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Lusin's theorem on fuzzy measure spaces☆

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Abstract

In this paper, we show that weakly null-additive fuzzy measures on metric spaces possess regularity. Lusin's theorem, which is well-known in classical measure theory, is generalized to fuzzy measure space by using the regularity and weakly null-additivity. A version of Egoroff's theorem for the fuzzy measure defined on metric spaces is given. An application of Lusin's theorem to approximation in the mean of measurable function on fuzzy measure spaces is presented.

Keywords: Non-additive measures; Fuzzy measure; Weakly null-additivity; Regularity; Lusin's theorem; Approximation in the mean

1. Introduction

The well-known Lusin's theorem in classical measure theory is very important and useful for discussing the continuity and the approximation of measurable function on metric spaces [8]. Song and Li [9] investigated the regularity of null-additive fuzzy measure on metric spaces and showed Lusin's theorem on fuzzy measure space under the null-additivity condition. These improved the previous results of Wu and Ha [11]. Further discussions for the regularity of fuzzy measures were made by Pap [7], Jiang et al. [2,3], and Wu and Wu [12].

In this paper, we shall use a weaker structural characteristic of fuzzy measures—weakly null-additivity—to discuss the above-mentioned problems. Our goal is to prove the Lusin's theorem on fuzzy measure space under the weakly null-additivity condition. The paper is organized as follows. In Section 2, a necessary and sufficient condition of weakly null-additivity of fuzzy measure is presented in Lemma 1. It constitutes the essential position in our discussion here. In Section 3,

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we prove that the weakly null-additivity implies regularity for a finite fuzzy measure defined on metric space. In Section 4, a version of Egoroff's theorem for the fuzzy measure defined on metric spaces is given. In Section 5, by using the regularity and Egoroff's theorem we shall prove that the well-known Lusin's theorem remains valid for those weakly null-additive fuzzy measures defined on a metric space. These are improvements and generalizations of the earlier results of Song and Li [9]. Lastly , as an application of Lusin's theorem, we shall describe the mean approximations of measurable function by continuous functions, or by polynomials, or by step functions in the sense of Sugeno and of Choquet integral, respectively.

2. Preliminaries

Throughout this paper, we suppose that (X,ρ) is a metric space, and that $\mathscr{O}$ and $\mathscr{C}$ are the classes of all open and closed sets in (X,ρ), respectively, and $\mathscr{B}$ is Borel σ-algebra on X, i.e., it is the smallest σ-algebra containing $\mathscr{O}$ [1]. Unless stated otherwise all the subsets mentioned are supposed to belong to $\mathscr{B}$.

A set function $\mu:\mathscr{B}\to[0,+\infty]$ is called *a fuzzy measure*, if it satisfies the following properties:
(FM1) $\mu(\emptyset)=0$;
(FM2) $A\subset B$ implies $\mu(A)\leqslant\mu(B)$ (monotonicity);
(FM3) $A_1\subset A_2\subset\cdots$ implies $\lim_{n\to\infty}\mu(A_n)=\mu(\bigcup_{n=1}^{\infty}A_n)$ (continuity from below);
(FM4) $A_1\supset A_2\supset\cdots$, and there exists n_0 with $\mu(A_{n_0})<+\infty$ imply

$$\lim_{n\to\infty}\mu(A_n)=\mu\left(\bigcap_{n=1}^{\infty}A_n\right)\quad\text{(continuity from above).}$$

In this paper, we always assume that μ is a finite fuzzy measure on $\mathscr{B}$, i.e., $\mu(X)<\infty$.

A fuzzy measure μ is called *null-additive*, if $\mu(E\cup F)=\mu(E)$ whenever $E,F\in\mathscr{B}$ and $\mu(F)=0$; *autocontinuous from above*, if $\lim_{n\to+\infty}\mu(E\cup F_n)=\mu(E)$ whenever $E\in\mathscr{B},\{F_n\}\subset\mathscr{B}$, and $\lim_{n\to+\infty}\mu(F_n)=0$ [10].

Definition 1 (Wang and Klir [10]). μ is called weakly null-additive, if for any $E,F\in\mathscr{B}$,

$$\mu(E)=\mu(F)=0\Rightarrow\mu(E\cup F)=0.$$

Obviously, the null-additivity of μ implies weakly null-additivity. If μ is autocontinuous from above, then it is null-additive [10], and hence it is weakly null-additive.

Lemma 1. *μ is weakly null-additive if and only if for any $\varepsilon>0$ and any double sequence $\{A_n^{(k)}\mid n\geqslant 1,k\geqslant 1\}\subset\mathscr{B}$ satisfying $A_n^{(k)}\searrow D_n$ $(k\to\infty)$, $\mu(D_n)=0$, $n=1,2,\ldots,$ there exists a subsequence $\{A_n^{(k_n)}\}$ of $\{A_n^{(k)}\mid n\geqslant 1,k\geqslant 1\}$ such that*

$$\mu\left(\bigcup_{n=1}^{\infty}A_n^{(k_n)}\right)<\varepsilon\quad(k_1<k_2<\cdots).$$

