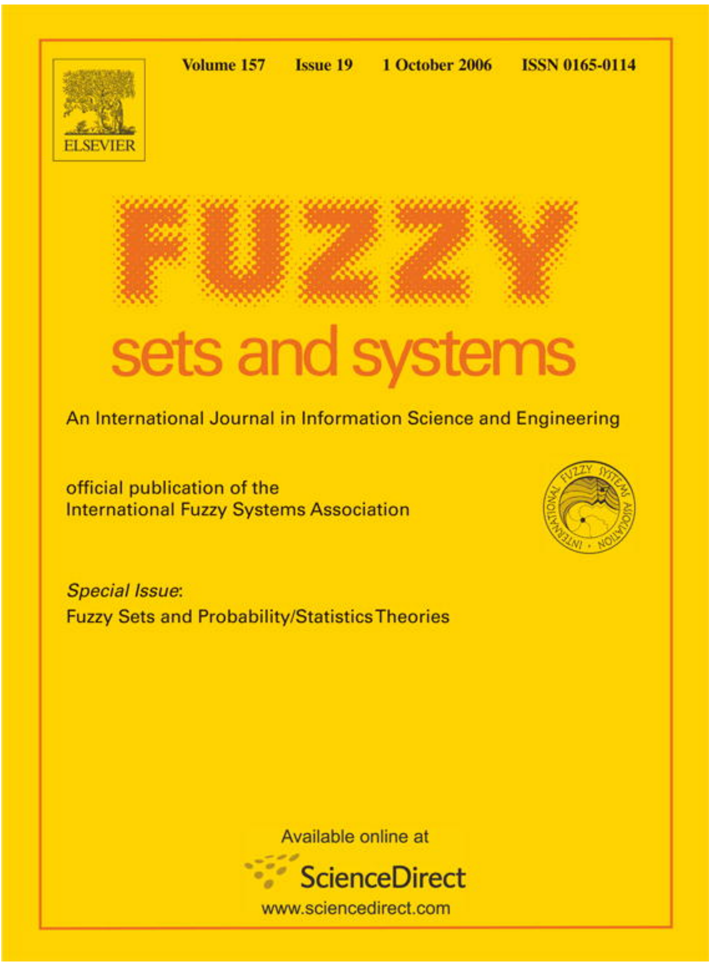
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A new evaluation of mean value for fuzzy numbers and its application to American put option under uncertainty

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Abstract

This paper discusses two topics on fuzzy random variables in decision making. One is a new evaluation method of fuzzy random variables, and the other is to present a mathematical model in financial engineering by fuzzy random variables. The evaluation method is introduced as mean values defined by fuzzy measures, and it is also applicable to fuzzy numbers and fuzzy stochastic process defined by fuzzy random variables. The other is to apply the method to an American put option with uncertainty formulated as an optimal stopping problem for fuzzy random variables, and the randomness and fuzziness are estimated by the probabilistic expectation and the mean values. The optimal expected price of the American put option is given by the mean values with decision maker's subjective parameters. Numerical examples are given to illustrate our idea.

Keywords: Fuzzy stochastic process; Mean values; Evaluation measures; Uncertainty modeling; American put option; Optimal stopping; Possibility–necessity weight, Pessimistic–optimistic index, λ-weighting function

1. Introduction

The fuzzy random variable, which is a random variable taking values in fuzzy numbers, is one of the successful hybrid notions of randomness and fuzziness as a representation of uncertainty in mathematical modeling. The study of fuzzy random variables was first proposed by Kwakernaak [9,10] and has been developed on the basis of the notions of random sets by Puri and Ralescu [12] and many authors. When we use fuzzy random variables in decision making problems, we finally need to evaluate fuzzy random variables. The most popular methods are the defuzzification and ordering of fuzzy numbers/fuzzy quantities [18,6,16], and many authors have examined the defuzzification method for fuzzy numbers/fuzzy random variables in various applications [1,7]. From the viewpoint of measure theory, Campos and Munoz [1] gave the following type evaluation of fuzzy numbers:

$$\int_0^1 h(\alpha)\,\mathrm{d}m(\alpha), \tag{1.1}$$

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where the function $h(\alpha)$ is an estimation of the α-cut of the fuzzy numbers and m is a probability measure. López-Díaz and Gil [11] studied this type of evaluation in a general form with randomness. When we use the defuzzification methods like (1.1) in decision making modeling, it is needed to discuss the meaning of the measure m on [0, 1] and to give its reasonable construction. In decision making with fuzzy numbers/fuzzy random variables, the meaning of estimation is important, and we discuss it from the viewpoint of measure theory.

This paper consists of two related topics on fuzzy random variables. The first part is devoted to a new evaluation method of fuzzy random variables for decision making modeling. In the second part, we apply the evaluation method to a mathematical model in financial engineering by fuzzy random variables. In this presented method, we estimate fuzzy random variables by probabilistic expectation and fuzzy measures, which are called *evaluation measures*, and the results are given by mean values with the decision maker's two subjective parameters, which are called a *possibility–necessity weight* for subjective estimation and a *pessimistic–optimistic index* for subjective decision. Particularly we focus on the estimation methods with the possibility measure and the necessity measure for its numerical computation in modeling. This method is also applicable to fuzzy numbers and fuzzy stochastic process defined by fuzzy random variables.

From the viewpoint of fuzzy random variables, this paper discusses an American put option model under uncertainty in financial engineering which is based on Black–Scholes stochastic mode [19]. The theory of option pricing in a financial market has been developing on the basis of the famous Black–Scholes log-normal stochastic differential models, however, there sometimes exists a difference between the actual prices and the value which derived from Black–Scholes theory. When the market are changing rapidly, the losses/errors often become bigger between the decision maker's expected price and the actual price. The difficulty comes from not only randomness of financial stochastic systems but also uncertainty which we cannot represent by only probability theory. When we deal with systems like financial markets, fuzzy logic works well because the markets contain the uncertain factors which are different from probabilistic essence and in which there exists a difficulty to identify actual price values exactly.

In this paper, probability is applied as the uncertainty such that something occurs or not with probability, and fuzziness is applied as the uncertainty such that we cannot specify the exact values because of a lack of knowledge regarding the present stock market. By introducing fuzzy logic to the log-normal stochastic processes for the financial market, we present a model with uncertainty of both randomness and fuzziness in output, which is a reasonable and natural extension of the original log-normal stochastic processes in Black–Scholes model. To valuate the American put option, we need to deal with optimal stopping in log-normal stochastic processes [4,8,13] and so on). In this paper, we discuss an optimal stopping model regarding log-normal stochastic processes with fuzziness from the viewpoint of Yoshida et al. [21], and the optimal stopping times mean exercise times for American option in the financial markets.

In the next section, we introduce a *fuzzy stochastic process* defined by fuzzy random variables. In Sections 2.2–2.4, this paper presents an evaluation method of fuzzy numbers and fuzzy random variables by mean values with fuzzy measures and we demonstrate its meaning particularly with the possibility measure and the necessity measure. We, in the sequel, discuss its validity in a financial model. In Section 3, American put option model with uncertainty is formulated and fuzzy prices of the American option are evaluated by the probabilistic expectation and mean values with the decision maker's two subjective parameters: a possibility–necessity weight and a pessimistic–optimistic index. An optimality equation for the optimal fuzzy price is derived by dynamic programming under a reasonable assumption. Next, we make sure the volatility of the presented mean values regarding the decision maker's optimal expected prices in the American put option by using the presented mean values. Finally, in Section 3.4, a numerical example is given to illustrate our idea throughout this paper.

