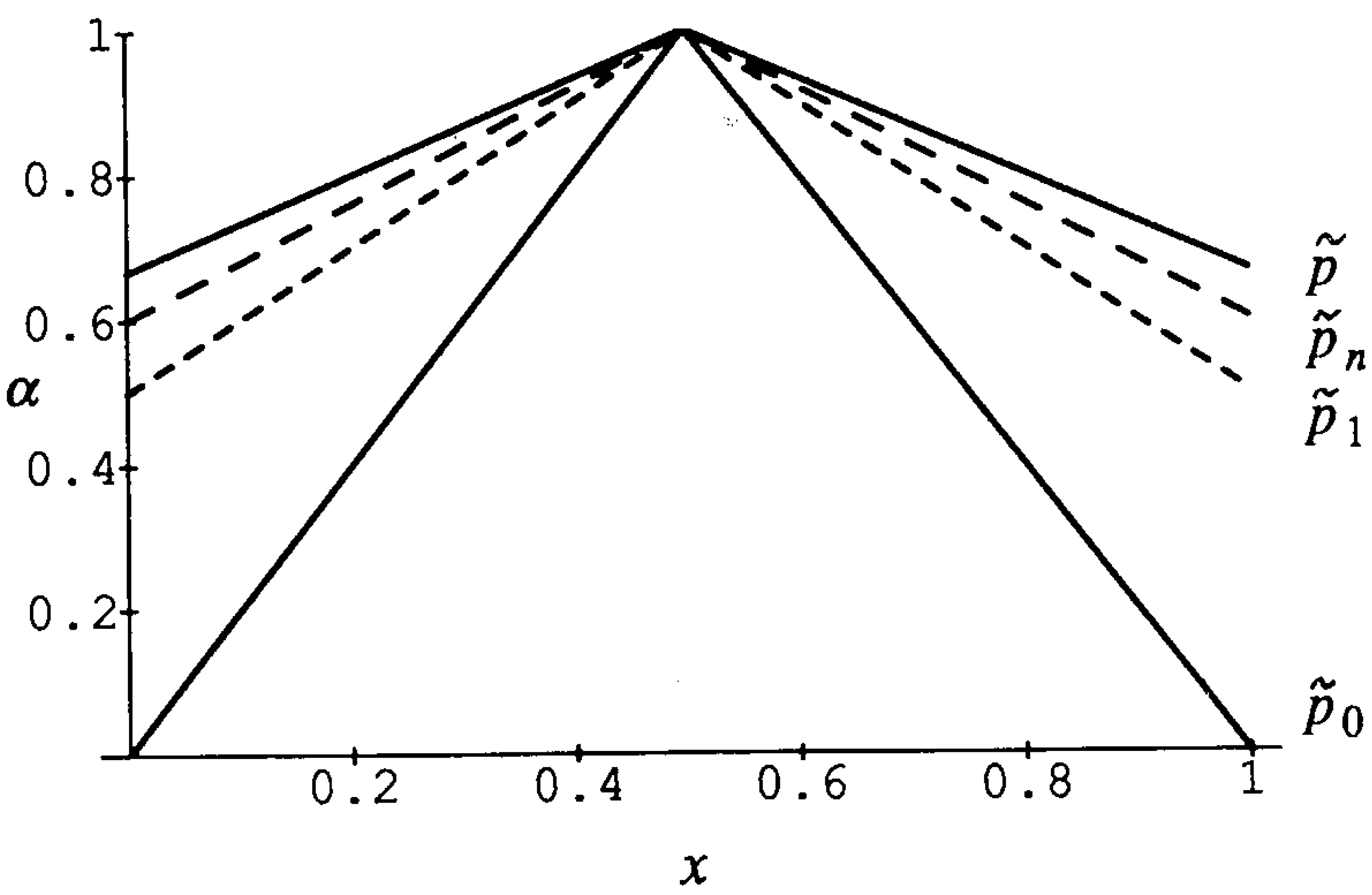

Fig. 2. The sequence of $\tilde{p}_n$ and the solution $\tilde{p}$.

and an initial fuzzy state

$$\tilde{p}_0(x) = 1 - 2\,|x - \tfrac{1}{2}|, \quad x \in X. \tag{3.2}$$

The fuzzy relation $\tilde{q}(x, y)$ is shown in Figure 1.

Under the above situation, we can easily check that the contraction coefficient β of (3.1) is $\frac{1}{2}$ and calculate that the sequence of fuzzy states defined by (1.1) is

$$\tilde{p}_n(x) = 1 - a_n\,|x - \tfrac{1}{2}|, \quad x \in X, \tag{3.3}$$

where $a_0 = 2$ and $a_n = 4a_{n-1}/(3a_{n-1} + 2)$. Then, the limit solution $\tilde{p}$ of (3.3) is

$$\tilde{p}(x) = 1 - \tfrac{2}{3}\,|x - \tfrac{1}{2}|, \quad x \in X, \tag{3.4}$$

which is also the unique solution of Theorem 1. Figure 2 shows the sequence of fuzzy states $\tilde{p}_n(x)$, $n \geq 0$, which are monotonically convergent to the unique solution $\tilde{p}(x)$.