Proof. *Necessity*: Suppose μ is weakly null-additive. Write $D=\bigcup_{n=1}^{\infty} D_n$, then by using the continuity from below of μ, we have $\mu(D)=0$ and $D_n \subset D$ $(n=1,2,\ldots)$. Since for any fixed $n=1,2,\ldots,$ $A_n^{(k)} \searrow D_n$ as $k \to \infty$, we have

$$A_n^{(k)} \cup D \searrow D_n \cup D = D \quad (k \to \infty)$$

for any fixed $n=1,2,\ldots$. For given $\varepsilon>0$, using the continuity from above of fuzzy measures, we have $\lim_{k\to+\infty} \mu(A_1^{(k)} \cup D)=\mu(D)=0$, therefore there exists k_1 such that $\mu(A_1^{(k_1)} \cup D)<\frac{\varepsilon}{2}$; For this k_1,

$$(A_1^{(k_1)} \cup A_2^{(k)}) \cup D \searrow (A_1^{(k_1)} \cup D_2) \cup D = A_1^{(k_1)} \cup D,$$

as $k \to \infty$. Therefore it follows, from the continuity from above of μ, that

$$\lim_{k\to+\infty} \mu((A_1^{(k_1)} \cup A_2^{(k)}) \cup D) = \mu(A_1^{(k_1)} \cup D).$$

Thus there exists k_2 $(>k_1)$, such that

$$\mu((A_1^{(k_1)} \cup A_2^{(k_2)}) \cup D) < \frac{\varepsilon}{2}.$$

Generally, there exist $k_1, k_2, \ldots, k_m$, such that

$$\mu((A_1^{(k_1)} \cup A_2^{(k_2)} \cup \cdots \cup A_m^{(k_m)}) \cup D) < \frac{\varepsilon}{2}.$$

Hence we obtain a sequence $\{k_n\}_{n=1}^{\infty}$ of numbers and a sequence $\{A_n^{(k_n)}\}_{n=1}^{\infty}$ of sets. By using the monotonicity and the continuity from below of μ, we have

$$\mu\left(\bigcup_{n=1}^{+\infty} A_n^{(k_n)}\right) \leqslant \mu\left(\left(\bigcup_{n=1}^{+\infty} A_n^{(k_n)}\right) \cup D\right) \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

Sufficiency: Let $E,F \in \mathscr{B}$ and $\mu(E)=\mu(F)=0$. We define a double sequence $\{A_n^{(k)} \mid n \geqslant 1, k \geqslant 1\}$ of sets satisfying the following conditions: $A_1^{(k)}=E$, $A_2^{(k)}=F$, $A_3^{(k)}=A_4^{(k)}=\cdots=\emptyset$, $\forall k \geqslant 1$ and let $D_1=E$, $D_2=F$, $D_n=\emptyset$, $\forall n \geqslant 3$. Then for any $\varepsilon>0$, by hypothesis, there exists a subsequence $\{A_n^{(k_n)}\}$ such that $\mu(\bigcup_{n=1}^{\infty} A_n^{(k_n)})<\varepsilon$, that is $\mu(E \cup F)<\varepsilon$. Therefore $\mu(E \cup F)=0$. This shows that μ is weakly null-additive. □

Remark 1. A weakly null-additive fuzzy measure may not be null-additive. In the following, a simple example indicates that the weakly null-additivity of fuzzy measure is really weaker than null-additivity and autocontinuity from above.

Example 1. Let $X=\{a,b\}$ and (X,ρ) be a metric space. Then $\mathscr{B}=\wp(X)$. Put

$$\mu(E) = \begin{cases} 1 & \text{if } E = X, \\ \frac{1}{2} & \text{if } E = \{b\}, \\ 0 & \text{if } E = \{a\} \text{ or } E = \emptyset. \end{cases}$$

Then μ is a fuzzy measure with weakly null-additivity. However μ is not null-additive and hence it is not autocontinuous from above either. In fact, $\mu(\{a\})=0$, but $\mu(\{a\}\cup\{b\})=1\neq\mu(\{b\})$.

3. Regularity of fuzzy measure

It is known that every probability measure P on a metric space is regular. Now we prove that this property is also enjoyed by those fuzzy measures with weakly null-additivity.

Definition 2 (Wu and Ha [11]). μ is called regular if, for every $A\in\mathscr{B}$ and $\varepsilon>0$, there exist a closed set F_ε and an open set G_ε of X, such that $F_\varepsilon\subset A\subset G_\varepsilon$ and $\mu(G_\varepsilon-F_\varepsilon)<\varepsilon$.

Theorem 1. *If μ is weakly null-additive, then μ is regular.*

Proof. Let $\mathscr{E}$ be the class of all set $E\in\mathscr{B}$ such that for any $\varepsilon>0$, there exist a closed set F_ε and an open set G_ε satisfying

$$F_\varepsilon\subset E\subset G_\varepsilon \quad\text{and}\quad \mu(G_\varepsilon-F_\varepsilon)<\varepsilon.$$

To prove the theorem, it is sufficient to show that $\mathscr{B}\subset\mathscr{E}$.

It is easy to verify that $\emptyset\in\mathscr{E}, X\in\mathscr{E}$ and $\mathscr{E}$ is closed under the formation of complements.