2. Estimation of fuzzy random variables by evaluation measures

2.1. Fuzzy random variables and fuzzy stochastic processes

First we introduce some notations of fuzzy numbers. Let $\mathbb{R}$ denote the set of all real numbers, and let $\mathcal{B}$ and $\mathcal{I}$ be the Borel σ-field and the set of all non-empty bounded closed intervals, respectively. A fuzzy number is denoted by its membership function $\tilde{a} : \mathbb{R} \mapsto [0, 1]$ which is normal, upper-semicontinuous, fuzzy convex and has a compact support. $\mathcal{R}$ denotes the set of all fuzzy numbers, and $\mathcal{R}_c$ also denotes the set of fuzzy numbers with continuous membership functions. Refer to Zadeh [22] regarding fuzzy set theory. In this paper, we identify fuzzy numbers with their corresponding membership functions. The α-cut of a fuzzy number $\tilde{a}(\in \mathcal{R})$ is given by $\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geqslant \alpha\}$

($\alpha \in (0, 1]$) and $\tilde{a}_0 := \mathrm{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\}$, where cl denotes the closure of an interval. The α-cut is also written by closed intervals $\tilde{a}_\alpha = [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$ ($\alpha \in [0, 1]$). Hence we introduce a partial order $\succcurlyeq$, so called the *fuzzy max order*, on fuzzy numbers $\mathcal{R}$: let $\tilde{a}, \tilde{b} \in \mathcal{R}$ be fuzzy numbers. $\tilde{a} \succcurlyeq \tilde{b}$ means that $\tilde{a}_\alpha^- \geqslant \tilde{b}_\alpha^-$ and $\tilde{a}_\alpha^+ \geqslant \tilde{b}_\alpha^+$ for all $\alpha \in [0, 1]$. Then $(\mathcal{R}, \succcurlyeq)$ becomes a lattice. For fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{R}$, we define the maximum $\tilde{a} \vee \tilde{b}$ with respect to the fuzzy max order $\succcurlyeq$ by the fuzzy number whose α-cuts are

$$(\tilde{a} \vee \tilde{b})_\alpha = [\max\{\tilde{a}_\alpha^-, \tilde{b}_\alpha^-\}, \max\{\tilde{a}_\alpha^+, \tilde{b}_\alpha^+\}], \quad \alpha \in [0, 1]. \tag{2.1}$$

An addition, a subtraction and a scalar multiplication for fuzzy numbers are defined as follows: for $\tilde{a}, \tilde{b} \in \mathcal{R}$ and $\zeta \geqslant 0$, the addition and subtraction $\tilde{a} \pm \tilde{b}$ of $\tilde{a}$ and $\tilde{b}$ and the scalar multiplication $\zeta\tilde{a}$ of ζ and $\tilde{a}$ are fuzzy numbers given by their α-cuts

$$(\tilde{a} + \tilde{b})_\alpha := [\tilde{a}_\alpha^- + \tilde{b}_\alpha^-, \tilde{a}_\alpha^+ + \tilde{b}_\alpha^+], \quad (\tilde{a} - \tilde{b})_\alpha := [\tilde{a}_\alpha^- - \tilde{b}_\alpha^+, \tilde{a}_\alpha^+ - \tilde{b}_\alpha^-] \quad \text{and} \quad (\zeta\tilde{a})_\alpha := [\zeta\tilde{a}_\alpha^-, \zeta\tilde{a}_\alpha^+], \tag{2.2}$$

where $\tilde{a}_\alpha = [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$ and $\tilde{b}_\alpha = [\tilde{b}_\alpha^-, \tilde{b}_\alpha^+]$ ($\alpha \in [0, 1]$).

Next we introduce fuzzy random variables. Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a σ-field of Ω and P is a non-atomic probability measure. A fuzzy-number-valued map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a *fuzzy random variable* if the maps $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are measurable for all $\alpha \in [0, 1]$, where $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] = \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geqslant \alpha\}$ [15]. A fuzzy random variable $\tilde{X}$ is called integrably bounded if both maps $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are integrable for all $\alpha \in [0, 1]$. Let $\tilde{X}$ be an integrably bounded fuzzy random variable. The expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number [12]

$$E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{E(\tilde{X})_\alpha}(x)\}, \quad x \in \mathbb{R}, \tag{2.3}$$

where $E(\tilde{X})_\alpha := [\int_\Omega \tilde{X}_\alpha^-(\omega)\,\mathrm{d}P(\omega), \int_\Omega \tilde{X}_\alpha^+(\omega)\,\mathrm{d}P(\omega)]$ ($\alpha \in [0, 1]$).

In this paper, we deal with a continuous-time fuzzy stochastic process defined by fuzzy random variables. Let $\{\tilde{X}_t\}_{t \geqslant 0}$ be a family of integrably bounded fuzzy random variables such that $E(\sup_{t \geqslant 0} \tilde{X}_{t,0}^+) < \infty$, where $\tilde{X}_{t,0}^+(\omega)$ is the right-end of the 0-cut of the fuzzy number $\tilde{X}_t(\omega)$. We assume that the map $t \mapsto \tilde{X}_t(\omega) (\in \mathcal{R})$ is continuous on $[0, \infty)$ for almost all $\omega \in \Omega$. $\{\mathcal{M}_t\}_{t \geqslant 0}$ is a family of non-decreasing sub-σ-fields of $\mathcal{M}$ which is right continuous, i.e. $\mathcal{M}_t = \bigcap_{r:r>t} \mathcal{M}_r$ for all $t \geqslant 0$, and fuzzy random variables $\tilde{X}_t$ are $\mathcal{M}_t$-adapted, i.e. random variables $\tilde{X}_{r,\alpha}^-$ and $\tilde{X}_{r,\alpha}^+$ ($0 \leqslant r \leqslant t$; $\alpha \in [0, 1]$) are $\mathcal{M}_t$-measurable. And $\mathcal{M}_\infty$ denotes the smallest σ-field containing $\bigcup_{t \geqslant 0} \mathcal{M}_t$. Then we call $(\tilde{X}_t, \mathcal{M}_t)_{t \geqslant 0}$ a *fuzzy stochastic process*, and then fuzzy stochastic processes are applied in Section 3 to describe stock price processes with fuzzy number values.

2.2. *Mean values of fuzzy numbers and fuzzy random variables by evaluation measures*

By using fuzzy measures, we present a method to estimate fuzzy numbers, and in Section 3 we apply it to American put option model with uncertainty of stock prices. Campos and Munoz [1] studied an evaluation of fuzzy numbers in the form of (1.1). In decision making with fuzzy numbers/fuzzy random variables, we discuss the meaning of the estimation from the viewpoint of measure theory, and then fuzzy measures are used to evaluate a confidence degree that a fuzzy number takes values in an interval.

Definition 2.1 (*Wang and Klir [17]*). A map $M : \mathcal{B} \mapsto [0, 1]$ is called a *fuzzy measure* on $\mathcal{B}$ if M satisfies the following (M.i), (M.ii) and (M.iii) (or (M.i), (M.ii) and (M.iv)):

(M.i) $M(\emptyset) = 0$ and $M(\mathbb{R}) = 1$;
(M.ii) $M(I_1) \leqslant M(I_2)$ holds for $I_1, I_2 \in \mathcal{B}$ satisfying $I_1 \subset I_2$;
(M.iii) $M(\bigcup_{n=0}^\infty I_n) = \lim_{n\to\infty} M(I_n)$ holds for $\{I_n\}_{n=0}^\infty \subset \mathcal{B}$ satisfying $I_n \subset I_{n+1}$ ($n = 0, 1, 2, \ldots$);
(M.iv) $M(\bigcap_{n=0}^\infty I_n) = \lim_{n\to\infty} M(I_n)$ holds for $\{I_n\}_{n=0}^\infty \subset \mathcal{B}$ satisfying $I_n \supset I_{n+1}$ ($n = 0, 1, 2, \ldots$).

In this paper, we use fuzzy measures M to evaluate a confidence degree that a fuzzy number takes values in an interval and we call them *evaluation measures*. First we deal with fuzzy numbers $\tilde{a}$ whose membership functions are continuous, i.e. $\tilde{a} \in \mathcal{R}_c$, and in the next section we discuss about general fuzzy numbers $\tilde{a} \in \mathcal{R}$ whose membership functions are upper-semicontinuous but are not necessarily continuous.