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// Errata

Corrigendum

A limit theorem in some dynamic fuzzy systems

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The result of the calculation of the numerical example of Section 3 is incorrect. The sequence $\{a_n\}_{n=0}^{\infty}$ is given by $a_0 = 2$ and $a_n = \dfrac{2a_{n-1}}{2a_{n-1}+1}$. Therefore the limit solution $\tilde{p}$ is

$$\tilde{p}(x) = 1 - \frac{1}{2}\left|x - \frac{1}{2}\right|, \quad x \in X, \tag{3.4}$$

and Figure 2 is

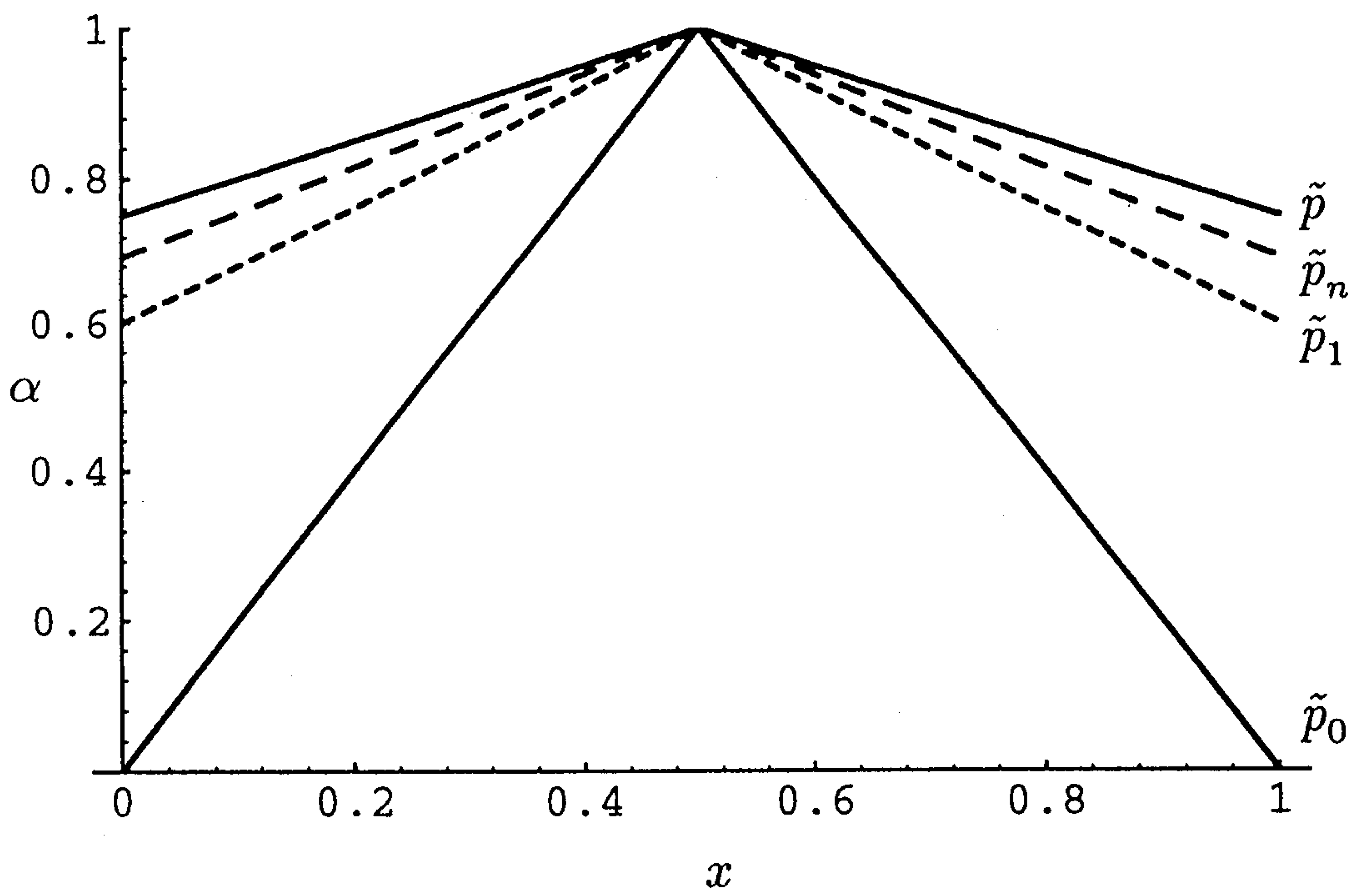

Fig. 2 : The sequence of $\tilde{p}_n$ and the solution $\tilde{p}$.