We shall now prove that $\mathscr{E}$ is also closed under the formation of countable unions. Let $\{E_n\}\subset\mathscr{E}$ and $\varepsilon>0$ be given. From the definition of $\mathscr{E}$ and $E_n\in\mathscr{E}$, we know that for every $n=1,2,\ldots,$ there exist a sequence $\{G_n^{(k)}\}_{k=1}^\infty$ of open sets and a sequence $\{F_n^{(k)}\}_{k=1}^\infty$ of closed sets such that

$$F_n^{(k)}\subset E_n\subset G_n^{(k)} \quad\text{and}\quad \mu(G_n^{(k)}-F_n^{(k)})<\frac{1}{k}$$

for $k=1,2,\ldots$. Without loss of generality, we can assume that for fixed $n=1,2,\ldots,$ as $k\to\infty$, $\{G_n^{(k)}\}_{k=1}^\infty$ is decreasing and $\{F_n^{(k)}\}_{k=1}^\infty$ is increasing. Therefore, for any fixed $n=1,2,\ldots,$ $\{G_n^{(k)}-F_n^{(k)}\}_{k=1}^\infty$ is a decreasing sequence of sets with respect to k and as $k\to\infty$

$$G_n^{(k)}-F_n^{(k)}\searrow\bigcap_{k=1}^\infty(G_n^{(k)}-F_n^{(k)}).$$

Denote $D_n=\bigcap_{k=1}^\infty(G_n^{(k)}-F_n^{(k)})$, then $G_n^{(k)}-F_n^{(k)}\searrow D_n$ as $k\to\infty$ and noting that $\mu(D_n)\leqslant\mu(G_n^{(k)}-F_n^{(k)})<\frac{1}{k}$, $k=1,2,\ldots,$ we have $\mu(D_n)=0$ $(n=1,2,\ldots)$. Applying Lemma 1 to the double sequence $\{G_n^{(k)}-F_n^{(k)}\}$ and the sequence $\{D_n\}_{n=1}^\infty$ of sets, then for any given $\varepsilon>0$, there exists a subsequence $\{G_n^{(k_n)}-F_n^{(k_n)}\}$ of $\{G_n^{(k)}-F_n^{(k)}\}$ such that

$$\mu\left(\bigcup_{n=1}^\infty(G_n^{(k_n)}-F_n^{(k_n)})\right)<\varepsilon.$$

Since

$$\bigcup_{n=1}^{\infty} G_n^{(k_n)} - \bigcup_{n=1}^{N} F_n^{(k_n)} \searrow \bigcup_{n=1}^{\infty} G_n^{(k_n)} - \bigcup_{n=1}^{\infty} F_n^{(k_n)}$$

as $N \to \infty$, and noting that $\bigcup_{n=1}^{\infty} G_n^{(k_n)} - \bigcup_{n=1}^{\infty} F_n^{(k_n)} \subset \bigcup_{n=1}^{\infty} (G_n^{(k_n)} - F_n^{(k_n)})$, by the continuity from above and monotonicity of μ, we have

$$\lim_{N \to +\infty} \mu\left(\bigcup_{n=1}^{\infty} G_n^{(k_n)} - \bigcup_{n=1}^{N} F_n^{(k_n)}\right) = \mu\left(\bigcup_{n=1}^{\infty} G_n^{(k_n)} - \bigcup_{n=1}^{\infty} F_n^{(k_n)}\right) < \varepsilon.$$

Therefore, there exists N_0 such that

$$\mu\left(\bigcup_{n=1}^{\infty} G_n^{(k_n)} - \bigcup_{n=1}^{N_0} F_n^{(k_n)}\right) < \varepsilon.$$

Denote

$$G_\varepsilon = \bigcup_{n=1}^{\infty} G_n^{(k_n)} \quad \text{and} \quad F_\varepsilon = \bigcup_{n=1}^{N_0} F_n^{(k_n)}$$

then G_ε is an open set, F_ε is a closed set and

$$F_\varepsilon \subset \bigcup_{n=1}^{\infty} E_n \subset G_\varepsilon \quad \text{and} \quad \mu(G_\varepsilon - F_\varepsilon) < \varepsilon.$$

Therefore $\bigcup_{n=1}^{\infty} E_n \in \mathscr{E}$. Thus we proved that $\mathscr{E}$ is a σ-algebra.

To complete the proof, it is enough to show that $\mathscr{E}$ contains all the open sets of X. For any closed set $F \in \mathscr{C}$, we denote $G_m = \{x \in X\colon\ \rho(x, F) < 1/m\}$ $(m = 1, 2, \ldots)$, where $\rho(x, F)$ is the distance of the set F from the point X, i.e. $\rho(x, F) = \inf\{\rho(x, y)\colon y \in F\}$, then for every $m = 1, 2, \ldots,$ G_m is open set. Noting that F is a closed set, we know $G_m \searrow F$ $(m \to \infty)$. It is follows from $G_m - F \searrow \emptyset$ $(m \to \infty)$ that $\lim_{m \to \infty} \mu(G_m - F) = 0$. Thus $\mathscr{C} \subset \mathscr{E}$. Since $\mathscr{E}$ is closed under the formation of complements, we have $\mathscr{O} \subset \mathscr{E}$. This shows that $\mathscr{E}$ is a σ-algebra containing $\mathscr{O}$. Therefore $\mathscr{B} \subset \mathscr{E}$. □