Fuzzy random variables have two kinds of uncertainty (randomness and fuzziness). In this paper, the randomness is evaluated by the probabilistic expectation, and the fuzziness is evaluated by λ-weighting functions and evaluation measures. Let $g : \mathcal{I} \mapsto \mathbb{R}$ be a map such that

$$g([x, y]) := \lambda x + (1 - \lambda) y, \quad [x, y] \in \mathcal{I}, \tag{2.4}$$

where λ is a constant satisfying $0 \leqslant \lambda \leqslant 1$. This scalarization is used for the estimation of fuzzy numbers, and λ is called a *pessimistic–optimistic index* and means the pessimistic degree in decision making [6]. Then we call g a *λ-weighting function*. Let a fuzzy number $\tilde{a} \in \mathcal{R}_c$. We introduce mean values of the fuzzy number $\tilde{a}$ with respect to λ-weighting functions g and an evaluation measure $M_{\tilde{a}}$, which depends on $\tilde{a}$, as follows:

$$\tilde{E}(\tilde{a}) = \int_0^1 M_{\tilde{a}}(\tilde{a}_\alpha) g(\tilde{a}_\alpha)\, \mathrm{d}\alpha \bigg/ \int_0^1 M_{\tilde{a}}(\tilde{a}_\alpha)\, \mathrm{d}\alpha, \tag{2.5}$$

where $\tilde{a}_\alpha = [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$ is the α-cut of the fuzzy number $\tilde{a}$. We note that (2.5) is normalized by $M(\tilde{a}_\alpha)(\alpha \in [0, 1])$. In a comparison with (1.1), $h(\alpha)$ is replaced with $g(\tilde{a}_\alpha)$ and the measure $\mathrm{d}m(\alpha)$ is taken as $M_{\tilde{a}}(\tilde{a}_\alpha)\, \mathrm{d}\alpha$. In (2.5), $M_{\tilde{a}}(\tilde{a}_\alpha)$ means a *confidence degree that the fuzzy number $\tilde{a}$ takes values in the interval $\tilde{a}_\alpha$ at each grade* α (see Example 2.1).

Example 2.1. Let a fuzzy number $\tilde{a} \in \mathcal{R}_c$. An evaluation measure $M_{\tilde{a}}$ is called the *possibility evaluation measure*, the *necessity evaluation measure* and the *credibility evaluation measure* induced from the fuzzy number $\tilde{a}$ if it is given by the following (2.6)–(2.8), respectively:

$$M_{\tilde{a}}^{\mathrm{P}}(I) := \sup_{x \in I} \tilde{a}(x), \quad I \in \mathcal{B}; \tag{2.6}$$

$$M_{\tilde{a}}^{\mathrm{N}}(I) := 1 - \sup_{x \notin I} \tilde{a}(x), \quad I \in \mathcal{B}; \tag{2.7}$$

$$M_{\tilde{a}}^{\mathrm{C}}(I) := \tfrac{1}{2}(M_{\tilde{a}}^{\mathrm{P}}(I) + M_{\tilde{a}}^{\mathrm{N}}(I)), \quad I \in \mathcal{B}. \tag{2.8}$$

We note that $M_{\tilde{a}}^{\mathrm{P}}$, $M_{\tilde{a}}^{\mathrm{N}}$ and $M_{\tilde{a}}^{\mathrm{C}}$ satisfy Definition 2.1 (M.i)–(M.iv) since $\tilde{a}$ is continuous and has a compact support. Since $M_{\tilde{a}}^{\mathrm{P}}(\tilde{a}_\alpha) = 1$ and $M_{\tilde{a}}^{\mathrm{N}}(\tilde{a}_\alpha) = 1 - \alpha$ and $M_{\tilde{a}}^{\mathrm{C}}(\tilde{a}_\alpha) = 1 - \alpha/2$ from (2.6)–(2.8), the corresponding mean values $\tilde{E}(\tilde{a})$ are reduced to

$$\tilde{E}^{\mathrm{P}}(\tilde{a}) := \int_0^1 g(\tilde{a}_\alpha)\, \mathrm{d}\alpha; \tag{2.9}$$

$$\tilde{E}^{\mathrm{N}}(\tilde{a}) := \int_0^1 2(1 - \alpha) g(\tilde{a}_\alpha)\, \mathrm{d}\alpha; \tag{2.10}$$

$$\tilde{E}^{\mathrm{C}}(\tilde{a}) := \int_0^1 \frac{2}{3}(2 - \alpha) g(\tilde{a}_\alpha)\, \mathrm{d}\alpha. \tag{2.11}$$

They are called a *possibility mean*, a *necessity mean* and a *credibility mean* of the fuzzy number $\tilde{a}$, respectively. Eq. (2.9) has been discussed in Fortemps and Roubens [6] and so on, however, an evaluation method

$$\int_0^1 2\alpha g(\tilde{a}_\alpha)\, \mathrm{d}\alpha = \int_0^1 \alpha g(\tilde{a}_\alpha)\, \mathrm{d}\alpha \bigg/ \int_0^1 \alpha\, \mathrm{d}\alpha, \tag{2.12}$$

which has been studied by Goetshel and Voxman [7] and Carlsson and Fullér [2], is different from our method (2.5) since $M_{\tilde{a}}(\tilde{a}_\alpha)$ in (2.5) is non-increasing in $\alpha \in [0, 1]$ from Definition 2.1 (M.ii) and the property of α-cuts. Fig. 1 illustrates the possibility mean and the necessity mean for a triangle-type fuzzy number

$$\tilde{a}(x) = \begin{cases} 0 & \text{if } x < c_1, \\ (x - c_1)/(c_2 - c_1) & \text{if } c_1 \leqslant x < c_2, \\ (x - c_3)/(c_2 - c_3) & \text{if } c_2 \leqslant x < c_3, \\ 0 & \text{if } x \geqslant c_3, \end{cases} \tag{2.13}$$

where c_1, c_2, c_3 are real numbers satisfying $c_1 < c_2 < c_3$.

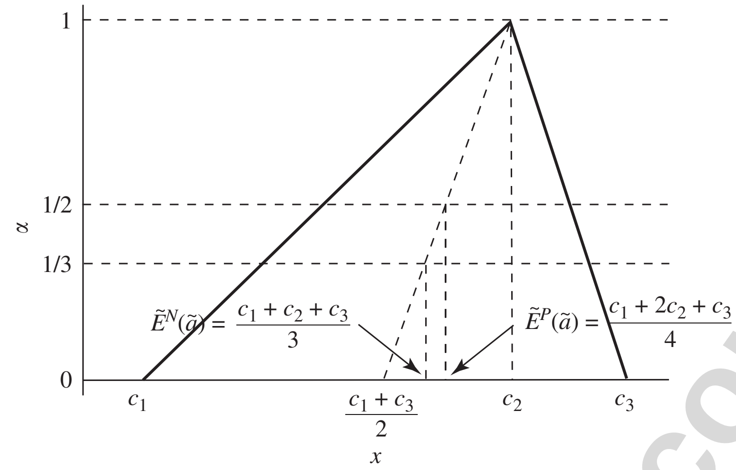

Fig. 1. The possibility mean and the necessity mean for a triangle-type fuzzy number.

2.3. General mean values by evaluation measures

Under the following regularity assumption, we extend estimation (2.5) to the mean value of a general fuzzy number $\tilde{a} \in \mathcal{R}$ whose membership functions are upper-semicontinuous but are not necessarily continuous.

Assumption M. There exists a non-increasing function $\rho : [0, 1] \mapsto [0, 1]$ such that

$$M_{\tilde{a}}(\tilde{a}_\alpha) = \rho(\alpha), \quad \alpha \in [0, 1] \quad \text{for all } \tilde{a} \in \mathcal{R}_c. \tag{2.14}$$

We note that ρ is independent of $\tilde{a} \in \mathcal{R}_c$ in (2.14) of Assumption M. Regarding the possibility evaluation measure, the necessity evaluation measure and the credibility evaluation measure, we may take $\rho(\alpha)$ in Assumption M as $\rho(\alpha) = M^{\mathrm{P}}_{\tilde{a}}(\tilde{a}_\alpha) = 1$ and $\rho(\alpha) = M^{\mathrm{N}}_{\tilde{a}}(\tilde{a}_\alpha) = 1 - \alpha$ and $\rho(\alpha) = M^{\mathrm{C}}_{\tilde{a}}(\tilde{a}_\alpha) = 1 - \alpha/2$, respectively (see (2.6)–(2.8)). From now on, we suppose Assumption M holds.