Corollary 1. *If μ is weakly null-additive, then for any $E \in \mathscr{B}$, there exist a sequence $\{F^{(k)}\}_{k=1}^{\infty}$ of closed sets and a sequence $\{G^{(k)}\}_{k=1}^{\infty}$ of open sets such that for every $k = 1, 2, \ldots,$ $F^{(k)} \subset E \subset G^{(k)}$,*

$$\mu(G^{(k)} - E) < \frac{1}{k} \quad \textit{and} \quad \mu(E - F^{(k)}) < \frac{1}{k}.$$

Note 1: Observe that we can assume in Corollary 1 that the sequence $\{F^{(k)}\}_{k=1}^{\infty}$ is increasing in k and the sequence $\{G^{(k)}\}_{k=1}^{\infty}$ is decreasing in k.

4. Egoroff's theorem

Egoroff's theorem on fuzzy measure spaces was discussed in [4–6,10]. Now we show a version of the Egoroff's theorem for the fuzzy measures defined on metric spaces. We assume that in this paper all functions considered are defined on X and are real-valued measurable with respect to $\mathscr{B}$. For a finite fuzzy measure μ on $\mathscr{B}$, we have obtained the following result [5]:

Theorem 2 (Egoroff's theorem). *If $\{f_n\}$ converges to f almost everywhere on X, then for any $\varepsilon > 0$ there exists $X_\varepsilon \in \mathscr{B}$ such that $\mu(X - X_\varepsilon) < \varepsilon$ and $\{f_n\}_n$ converges to f uniformly on X_ε.*

The following corollary gives an alternative form of Egoroff's theorem.

Corollary 2. *If $\{f_n\}$ converges to f almost everywhere on X, then there exists an increasing sequence $\{X_m\}_{m=1}^{\infty} \subset \mathscr{B}$ such that $\mu(X - \bigcup_{m=1}^{\infty} X_m) = 0$ and f_n converges to f on X_m uniformly for any fixed $m = 1, 2, \ldots$.*

When μ is a weakly null-additive fuzzy measure on metric space, we can obtain a slightly stronger conclusion:

Theorem 3. *Let μ be weakly null-additive fuzzy measure on $\mathscr{B}$. If $\{f_n\}$ converges to f almost everywhere on X, then for any $\varepsilon > 0$ there exists a closed subset $F_\varepsilon \in \mathscr{C}$ such that $\mu(X - F_\varepsilon) < \varepsilon$ and $\{f_n\}_n$ converges to f uniformly on F_ε.*

Proof. Since $\{f_n\}$ converges to f almost everywhere on X, by using Corollary 2 there exists an increasing sequence $\{X_m\}_{m=1}^{\infty} \subset \mathscr{B}$ such that f_n converges to f on X_m uniformly for any fixed $m = 1, 2, \ldots$ and $\mu(X - \bigcup_{m=1}^{\infty} X_m) = 0$. Denote $H = X - \bigcup_{m=1}^{\infty} X_m$, then $\mu(H) = 0$.

From Corollary 1, for every fixed X_m $(m = 1, 2, \ldots)$, there exists a sequence $\{F_m^{(k)}\}_{k=1}^{\infty}$ of closed sets satisfying $F_m^{(k)} \subset X_m$ and $\mu(X_m - F_m^{(k)}) < 1/k$ for any $k = 1, 2, \ldots$. Without loss of generality, we can assume that for fixed $m = 1, 2, \ldots$, $\{X_m - F_m^{(k)}\}_{k=1}^{\infty}$ is decreasing (as $k \to \infty$). Thus

$$X_m - F_m^{(k)} \searrow \bigcap_{k=1}^{\infty} (X_m - F_m^{(k)})$$

as $k \to \infty$. Write $D_m = (\bigcap_{k=1}^{\infty} (X_m - F_m^{(k)})) \cup H$ $(m = 1, 2, \ldots)$, then $(X_m - F_m^{(k)}) \cup H \searrow D_m$ as $k \to \infty$. Noting that for any $m = 1, 2, \ldots$, $\mu(\bigcap_{k=1}^{\infty} (X_m - F_m^{(k)})) = \lim_{k \to +\infty} \mu(X_m - F_m^{(k)}) = 0$, and by the weakly null-additivity of μ, we get $\mu(D_m) = \mu((\bigcap_{k=1}^{\infty} (X_m - F_m^{(k)})) \cup H) = 0$ $(m = 1, 2, \ldots)$. Applying Lemma 1 to the double sequence $\{(X_m - F_m^{(k)}) \cup H\}$ of sets and the sequence $\{D_m\}_{m=1}^{\infty}$ of sets , then for any $\varepsilon > 0$, there exists a subsequence $\{(X_m - F_m^{(k_m)}) \cup H\}$ of $\{(X_m - F_m^{(k)}) \cup H\}$, such that

$$\mu\left(\bigcup_{m=1}^{\infty} ((X_m - F_m^{(k_m)}) \cup H)\right) < \varepsilon.$$

Since $X-\bigcup_{m=1}^{\infty}F_m^{(k_m)}\subset\bigcup_{m=1}^{\infty}(X_m-F_m^{(k)})\cup H$, we have

$$\mu\left(X-\bigcup_{m=1}^{\infty}F_m^{(k_m)}\right)<\varepsilon.$$

On the other hand, from $X-\bigcup_{m=1}^{N}F_m^{(k_m)}\searrow X-\bigcup_{m=1}^{\infty}F_m^{(k_m)}$ as $N\to\infty$ and the continuity from above of μ, we have $\lim_{N\to+\infty}\mu(X-\bigcup_{m=1}^{N}F_m^{(k_m)})=\mu(X-\bigcup_{m=1}^{\infty}F_m^{(k_m)})<\varepsilon$. Therefore there exists N_0 such that $\mu(X-\bigcup_{m=1}^{N_0}F_m^{(k_m)})<\varepsilon$.

Denote $F_\varepsilon=\bigcup_{m=1}^{N_0}F_m^{(k_m)}$, then F_ε is a closed set, $\mu(X-F_\varepsilon)<\varepsilon$ and from $F_\varepsilon\subset\bigcup_{m=1}^{N_0}X_m$, we know that $\{f_n\}_n$ converges to f uniformly on F_ε. □

5. Lusin's theorem

In this section, we shall further generalize the well-known Lusin's theorem in classical measure theory to fuzzy measure space by using the results obtained in Sections 2–4.

Theorem 4 (Lusin's theorem). *Let μ be weakly null additive fuzzy measure on $\mathscr{B}$. If f is a real-valued measurable function on X, then, for every $\varepsilon>0$, there exists a closed subset $F_\varepsilon\in\mathscr{C}$ such that f is continuous on F_ε and $\mu(X-F_\varepsilon)<\varepsilon$.*

Proof. We prove the theorem stepwise in the following two situations.

(a) Suppose that f is a simple function, i.e. $f(x)=\sum_{n=1}^{s}c_k\chi_{E_n}(x)$ $(x\in X)$, where $\chi_{E_n}(x)$ is the characteristic function of the set E_n and $X=\bigcup_{n=1}^{s}E_n$ (a disjoint finite union). For every fixed E_n $(n=1,2,\ldots,s)$, by Corollary 1, there exists the sequence $\{F_n^{(k)}\}_{k=1}^{\infty}$ of closed sets such that

$$F_n^{(k)}\subset E_n \quad\text{and}\quad \mu(E_n-F_n^{(k)})<\frac{1}{k}$$

for any $k=1,2,\ldots$. We may assume that $\{F_n^{(k)}\}_{k=1}^{\infty}$ is increasing in k for each fixed n, without any loss of generality.

For any $\varepsilon>0$, applying Lemma 1 to the double sequence $\{E_n-F_n^{(k)}\}$ $(n=1,2,\ldots,s,k=1,2,\ldots)$ of sets, there exists a subsequence $\{E_n-F_n^{(k_n)}\}$ of $\{E_n-F_n^{(k)}\}$ such that

$$\mu\left(\bigcup_{n=1}^{s}(E_n-F_n^{(k_n)})\right)<\varepsilon.$$

Put $F_\varepsilon=\bigcup_{n=1}^{s}F_n^{(k_n)}$, then f is continuous on the closed subset F_ε of X, and

$$\mu(X-F_\varepsilon)\leqslant\mu\left(\bigcup_{n=1}^{s}E_n-\bigcup_{n=1}^{s}F_n^{(k_n)}\right)\leqslant\mu\left(\bigcup_{n=1}^{s}(E_n-F_n^{(k_n)})\right)<\varepsilon.$$

(b) Let f be a real-valued measurable function. Then there exists a sequence $\{\varphi_n(x)\}_{n=1}^{\infty}$ of simple functions such that $\varphi_n\to f$ $(n\to\infty)$ on X. By the result obtained in (a), for each simple function

φ_n and every $k=1,2,\ldots$, there exists closed set $X_n^{(k)}\subset X$ such that φ_n is continuous on $X_n^{(k)}$ and $\mu(X-X_n^{(k)})<\frac{1}{k}$ $(k=1,2,\ldots)$. There is no loss of generality in assuming the sequence $\{X_n^{(k)}\}_{k=1}^{\infty}$ of closed sets is increasing with respect to k for any fixed n (otherwise, we can take $\bigcup_{i=1}^{k} X_n^{(i)}$ instead of $X_n^{(k)}$ and noting that φ_n is a simple function, it remains continuous on $\bigcup_{i=1}^{k} X_n^{(i)}$). Therefore $X-X_n^{(k)}\searrow\bigcap_{k=1}^{\infty}(X-X_n^{(k)})$ as $k\to\infty$, and thus, we have

$$\mu\left(\bigcap_{k=1}^{\infty}(X-X_n^{(k)})\right)=\lim_{n\to+\infty}\mu(X-X_n^{(k)})=0 \quad (n=1,2,\ldots).$$