Let $\tilde{a} \in \mathcal{R}$. We define the *mean values* for the general fuzzy number $\tilde{a} \in \mathcal{R}$ by

$$\tilde{E}(\tilde{a}) := \lim_{n\to\infty} \tilde{E}(\tilde{a}^n), \tag{2.15}$$

where $\tilde{E}(\tilde{a}^n)$ are defined by (2.5) and $\{\tilde{a}^n\}_{n=1}^{\infty} (\subset \mathcal{R}_c)$ is a sequence of fuzzy numbers whose membership functions are continuous and satisfy that $\tilde{a}^n \downarrow \tilde{a}$ pointwise as $n \to \infty$. The limiting value (2.15) is called well-defined if it is independent of the selection of the sequences $\{\tilde{a}^n\}_{n=1}^{\infty} \subset \mathcal{R}_c$. From (2.9)–(2.11), by the bounded convergence theorem we obtain the mean values defined by the *possibility evaluation measure* and the *necessity evaluation measure* and the *credibility evaluation measure* as follows: for general fuzzy numbers $\tilde{a} \in \mathcal{R}$,

$$\tilde{E}^{\mathrm{P}}(\tilde{a}) = \int_0^1 g(\tilde{a}_\alpha)\,\mathrm{d}\alpha; \tag{2.16}$$

$$\tilde{E}^{\mathrm{N}}(\tilde{a}) = \int_0^1 2(1-\alpha) g(\tilde{a}_\alpha)\,\mathrm{d}\alpha; \tag{2.17}$$

$$\tilde{E}^{\mathrm{C}}(\tilde{a}) = \int_0^1 \frac{2}{3}(2-\alpha) g(\tilde{a}_\alpha)\,\mathrm{d}\alpha. \tag{2.18}$$

We note that (2.16)–(2.18) are well-defined. The following gives a counterexample to define the mean values (2.5) directly for general fuzzy numbers.

Remark 2.1. We consider the following numerical example (see Fig. 2). Take $\lambda = \frac{1}{2}$ for g in (2.4). Let

$$\tilde{a}(x) := \begin{cases} 0, & x < 1, \\ (x+1)/4, & 1 \leqslant x < 3, \\ (-x+5)/2, & 3 \leqslant x \leqslant 4, \\ 0, & x > 4, \end{cases} \tag{2.19}$$

and

$$\tilde{a}^n(x) := \begin{cases} 0, & x < 1 - 1/n \\ (nx - n + 1)/2, & 1 - 1/n \leqslant x < 1 \\ (x+1)/4, & 1 \leqslant x < 3 \\ (-x+5)/2, & 3 \leqslant x \leqslant 4 \\ (-nx + 4n + 1)/2, & 4 < x \leqslant 4 + 1/n \\ 0, & x > 4 + 1/n \end{cases} \quad \text{for } n \geqslant 1. \tag{2.20}$$

Then $\tilde{a} \in \mathcal{R}$ and $\tilde{a} \notin \mathcal{R}_c$, and $\tilde{a}^n \in \mathcal{R}_c$ for $n \geqslant 1$. Further, we can easily check $\tilde{a}^n \downarrow \tilde{a}$ as $n \to \infty$. Now we compare the necessity mean values of $\tilde{a}^n$ and $\tilde{a}$. From (2.20) and (2.7) we have

$$\tilde{a}^n_\alpha = [\tilde{a}^{n,-}_\alpha, \tilde{a}^{n,+}_\alpha] = \begin{cases} [(2\alpha + n - 1)/n, (-2\alpha + 4n + 1)/n], & 0 \leqslant \alpha < \frac{1}{2}, \\ [4\alpha - 1, -2\alpha + 5], & \frac{1}{2} \leqslant \alpha \leqslant 1, \end{cases}$$

and

$$M^{\mathrm{N}}_{\tilde{a}^n}(\tilde{a}^n_\alpha) = 1 - \alpha, \quad \alpha \in [0, 1].$$

Then

$$\begin{aligned} \tilde{E}^{\mathrm{N}}(\tilde{a}^n) &= \int_0^1 M^{\mathrm{N}}_{\tilde{a}^n}(\tilde{a}^n_\alpha) g(\tilde{a}^n_\alpha)\,\mathrm{d}\alpha \Big/ \int_0^1 M^{\mathrm{N}}_{\tilde{a}^n}(\tilde{a}^n_\alpha)\,\mathrm{d}\alpha \\ &= \int_0^1 M^{\mathrm{N}}_{\tilde{a}^n}(\tilde{a}^n_\alpha) \frac{\tilde{a}^{n,-}_\alpha + \tilde{a}^{n,+}_\alpha}{2}\,\mathrm{d}\alpha \Big/ \int_0^1 M^{\mathrm{N}}_{\tilde{a}^n}(\tilde{a}^n_\alpha)\,\mathrm{d}\alpha \\ &= 2\left(\int_0^{1/2} (1-\alpha) \frac{(2\alpha + n - 1)/n + (-2\alpha + 4n + 1)/n}{2}\,\mathrm{d}\alpha \right. \\ &\quad \left. + \int_{1/2}^1 (1-\alpha) \frac{(4\alpha - 1) + (-2\alpha + 5)}{2}\,\mathrm{d}\alpha \right) \\ &= \frac{61}{24} \quad \text{for all } n \geqslant 1. \end{aligned}$$

Therefore we obtain $\tilde{E}^{\mathrm{N}}(\tilde{a}^n) = \frac{61}{24}$ for all $n \geqslant 1$. Next we calculate the formal mean value of $\tilde{a}$, which is not defined in this paper, in the same way as $\tilde{E}^{\mathrm{N}}(\tilde{a}^n)$, and we compare them. From (2.19), we have

$$\tilde{a}_\alpha = [\tilde{a}^-_\alpha, \tilde{a}^+_\alpha] = \begin{cases} [1, 4], & 0 \leqslant \alpha < \frac{1}{2}, \\ [4\alpha - 1, -2\alpha + 5], & \frac{1}{2} \leqslant \alpha \leqslant 1. \end{cases}$$

From (2.7), the necessity evaluation measure of $\tilde{a}$, which does not satisfies Definition 2.1 (vi), is

$$M^{\mathrm{N}}_{\tilde{a}}(\tilde{a}_\alpha) = 1 - \sup_{x \notin \tilde{a}_\alpha} \tilde{a}(x) = \begin{cases} 1, & 0 \leqslant \alpha < \frac{1}{2}, \\ 1 - \alpha, & \frac{1}{2} \leqslant \alpha \leqslant 1. \end{cases}$$

Then

$$\begin{aligned} &\int_0^1 M^{\mathrm{N}}_{\tilde{a}}(\tilde{a}_\alpha) g(\tilde{a}_\alpha)\,\mathrm{d}\alpha \Big/ \int_0^1 M^{\mathrm{N}}_{\tilde{a}}(\tilde{a}_\alpha)\,\mathrm{d}\alpha \\ &\quad = \int_0^1 M^{\mathrm{N}}_{\tilde{a}}(\tilde{a}_\alpha) \frac{\tilde{a}^-_\alpha + \tilde{a}^+_\alpha}{2}\,\mathrm{d}\alpha \Big/ \int_0^1 M^{\mathrm{N}}_{\tilde{a}}(\tilde{a}_\alpha)\,\mathrm{d}\alpha \\ &\quad = \frac{8}{5}\left(\int_0^{1/2} \frac{1+4}{2}\,\mathrm{d}\alpha \int_{1/2}^1 (1-\alpha) \frac{(4\alpha - 1) + (-2\alpha + 5)}{2}\,\mathrm{d}\alpha\right) \\ &\quad = \frac{38}{15}. \end{aligned} \tag{2.21}$$