Now we consider the double sequence $\{X-X_n^{(k)}\mid n\geqslant 1,k\geqslant 1\}$ of sets. By using Lemma 1, for every m $(m=1,2,\ldots)$, we may take a subsequence $\{X-X_n^{(k_n^{(m)})}\}_{n=1}^{\infty}$ of $\{X-X_n^{(k)}\mid n\geqslant 1,k\geqslant 1\}$ such that

$$\mu\left(\bigcup_{n=1}^{\infty}(X-X_n^{(k_n^{(m)})})\right)<\frac{1}{m},$$

namely, $\mu(X-\bigcap_{n=1}^{\infty}X_n^{(k_n^{(m)})})<1/m$. Since the double sequence $\{X-X_n^{(k)}\mid n\geqslant 1,k\geqslant 1\}$ of sets is decreasing in k for fixed n, without any loss of generality, we can assume that for fixed $n (n=1,2,\ldots)$, $k_n^{(1)}<k_n^{(2)}<\cdots<k_n^{(m)}\ldots$. Write $H_m=\bigcap_{n=1}^{\infty}X_n^{(k_n^{(m)})}$ $(m=1,2,\ldots)$, then we obtain a sequence $\{H_m\}_{m=1}^{\infty}$ of closed sets satisfying $H_1\subset H_2\subset\cdots$ and $\mu(X-\bigcup_{m=1}^{\infty}H_m)=\lim_{n\to+\infty}\mu(X-H_m)=0$. Noting that φ_n is continuous on $X_n^{(k_n^{(m)})}$ and $H_m\subset X_n^{(k_n^{(m)})}$ $(n=1,2,\ldots)$, therefore for each H_m, φ_n is continuous on H_m for every $n=1,2,\ldots$.

On the other hand, since $\varphi_n\to f$ $(n\to\infty)$ on X, by Theorem 3, there exists an increasing sequence $\{X_m\}_{m=1}^{\infty}$ of closed sets satisfying $X-X_m\searrow X-\bigcup_{m=1}^{\infty}X_m$ $(n\to+\infty)$, $\mu(X-\bigcup_{m=1}^{\infty}X_m)=0$, and $\{\varphi_n\}$ converges to f uniformly on closed set X_m for every $m=1,2,\ldots$.

Considering the sequence $\{(X-H_m)\cup(X-X_m)\}_{m=1}^{\infty}$ of sets, then, as $m\to+\infty$

$$(X-H_m)\cup(X-X_m)\searrow\left(X-\bigcup_{m=1}^{\infty}H_m\right)\cup\left(X-\bigcup_{m=1}^{\infty}X_m\right).$$

By using the continuity from above and weakly null-additivity of fuzzy measures, we have

$$\lim_{m\to+\infty}\mu((X-H_m)\cup(X-X_m))=\mu\left(\left(X-\bigcup_{m=1}^{\infty}H_m\right)\cup\left(X-\bigcup_{m=1}^{\infty}X_m\right)\right)=0.$$

That is, $\lim_{m\to+\infty}\mu(X-H_m\cap X_m)=0$. Therefore, for given $\varepsilon>0$, we can take m_0 such that $\mu(X-H_{m_0}\cap X_{m_0})<\varepsilon$. Put $F_\varepsilon=H_{m_0}\cap X_{m_0}$, then F_ε is a closed set and $\mu(X-F_\varepsilon)<\varepsilon$. Now we show that f is continuous on F_ε. In fact, $F_\varepsilon\subset H_{m_0}$ and φ_n is continuous on H_{m_0}, therefore φ_n is continuous on F_ε for every $n=1,2,\ldots$. Noting that $\{\varphi_n\}$ converges to f on F_ε uniformly, then f is continuous on F_ε. □

Remark 2. Song and Li [9] have obtained the conclusions of Theorems 1, 3 and 4 under the null-additivity condition. As shown in Example 1, weekly null-additivity is really weaker than null-additivity and autocontinuity from above. Therefore, Theorems 1, 3, and 4 in this paper are improvements of the related results in Song and Li [9] and, Wu and Ha [11].

6. Applications of Lusin's theorem

Now we present some applications of Lusin's theorem to the mean approximation of measurable function by continuous functions, or by polynomials, or by step functions in the sense of Sugeno and of Choquet integral, respectively.

Consider a nonnegative real-valued measurable function f on $(X,\mathscr{B})$. The *Sugeno(fuzzy) integral* of f on X with respect to μ, denoted by $(S)\int f\,\mathrm{d}\mu$, is defined by

$$(S)\int f\,\mathrm{d}\mu = \sup_{0\leqslant\alpha<+\infty}[\alpha\wedge\mu(\{x: f(x)\geqslant\alpha\})].$$

The *Choquet integral* of f on X with respect to μ, denoted by $(C)\int f\,\mathrm{d}\mu$, is defined by

$$(C)\int f\,\mathrm{d}\mu = \int_0^\infty \mu(\{x: f(x)>t\})\,\mathrm{d}t,$$

where the right side integral is Lebesgue integral.