Therefore $\tilde{E}^{\mathrm{N}}(\tilde{a}^n) = \frac{61}{24} \neq \frac{38}{15} = (2.21)$. From Fig. 2, it is natural that the mean values $\tilde{E}^{\mathrm{N}}(\tilde{a}^n)$ converges to the mean value of $\tilde{a}$ as $n \to \infty$. These results show that it is not reasonable to give the mean values (2.21) directly for

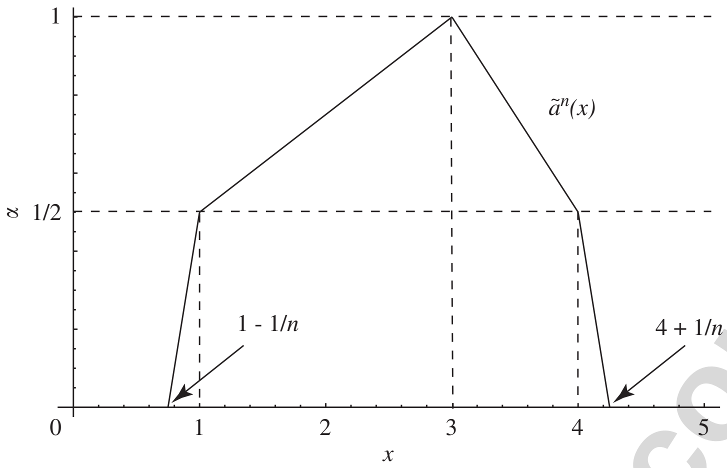

Fig. 2. The sequence of fuzzy numbers $\{\tilde{a}^n\}_{n=1}^{\infty}$ given by (2.20) ($n = 4$).

fuzzy numbers $\tilde{a} \in \mathcal{R}$ with discontinuous membership functions. Possibility mean value, necessity mean value and credibility mean value vary when the fuzzy sets $\tilde{a}$ have discontinuous membership functions, but they are invariant, $\rho(\alpha) = 1, 1 - \alpha, 1 - \alpha/2$, when fuzzy sets $\tilde{a}$ have continuous membership functions. From this example, we find that it is difficult to define the mean values directly for general fuzzy numbers of $\mathcal{R}$ in the form of (2.5). Therefore, we define the mean values by (2.15) through a sequence of fuzzy numbers $\{\tilde{a}^n\}_{n=1}^{\infty} (\subset \mathcal{R}_c)$ with continuous membership functions.

Similar to (2.16)–(2.28), under Assumption M we obtain the following representation regarding a general mean value (2.15) through the dominated convergence theorem: for general fuzzy numbers $\tilde{a} \in \mathcal{R}$,

$$\tilde{E}(\tilde{a}) = \int_0^1 \rho(\alpha) g(\tilde{a}_\alpha) \, \mathrm{d}\alpha \Big/ \int_0^1 \rho(\alpha) \, \mathrm{d}\alpha. \tag{2.22}$$

The mean value $\tilde{E}(\cdot)$ has the following natural properties for fuzzy numbers.

Theorem 2.1. *Suppose Assumption M holds. For fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{R}$, $\theta \in \mathbb{R}$ and $\zeta \geqslant 0$, the following* (i)–(iv) *hold.*

(i) $\tilde{E}(\tilde{a} + 1_{\{\theta\}}) = \tilde{E}(\tilde{a}) + \theta$.
(ii) $\tilde{E}(\zeta\tilde{a}) = \zeta\tilde{E}(\tilde{a})$.
(iii) $\tilde{E}(\tilde{a} + \tilde{b}) = \tilde{E}(\tilde{a}) + \tilde{E}(\tilde{b})$.
(iv) *If $\tilde{a} \succcurlyeq \tilde{b}$, then $\tilde{E}(\tilde{a}) \geqslant \tilde{E}(\tilde{b})$ holds, where $\succcurlyeq$ is the fuzzy max order.*

Proof. (i) Since $g[(\tilde{a} + 1_{\{\theta\}})_\alpha] = g[\tilde{a}_\alpha + \{\theta\}] = g[\tilde{a}_\alpha] + \theta$ for all $\alpha \in [0, 1]$, we obtain (i). (ii) Since $g[(\zeta\tilde{a})_\alpha] = \zeta g[\tilde{a}_\alpha]$ for all $\alpha \in [0, 1]$, we also have (ii). (iii) Since $g[(\tilde{a} + \tilde{b})_\alpha] = g[\tilde{a}_\alpha] + g[\tilde{b}_\alpha]$ for all $\alpha \in [0, 1]$, we get (iii). (iv) Let $\tilde{a} \succcurlyeq \tilde{b}$. Since $\tilde{a}_\alpha^- \geqslant \tilde{b}_\alpha^-$ and $\tilde{a}_\alpha^+ \geqslant \tilde{b}_\alpha^+$ for all $\alpha \in [0, 1]$, we have $g(\tilde{a}_\alpha) \geqslant g(\tilde{b}_\alpha)$. Therefore we obtain (iv). □

2.4. Mean values with possibility–necessity weights

For a fuzzy number $\tilde{a} \in \mathcal{R}$ and a parameter $v \in [0, 1]$, we introduce a mean value

$$\tilde{E}^v(\tilde{a}) := v\tilde{E}^{\mathrm{P}}(\tilde{a}) + (1 - v)\tilde{E}^{\mathrm{N}}(\tilde{a}). \tag{2.23}$$

Then, v is called a *possibility–necessity weight*, and (2.23) means mean values with the possibility–necessity weight v. We note that (2.23) is well-defined. The possibility mean $\tilde{E}^{\mathrm{P}}(\cdot)$, the necessity mean $\tilde{E}^{\mathrm{N}}(\cdot)$, and the credibility mean $\tilde{E}^{\mathrm{C}}(\cdot)$ are represented by the mean values (2.23) with the corresponding possibility–necessity weights $v = 1, 0, \frac{2}{3}$, respectively. In this paper, we focus on this type of mean value (2.23) for numerical computation and we apply it to a mathematical model with fuzzy random variables in Section 3. Hence (2.23) satisfies Assumption M with $\rho(\alpha) = v + 2(1-v)(1-\alpha)$. The following theorem is trivial from (2.16), (2.17) and (2.23), but it is convenient for numerical calculations in applications.

Theorem 2.2. *Let a fuzzy number* $\tilde{a} \in \mathcal{R}$ *and* $v, \lambda \in [0, 1]$. *Then, the mean value* $\tilde{E}^v(\cdot)$ *with the possibility–necessity weight* v *and the pessimistic–optimistic index* λ *is represented by*

$$\tilde{E}^v(\tilde{a}) = \int_0^1 (v + 2(1-v)(1-\alpha)) g(\tilde{a}_\alpha)\, \mathrm{d}\alpha = \int_0^1 (v + 2(1-v)(1-\alpha))(\lambda \tilde{a}_\alpha^- + (1-\lambda)\tilde{a}_\alpha^+)\, \mathrm{d}\alpha, \tag{2.24}$$

where λ*-weighting function* g *is given by* (2.4).

Now we calculate the mean values $\tilde{E}^v(\cdot)$ of triangle-type fuzzy numbers and trapezoidal-type fuzzy numbers.

Example 2.2. Let $v, \lambda \in [0, 1]$. Let $\tilde{a} \in \mathcal{R}_c$ be a triangle-type fuzzy number (2.13) and let $\tilde{b} \in \mathcal{R}_c$ be a trapezoidal-type fuzzy number (2.25):

$$\tilde{b}(x) = \begin{cases} 0 & \text{if } x < c_1, \\ (x - c_1)/(c_2 - c_1) & \text{if } c_1 \leqslant x < c_2, \\ 1 & \text{if } c_2 \leqslant x < c_3, \\ (x - c_4)/(c_3 - c_4) & \text{if } c_3 \leqslant x < c_4, \\ 0 & \text{if } x \geqslant c_4, \end{cases} \tag{2.25}$$

where c_1, c_2, c_3, c_4 are real numbers satisfying $c_1 < c_2 < c_3 < c_4$. Then we can easily calculate the corresponding mean values $\tilde{E}^v(\tilde{a})$ and $\tilde{E}^v(\tilde{b})$ as follows:

$$\tilde{E}^v(\tilde{a}) = \frac{v(\lambda c_1 + c_2 + (1-\lambda)c_3)}{2} + \frac{(1-v)(2\lambda c_1 + c_2 + 2(1-\lambda)c_3)}{3}, \tag{2.26}$$

$$\tilde{E}^v(\tilde{b}) = \frac{v(\lambda c_1 + \lambda c_2 + (1-\lambda)c_3 + (1-\lambda)c_4)}{2} + \frac{(1-v)(2\lambda c_1 + \lambda c_2 + (1-\lambda)c_3 + 2(1-\lambda)c_4)}{3}. \tag{2.27}$$

Finally, fuzzy random variables have two kinds of uncertainty (randomness and fuzziness). In this paper, the randomness is estimated by the probabilistic expectation, and the fuzziness is estimated by the mean value $\tilde{E}^v(\cdot)$ with a possibility–necessity weight v and a pessimistic–optimistic index λ, where v means the decision maker's subjective evaluation weight between the possibility mean $\tilde{E}^{\mathrm{P}}(\cdot)$ and the necessity mean $\tilde{E}^{\mathrm{N}}(\cdot)$ and λ means the pessimistic degree in the writer's decision making. In the same idea as Theorem 2.2, we introduce the mean value of fuzzy random variables $\tilde{X}$ as follows:

$$E(\tilde{E}^v(\tilde{X})) := E\left(\int_0^1 (v + 2(1-v)(1-\alpha)) g(\tilde{X}_\alpha(\cdot))\, \mathrm{d}\alpha\right), \tag{2.28}$$

where $\tilde{X}_\alpha(\cdot) = [\tilde{X}_\alpha^-(\cdot), \tilde{X}_\alpha^+(\cdot)]$ and $E(\cdot)$ is the probabilistic expectation. In Section 3, we apply this evaluation method to American put option described by fuzzy random variables and we discuss the actual meaning of the mean value $\tilde{E}^v(\cdot)$.

3. American put option model with uncertainty

3.1. Fuzzification of Black–Scholes financial engineering model

In this section, by using fuzzy random variables, we introduce American put option model with uncertainty on the basis of Yoshida [19,20], which have studied an European and American options model with fuzzy goals. We discuss fuzzy prices of the American option, and in the next section we estimate them by the probabilistic expectation and

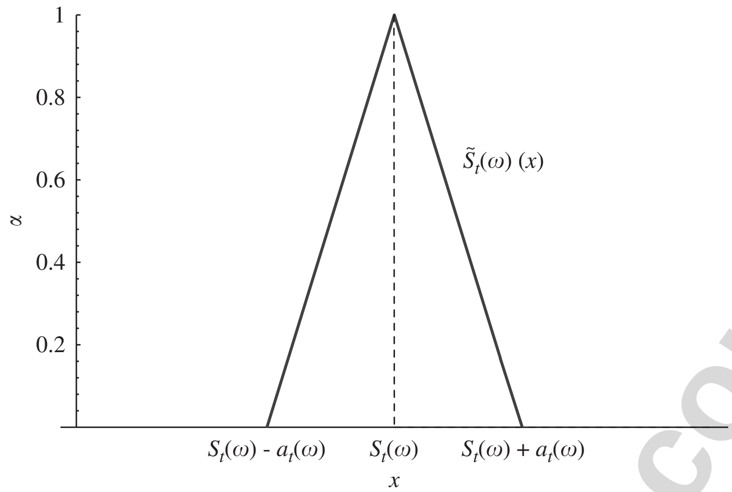

Fig. 3. Fuzzy random variable $\tilde{S}_t(\omega)(x)$.

the mean values $\tilde{E}^{\nu}(\cdot)$. First we describe some notations regarding bond price processes, stock price processes with fuzziness and American put option where there is no arbitrage opportunities [4,8]. Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a σ-field of Ω and P is a non-atomic probability measure. $\mathbb{R}$ denotes the set of all real numbers. Let μ be the *appreciation rate* and let σ be the *volatility* ($\mu \in \mathbb{R}$, $\sigma > 0$). Let $\{B_t\}_{t \geqslant 0}$ be a standard Brownian motion on $(\Omega, \mathcal{M}, P)$. $\{\mathcal{M}_t\}_{t \geqslant 0}$ denotes a family of nondecreasing right-continuous complete sub-σ-fields of $\mathcal{M}$ such that $\mathcal{M}_t$ is generated by $B_s (0 \leqslant s \leqslant t)$. We consider two assets, a bond price $\{R_t\}_{t \geqslant 0}$ and a stock price $\{S_t\}_{t \geqslant 0}$, where the bond price process $\{R_t\}_{t \geqslant 0}$ is riskless and the stock price process $\{S_t\}_{t \geqslant 0}$ is risky. Let r $(r \geqslant 0)$ be the instantaneous interest rate, i.e. *interest factor*, on a bond. Let a *bond price process* $\{R_t\}_{t \geqslant 0}$ be

$$R_t = \mathrm{e}^{rt}, \quad t \geqslant 0. \tag{3.1}$$

Let a *stock price process* $\{S_t\}_{t \geqslant 0}$ satisfy the log-normal stochastic differential equation: S_0 is a positive constant, and

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}B_t, \quad t \geqslant 0. \tag{3.2}$$

It is known [4] that there exists an equivalent probability measure Q such that $\{S_t/R_t\}_{t \geqslant 0}$ is a martingale under Q, by setting $\mathrm{d}Q/\mathrm{d}P|_{\mathcal{M}_t} = \exp(((r-\mu)/\sigma)B_t - \frac{1}{2}((r-\mu)/\sigma)^2 t)$, $t \geqslant 0$. Under Q, $W_t := B_t - ((r-\mu)/\sigma)t$ is a standard Brownian motion and it holds that $\mathrm{d}S_t = rS_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t$. By Ito's formula, we have

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t\right), \quad t \geqslant 0. \tag{3.3}$$

Next, we consider American put option with fuzzy prices and we discuss its properties. Let $\{a_t\}_{t \geqslant 0}$ be an $\mathcal{M}_t$-adapted stochastic process such that the map $t \mapsto a_t(\omega)$ is continuous on $[0, \infty)$ and $0 < a_t(\omega) \leqslant S_t(\omega)$ for almost all $\omega \in \Omega$. We give a *fuzzy stochastic process of the stock price process* $\{\tilde{S}_t\}_{t \geqslant 0}$ by the following fuzzy random variables:

$$\tilde{S}_t(\omega)(x) := L((x - S_t(\omega))/a_t(\omega)) \tag{3.4}$$

for $t \geqslant 0$, $\omega \in \Omega$ and $x \in \mathbb{R}$, where $L(x) := \max\{1 - |x|, 0\}$ $(x \in \mathbb{R})$ is the triangle type shape function Fig. 3 and $\{S_t\}_{t \geqslant 0}$ is defined by (3.2). Hence, $a_t(\omega)$ is a spread of triangular fuzzy numbers $\tilde{S}_t(\omega)$ and corresponds to the amount of fuzziness in the process. The α-cuts of (3.4) are

$$\tilde{S}_{t,\alpha}(\omega) = [\tilde{S}^-_{t,\alpha}(\omega), \tilde{S}^+_{t,\alpha}(\omega)] = [S_t(\omega) - (1-\alpha)a_t(\omega), S_t(\omega) + (1-\alpha)a_t(\omega)]. \tag{3.5}$$

Let K $(K > 0)$ be a *strike price*. We define the fuzzy price process by a fuzzy stochastic process $\{\tilde{P}_t\}_{t \geqslant 0}$ [19,20]:

$$\tilde{P}_t(\omega) := \mathrm{e}^{-rt}(1_{\{K\}} - \tilde{S}_t(\omega)) \vee 1_{\{0\}} \quad \text{for } t \geqslant 0, \quad \omega \in \Omega, \tag{3.6}$$

where $\vee$ is given by (3.1), and $1_{\{K\}}$ and $1_{\{0\}}$ denote the crisp numbers K and zero, respectively. Now we analyze (3.6) by α-cuts technique of fuzzy numbers. The α-cuts of (3.6) are

$$\tilde{P}_{t,\alpha}(\omega) = [\tilde{P}^-_{t,\alpha}(\omega), \tilde{P}^+_{t,\alpha}(\omega)] := [\mathrm{e}^{-rt} \max\{K - \tilde{S}^+_{t,\alpha}(\omega), 0\}, \mathrm{e}^{-rt} \max\{K - \tilde{S}^-_{t,\alpha}(\omega), 0\}], \tag{3.7}$$

and we obtain $E(\max_{t \geqslant 0} \sup_{\alpha \in [0,1]} \tilde{P}^+_{t,\alpha}) \leqslant K < \infty$, where $E(\cdot)$ is the expectation with respect to some risk-neutral equivalent martingale measure [4,8].