We say that a measurable function sequence $\{f_n\}_n$ *converges to f in fuzzy measure μ*, and denote it by $f_n \xrightarrow{\mu} f$, if for any $\varepsilon>0$, $\lim_{n\to\infty}\mu(\{x: |f_n(x)-f(x)|\geqslant\varepsilon\})=0$.

Theorem 5. *Let μ be a weakly null-additive fuzzy measure on $\mathscr{B}$. If f is a real-valued measurable function on X, then there exists a continuous function sequence $\{\psi_n\}_n$ on X such that $\psi_n \xrightarrow{\mu} f$. Furthermore, if $|f|\leqslant M$, then $|\psi_n|\leqslant M$, $n=1,2,\ldots$.*

Proof. For every $n=1,2,\ldots$, using Theorem 4 (Lusin's theorem), we can obtain a closed subset F_n of X such that f is continuous on F_n and $\mu(X-F_n)<\frac{1}{n}$. By Tietze's extension theorem [8], for every $n=1,2,\ldots$, there exists continuous function ψ_n on X such that $\psi_n(x)=f(x)$ for $x\in F_n$, and if $|f|\leqslant M$, then $|\psi_n|\leqslant M$. Now we show that $\{\psi_n\}_n$ converges to f in fuzzy measure. In fact, for any $\varepsilon>0$, we have $\{x: |\psi_n(x)-f(x)|\geqslant\varepsilon\}\subset X-F_n$, and therefore $\mu(\{x: |\psi_n(x)-f(x)|\geqslant\varepsilon\})\leqslant\mu(X-F_n)<\frac{1}{n}, n=1,2,\ldots$. Thus we have $\lim_{n\to\infty}\mu(\{x: |\psi_n(x)-f(x)|\geqslant\varepsilon\})=0$. □

The following result can be thought as to be the mean approximation theorem on fuzzy measure spaces $(X,\mathscr{B},\mu)$.

Theorem 6. *Let μ be a weakly null-additive fuzzy measure on $\mathscr{B}$. If f is a real-valued measurable function on X, then there exists a continuous function sequence $\{\psi_n\}_n$ on X such that*

$$\lim_{n\to+\infty}(S)\int|\psi_n-f|\,\mathrm{d}\mu=0,$$

Furthermore, if $|f|\leqslant M$, *then* $|\psi_n|\leqslant M$ $(n=1,2,\dots)$ *and*

$$\lim_{n\to+\infty}(C)\int|\psi_n-f|\,\mathrm{d}\mu=0.$$

Proof. From Theorem 5, there exists a continuous function sequence $\{\psi_n\}_n$ on X such that $\psi_n\xrightarrow{\mu}f$. By using Theorem 7.4 in [10], we can directly obtain $\lim_{n\to+\infty}(S)\int|\psi_n-f|\,\mathrm{d}\mu=0$.

If $|f|\leqslant M$, then from Theorem 5, $|\psi_n|\leqslant M$ $(n=1,2,\dots)$. Put

$$g_n(t)=\mu(\{x\colon|\psi_n(x)-f(x)|>t\}),\quad t\in[0,+\infty)$$

since $\psi_n\xrightarrow{\mu}f$, we have $g_n(t)\xrightarrow{a.e.}0$ on $[0,+\infty)$ as $n\to\infty$. Note that $|g_n(t)|\leqslant\mu(X)<\infty$, and $g_n(t)=0$ for any $t>2M$ $(n=1,2,\dots)$. Applying the Bounded Convergence Theorem in Lebesgue integral theory [8] to the function sequence $\{g_n(t)\}_n$, we have

$$\int_0^\infty g_n(t)dt=\int_0^{2M}g_n(t)\,\mathrm{d}t\to 0\quad(n\to\infty).$$

That is, $\lim_{n\to+\infty}(C)\int|\psi_n-f|\,\mathrm{d}\mu=0$. □

In the following, we discuss the mean approximation of measurable function either by polynomials or by step functions on fuzzy measure space $(R^1,\mathscr{B},\mu)$.

Theorem 7. *Let* μ *be a weakly null-additive fuzzy measure on* $\mathscr{B}$. *If* f *is a real-valued measurable function on* $[a,b]$, *then there exists a sequence* $\{P_n\}_n$ *of polynomials on* $[a,b]$ *such that* $P_n\xrightarrow{\mu}f$. *Furthermore, if* $|f|\leqslant M$, *then* $|P_n|\leqslant M+1$, $n=1,2,\dots$.

Proof. Considering the problem on the reduced fuzzy measure space $([a,b],[a,b]\cap\mathscr{B},\mu)$, then we can from Theorem 5 obtain a continuous function sequence $\{\psi_n\}_n$ on $[a,b]$ such that $\psi_n\xrightarrow{\mu}f$ on $[a,b]$. Therefore, there exists a subsequence $\{\psi_{n_k}\}_k$ of $\{\psi_n\}_n$, such that

$$\mu\left(\left\{x\colon|\psi_{n_k}(x)-f(x)|\geqslant\frac{1}{2k}\right\}\right)<\frac{1}{k},$$

for any $k=1,2,\dots$.