3.2. Estimation of American put option under uncertainty by evaluation measures

In this section, we estimate fuzzy prices $\tilde{P}_t$ by the probabilistic expectation and the mean values $\tilde{E}^\nu(\cdot)$ which introduced in Section 2. We deal with a model with an *expiration date T*. A map $\tau : \Omega \mapsto [0, T]$ is called a *stopping time* if $\{\omega \in \Omega \mid \tau(\omega) \leqslant t\} \in \mathcal{M}_t$ for all $t \in [0, T]$. An *exercise time* in American put option is given by a stopping time τ with values in $[0, T]$. For an exercise time τ from (3.6), we define $\tilde{P}_\tau(\omega) := \tilde{P}_t(\omega)$ if $\tau(\omega) = t$ for $t \in [0, T]$ and $\omega \in \Omega$. Then $\tilde{P}_\tau$ is a fuzzy random variable. Now we apply the estimation method (2.28) to American put option $\tilde{P}_\tau$. Let $\nu \in [0, 1]$ and let τ be an exercise time. From (2.28) and (3.7), the mean value of the fuzzy number $\tilde{P}_\tau(\omega)$ is given by

$$\boldsymbol{P}(\tau) := E(\tilde{E}^\nu(\tilde{P}_\tau)(\cdot)) = E\left(\int_0^1 (\nu + 2(1-\nu)(1-\alpha)) g(\tilde{P}_{\tau,\alpha}(\cdot))\, \mathrm{d}\alpha\right), \tag{3.8}$$

where $\tilde{P}_{\tau,\alpha}(\cdot)$ is the α-cut of $\tilde{P}_\tau(\cdot)$ (see (3.7)). Then $\boldsymbol{P}(\tau)$ means an estimation of the fuzzy expected price of American put option when τ is an exercise time. On the other hand, for an exercise time τ, the probabilistic expectation of the fuzzy random variable $\tilde{P}_\tau$ is a fuzzy number (see (2.3))

$$E(\tilde{P}_\tau)(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{E(\tilde{P}_\tau)_\alpha}(x)\}, \quad x \in \mathbb{R}, \tag{3.9}$$

where $E(\tilde{P}_\tau)_\alpha = [\int_\Omega \tilde{P}^-_{\tau,\alpha}(\omega)\, \mathrm{d}P(\omega), \int_\Omega \tilde{P}^+_{\tau,\alpha}(\omega)\, \mathrm{d}P(\omega)]$. Thus, by (3.6) we have another estimation regarding the fuzzy price process $\{\tilde{P}_t\}_{t \in [0,T]}$ of American put option at time τ:

$$\tilde{E}^\nu(E(\tilde{P}_\tau)) = \int_0^1 (\nu + 2(1-\nu)(1-\alpha)) g(E(\tilde{P}_\tau)_\alpha)\, \mathrm{d}\alpha. \tag{3.10}$$

However, the following lemma, which is trivial by Fubini's theorem, implies that (3.10) coincides with (3.8).

Lemma 3.1. *Let τ be an exercise time. Then it holds that $\boldsymbol{P}(\tau) = \tilde{E}^\nu(E(\tilde{P}_\tau))$. Namely,*

$$E\left(\int_0^1 (\nu + 2(1-\nu)(1-\alpha)) g(\tilde{P}_{\tau,\alpha}(\cdot))\, \mathrm{d}\alpha\right) = \int_0^1 (\nu + 2(1-\nu)(1-\alpha)) g(E(\tilde{P}_\tau)_\alpha)\, \mathrm{d}\alpha.$$

In American put option, we maximize the expected prices $\boldsymbol{P}(\tau)$ by exercise times τ. Put the *optimal expected price* by

$$\boldsymbol{V} := \sup_{\tau : \tau \leqslant T} \boldsymbol{P}(\tau). \tag{3.11}$$

In Section 3.3, this paper discusses the following optimal stopping problem regarding American put option with fuzziness.

Problem P. Find stopping times $\tau^* (\tau^* \leqslant T)$ and the optimal expected price $\boldsymbol{V}$ such that

$$\boldsymbol{P}(\tau^*) = \boldsymbol{V}. \tag{3.12}$$

Then, τ^* are called *optimal exercise times*.

3.3. The mean values of the optimal expected price and the optimal exercise time

In this section, we discuss the optimal fuzzy price $\boldsymbol{V}$ and the optimal exercise time τ^* by dynamic programming approach [21,19]. Now we introduce a natural assumption from Yoshida [20] to reduce the computation complexity. The economic meaning of Assumption S has been discussed in [20].

Assumption S. The stochastic process $\{a_t\}_{t\in[0,T]}$ is represented by

$$a_t(\omega) := cS_t(\omega), \quad t \in [0, T], \quad \omega \in \Omega, \tag{3.13}$$

where c is a constant satisfying $0 < c < 1$.

By putting $b(\alpha) := 1 \pm (1-\alpha)c$ ($\alpha \in [0,1]$), from (3.5) and (3.13) we have

$$\tilde{S}^{\pm}_{t,\alpha}(\omega) = S_t(\omega) \pm (1-\alpha)a_t(\omega) = b(\alpha)S_t(\omega), \quad \omega \in \Omega \tag{3.14}$$

for $t \in [0, T]$ and $\alpha \in [0, 1]$. Then, from (3.6) we obtain the fuzzy price process:

$$\tilde{P}^{\pm}_{\tau,\alpha}(\omega) = \mathrm{e}^{-r\tau(\omega)} \max\{K - b^{\mp}(\alpha)S_\tau(\omega), 0\}, \quad \omega \in \Omega. \tag{3.15}$$

For an exercise time τ, we define random variables

$$\Pi_\tau(\omega) := \tilde{E}^\nu(\tilde{P}_\tau)(\omega) = \int_0^1 (\nu + 2(1-\nu)(1-\alpha))g(\tilde{P}_{\tau,\alpha}(\omega))\,\mathrm{d}\alpha, \quad \omega \in \Omega. \tag{3.16}$$

Then we can easily calculate the following lemma.