Since ψ_{n_k} is continuous function on $[a,b]$ $(k=1,2,\dots)$, by using Weierstrass's theorem [8], for every $k=1,2,\dots$, there exists a polynomial P_k on $[a,b]$ such that for all $x\in[a,b]$

$$|P_k(x)-\psi_{n_k}(x)|<\frac{1}{2k}.$$

Thus, for every $k=1,2,\dots$, we have

$$\left\{x\colon|P_k(x)-\psi_{n_k}(x)|\geqslant\frac{1}{2k}\right\}=\emptyset.$$

Noting that

$$\left\{x: |P_k(x) - f(x)| \geqslant \frac{1}{k}\right\} \subset A_k \cup B_k = \left\{x: |\psi_{n_k}(x) - f(x)| \geqslant \frac{1}{2k}\right\},$$

where

$$A_k = \left\{x: |P_k(x) - \psi_{n_k}(x)| \geqslant \frac{1}{2k}\right\}$$

and

$$B_k = \left\{x: |\psi_{n_k}(x) - f(x)| \geqslant \frac{1}{2k}\right\},$$

therefore we have

$$\mu\left(\left\{x: |P_k(x) - f(x)| \geqslant \frac{1}{k}\right\}\right) < \frac{1}{k}.$$

Now we show that $P_n \xrightarrow{\mu} f$ on [a, b]. In fact, for any given $\varepsilon > 0$, we take n_0 such that $1/n_0 < \varepsilon$, then $n \geqslant n_0$,

$$\{x: |P_n(x) - f(x)| \geqslant \varepsilon\} \subset \left\{x: |P_n(x) - f(x)| \geqslant \frac{1}{n}\right\},$$

and therefore,

$$\mu(\{x: |P_n(x) - f(x)| \geqslant \varepsilon\}) \leqslant \mu\left(\left\{x: |P_n(x) - f(x)| \geqslant \frac{1}{n}\right\}\right) < \frac{1}{n},$$

where $n > n_0$. This shows $P_n \xrightarrow{\mu} f$.

In the proof above, if $|f| \leqslant M$, then $|\psi_{n_k}| \leqslant M$. Since for every P_k, $|P_k(x) - \psi_{n_k}(x)| < \frac{1}{2k}$ for all $x \in [a, b]$, we have $|P_n| \leqslant M + 1$, $n = 1, 2, \ldots$. □

Theorem 8. *Let μ be a weakly null-additive fuzzy measure on $\mathscr{B}$. If f is a real-valued measurable function on $[a, b]$, then there exists a sequence $\{P_n\}_n$ of polynomials on $[a, b]$ such that*

$$\lim_{n\to+\infty} (S)\int |P_n - f| \,\mathrm{d}\mu = 0.$$

Furthermore, if $|f| \leqslant M$, then $|P_n| \leqslant M + 1$ $(n = 1, 2, \ldots)$ and

$$\lim_{n\to+\infty} (C)\int |P_n - f| \,\mathrm{d}\mu = 0.$$

Proof. It is similar to the proof of Theorem 6. □

Similarly, we can obtain the following result:

Theorem 9. *Let* μ *be a weakly null-additive fuzzy measure on* $\mathscr{B}$. *If* f *is a real-valued measurable function on* $[a,b]$, *then there exists a sequence* $\{s_n\}_n$ *of step functions on* $[a,b]$ *such that* $s_n \xrightarrow{\mu} f$ *and*

$$\lim_{n\to+\infty} (S)\int |s_n - f|\,\mathrm{d}\mu = 0;$$

Furthermore, if $|f|$ *is Choquet integrable, i.e.,* $(C)\int |f|\,\mathrm{d}\mu < \infty$, *then* $|s_n|$ *is also Choquet integrable and*

$$\lim_{n\to+\infty} (C)\int |s_n - f|\,\mathrm{d}\mu = 0.$$

Corollary 3. *If* μ *is null-additive fuzzy measure on* $\mathscr{B}$, *then the conclusions of Theorems* 5–9 *hold.*

7. Concluding remarks

We have proved Lusin's theorem on finite fuzzy measure space under the weakly null-additivity condition. As we have seen, the weakly null-additivity, including its a necessary and sufficient condition presented in Lemma 1, and the regularity of fuzzy measures play important roles in our discussions.

It should be pointed out that our discussion on the weakly null-additivity is nothing but sufficient, not necessary for Theorem 1, 4, 5 and 6. Compared with the null-additivity and autocontinuity, it is a weaker requirement, we need still further discussion.

Example 2. Let $X = \{a,b\}$ and (X,ρ) be a metric space. Then $\mathscr{B} = \wp(X)$. Put

$$\mu(E) = \begin{cases} 1 & \text{if } E = X, \\ 0 & \text{if } E \neq X. \end{cases}$$

Then fuzzy measure μ is not weakly null-additive. But μ is regular and any measurable function is continuous on X, and hence Lusin's theorem holds on $(X,\mathscr{B},\mu)$.

We do not know whether the weakly null-additivity condition may be abandoned in our discussion. In our further research, we intend to address this issue and to investigate whether Lusin's theorem remains valid on finite fuzzy measure spaces $(X,\mathscr{B},\mu)$ without any additional condition as the Egoroff's theorem we have proved in [5].

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