Lemma 3.2. *Let $\nu(\in [0,1])$ be a possibility–necessity weight and let $\lambda(\in [0,1])$ be a pessimistic–optimistic index. Let τ be a stopping time satisfying $\tau \leqslant T$. Then, there exists a function f on $(0,\infty)$ such that*

$$\Pi_\tau(\omega) = \mathrm{e}^{-r\tau(\omega)} f(S_\tau(\omega)), \quad \omega \in \Omega, \tag{3.17}$$

where f is given by

$$f(y) := \begin{cases} K - y - cy(4-\nu)(2\lambda-1)/6 + \lambda\varphi^1(y) & \text{if } 0 < y < K, \\ (1-\lambda)\varphi^2(y) & \text{if } y \geqslant K, \end{cases} \tag{3.18}$$

with the following functions φ^1, φ^2 on $(0,\infty)$:

$$\varphi^1(y) := \frac{2-\nu}{2cy}\max\{0, -K+y+cy\}^2 - \frac{1-\nu}{3(cy)^2}\max\{0, -K+y+cy\}^3, \quad y > 0,$$

$$\varphi^2(y) := \frac{2-\nu}{2cy}\max\{0, K-y+cy\}^2 - \frac{1-\nu}{3(cy)^2}\max\{0, K-y+cy\}^3, \quad y > 0.$$

Finally we give an optimality characterization for the optimal expected price $\boldsymbol{V}$ in (3.12). To characterize the optimal fuzzy price $\boldsymbol{V}$, we put

$$\boldsymbol{V}(y,t) = \sup_{\tau:\text{ stopping times},\ t\leqslant\tau\leqslant T} E(\mathrm{e}^{-rt}\Pi_\tau | S_t = y) \tag{3.19}$$

for $t \in [0, T]$ and an initial stock price y ($y > 0$). Then we note that $\boldsymbol{V} = \boldsymbol{V}(y, 0)$. Define a differential operator

$$\mathcal{L} := \frac{1}{2}\sigma^2 y^2 \frac{\partial^2}{\partial y^2} + ry\frac{\partial}{\partial y} + \frac{\partial}{\partial t}, \quad (y,t) \in [0,\infty) \times [0,T). \tag{3.20}$$

Since the stock price process $\{S_t\}_{t\in[0,T]}$ is Markov by (3.3), we obtain the following optimality equations [19].

Theorem 3.1 (*Free boundary problem*). *The fuzzy price* $\boldsymbol{V}(y, t)$ *satisfies the following equations*:

$$\mathcal{L}(\mathrm{e}^{-rt}\boldsymbol{V}(y,t)) \leqslant 0 \quad \text{on } [0,\infty)\times[0,T) \text{ in the sense of Schwartz distributions}, \tag{3.21}$$

$$\mathcal{L}(\mathrm{e}^{-rt}\boldsymbol{V}(y,t)) = 0 \quad \text{on } D, \tag{3.22}$$

$$\boldsymbol{V}(y,t) \geqslant f(y) \quad \text{on } [0,\infty)\times[0,T), \tag{3.23}$$

$$\boldsymbol{V}(y,T) = f(y) \quad \text{on } [0,\infty), \tag{3.24}$$

where $D := \{(y,t) \in [0,\infty)\times[0,T) \mid \boldsymbol{V}(y,t) > f(y)\}$. *The corresponding optimal exercise time is*

$$\begin{aligned}\tau^*(\omega) &= \inf\{t \in [0,T] \mid \boldsymbol{V}(S_t(\omega),t) = f(S_t(\omega))\}\\ &= \min\{\inf\{t \geqslant 0 \mid (S_t(\omega),t) \in B\}, T\},\end{aligned} \tag{3.25}$$

$\omega \in \Omega$, *where* $B := ([0,\infty)\times[0,T]) \setminus D$.

In Theorem 3.1, the optimal exercise time $\tau^*(\omega)$ the first hitting time of the *stopping region* B by the stock price process $\{S_t\}_{t \geqslant 0}$. Hence, (3.23) means that f is the lower bound of the optimal fuzzy price $\boldsymbol{V}$. Properties (3.21) and (3.22) are called *superharmonic* and *harmonic*, respectively, in theory of Markov processes [14]. Eq. (3.21) means that the optimal fuzzy price $\boldsymbol{V}$ is superharmonic over all the state space, and (3.22) means that the optimal fuzzy price $\boldsymbol{V}$ is harmonic on D, which is called *continuation region* and which is also outside the stopping region B.

3.4. A numerical example

Now we give a numerical example to illustrate our idea in the previous sections.

Example 3.1. Now we give a numerical example to illustrate our idea in Sections 2 and 3. Put an expiration date $T = 7$, an interest rate of a bond $r = 0.05$, a fuzzy factor $c = 0.05$, an initial stock price $y = 25$ and a strike price $K = 30$. If we take a λ-weighting function g with $\lambda = \frac{1}{2}$ in (2.4), Fig. 4 shows the corresponding optimal expected price $\boldsymbol{V} = \boldsymbol{V}(y, 0)$ for each initial stock price y in the case of the possibility mean, and the corresponding optimal exercise time is given by (3.25).

The optimal expected price $\boldsymbol{V}(y) = \boldsymbol{V}(y, 0)$ at initial stock price $y = 25$ changes corresponding to the possibility–necessity index v $(0 \leqslant v \leqslant 1)$ in (2.23) and the pessimistic–optimistic index λ $(0 \leqslant \lambda \leqslant 1)$ in the λ-weighting function (2.4), where v means the decision maker's subjective evaluation weight between the possibility mean $\tilde{E}^{\mathrm{P}}(\cdot)$ and the necessity mean $\tilde{E}^{\mathrm{N}}(\cdot)$ and while λ means the pessimistic degree in the writer's decision making (see Table 1).

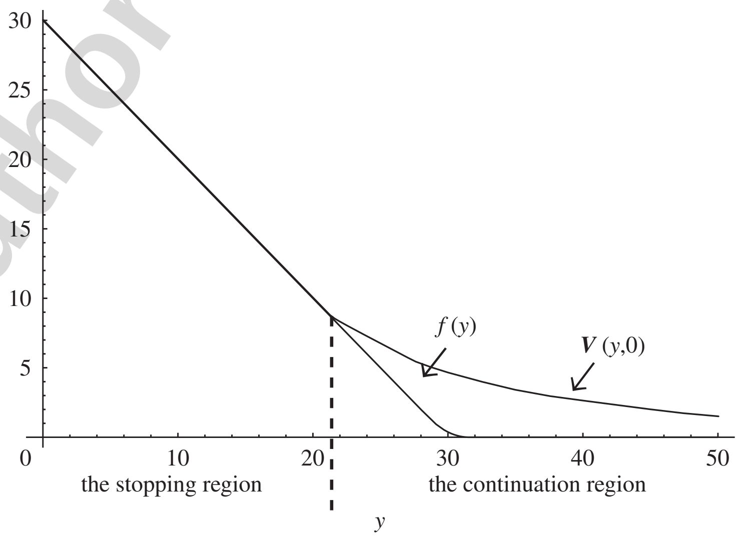

Fig. 4. The optimal expected price $\boldsymbol{V}$ and the function $f(y)$. ($v = 1$, $\lambda = \frac{1}{2}$, $c = 0.05$, $K = 30$, y: initial stock price).

Table 1
The optimal expected price $\boldsymbol{V}(y) = \boldsymbol{V}(y, 0)$ ($\lambda = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$; $c = 0.05$; $K = 30$; $y = 25$)

	$\lambda = \frac{1}{3}$	$\lambda = \frac{1}{2}$	$\lambda = \frac{2}{3}$
The possibility mean $\tilde{E}^{\mathrm{P}}$, $v = 1$	6.52454	6.41025	6.30111
The necessity mean $\tilde{E}^{\mathrm{N}}$, $v = 0$	6.56457	6.41025	6.26727
The credibility mean $\tilde{E}^{\mathrm{C}}$, $v = \frac{2}{3}$	6.53789	6.41025	6.28988

The presented method of the mean value $\tilde{E}^{v}(\cdot)$ is applicable to decision making problem. Then, the a possibility–necessity weight v and a pessimistic–optimistic index λ are simple and effective parameters in applications. The decision maker can choose strategies τ, v and λ, by taking into account of the results in Table 1 which are changing corresponding to the controllable weight v and index λ.

4. Concluding remarks

There are many defuzzification methods have been known and studied [16]. However, in actual applications with fuzziness, we need to choose one of them to be fit for use. In this paper, we have discussed what kinds of evaluation methods in form (1.1) are applicable for decision making problems. We have found the mean value $\tilde{E}^{v}(\cdot)$ is one of simple and useful methods. The mean value $\tilde{E}^{v}(\cdot)$ has been examined in the American put option model under randomness and fuzziness, and we have seen the results are reasonable and valid in applications.

The mean value by evaluation measures for fuzzy numbers, fuzzy random variables and fuzzy stochastic processes is valid for various decision making modeling. We hope that the presented method will be examined in the other types of decision making problems and it will be improved from application aspects.

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