A fuzzy stopping problem with the concept of perception : The finite and infinite horizon cases

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Abstract

In this paper, we try the perceptive analysis of the optimal stopping problem in which both fuzzy and probabilistic uncertainty is accommodated. We establish the recursive equation for computing the expected value of the optimal stopped fuzzy perception reward. Also, a numerical example is given.

Keywords : Fuzzy stopping problem, fuzzy perception, fuzzy random variable, fuzzy perception reward, optimal stopping time.

1. Introduction and notation

The stopping problem in a stochastic environment is described by real random variables for which an optimal stopping time is given (cf. [2]). However, in practice, we are often faced with the case that the values of random variables are partially observed by dimness of perception or measurement imprecision. For example, in the classical stopping problem of selling or buying an asset (cf. [4]), the price of the asset may not be observed exactly. Usually it is linguistically and roughly perceived e.g. about \$10000, the price considerably larger than \$10000, etc. When there will be long periods of time before our actual decision for the problem, we are still wrapped in a fog of dimness. But immediately before our decision, the fog mist is cleared up and we can know the very value of the price so that the optimal procedure could be taken. Then, under dimness of perception or measurement imprecision, how can we estimate in advance the future reward obtained from the optimal procedure. A possible way of handling such a case is by using the fuzzy set (cf. [11]), whose membership functions can describe the perception value of price. Motivated by the example of the above, in this paper we try the perceptive analysis ([1, 12]) of the stopping problem in which fuzzy perception is accommodated, and we investigate the method of computing the fuzzy perception reward when stopped optimally.

In remainder of this section, we will give some notation and the definition of a fuzzy perception function referring Baswell and Tayler [1], by which the perceptive stopping problem is formulated in the sequel. For non-perception approaches to fuzzy stopping problems, refer to our previous works [5, 10]. Recently Zadeh wrote a summary paper of perception-based theory [12].

For any set A, the fuzzy set on A will be denoted by its membership function $\widetilde{a} : A \to [0, 1]$. The α-cut of $\widetilde{a}$ is given by $\widetilde{a}_\alpha := \{x \in A \mid \widetilde{a}(x) \geq \alpha\}(\alpha \in (0, 1])$ and $\widetilde{a}_0 := \mathrm{cl}\{x \in A \mid \widetilde{a}(x) > 0\}$, where $\mathrm{cl}\{B\}$ is the closure of a set B. For the theory of fuzzy sets, we refer to Zadeh [11] and Dubois and Prade [3].

Let $\mathbb{R}$ be the set of all real numbers and $\widetilde{\mathbb{R}}$ the set of all fuzzy numbers, i.e., $\widetilde{r} \in \widetilde{\mathbb{R}}$ means that $\widetilde{r} : \mathbb{R} \to [0, 1]$ is normal, upper-semicontinuous and fuzzy convex and has a compact support.

Let $\mathbb{C}$ be the set of all bounded and closed intervals of $\mathbb{R}$. Then, obviously for any $\widetilde{r} \in \widetilde{\mathbb{R}}$, it holds that $\widetilde{r}_\alpha \in \mathbb{C}$ $(\alpha \in [0, 1])$. So, we write $\widetilde{r}_\alpha = [\widetilde{r}^-_\alpha, \widetilde{r}^+_\alpha]$ $(\alpha \in [0, 1])$.

A partial order relation $\preccurlyeq$ on $\widetilde{\mathbb{R}}$, called the fuzzy max order (cf. [8]), is defined as follows: For $\widetilde{s}, \widetilde{r} \in \widetilde{\mathbb{R}}$, $\widetilde{s} \preccurlyeq \widetilde{r}$, if $\widetilde{s}^-_\alpha \leq \widetilde{r}^-_\alpha$ and $\widetilde{s}^+_\alpha \leq \widetilde{r}^+_\alpha$ for all $\alpha \in [0, 1]$, where $\widetilde{s}_\alpha = [\widetilde{s}^-_\alpha, \widetilde{s}^+_\alpha]$ and $\widetilde{r}_\alpha = [\widetilde{r}^-_\alpha, \widetilde{r}^+_\alpha]$.

Here, we define $\widetilde{\max}\{\widetilde{s}, \widetilde{r}\} \in \widetilde{\mathbb{R}}$ by

$$\widetilde{\max}\{\widetilde{s}, \widetilde{r}\}(y) := \sup_{\substack{x_1, x_2 \in \mathbb{R} \\ y = x_1 \vee x_2}} \{\widetilde{s}(x_1) \wedge \widetilde{r}(x_2)\} \quad (y \in \mathbb{R}), \tag{1.1}$$

where $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ for any $a, b \in \mathbb{R}$.

Then, it is well-known (cf. [8]) that $\widetilde{s} \preccurlyeq \widetilde{r}$ if and only if $\widetilde{r} = \widetilde{\max}\{\widetilde{s}, \widetilde{r}\}$.

Let $(\Omega, \mathcal{M}, P)$ be a probability space. A map $\widetilde{X} : \Omega \to \widetilde{\mathbb{R}}$ is called a fuzzy perception function if for each $\alpha \in [0, 1]$ the maps $\Omega \ni \omega \mapsto \widetilde{X}^-_\alpha(\omega)$ and $\Omega \ni \omega \mapsto \widetilde{X}^+_\alpha(\omega)$ are $\mathcal{M}$-measurable for all $\alpha \in [0, 1]$, where $\widetilde{X}_\alpha(\omega) = [\widetilde{X}^-_\alpha(\omega), \widetilde{X}^+_\alpha(\omega)] := \{x \in \mathbb{R} \mid \widetilde{X}(\omega)(x) \geq \alpha\}$.

Let $\mathscr{X}$ be the set of all integrable random variables on $(\Omega, \mathcal{M}, P)$. For any fuzzy perception function $\widetilde{X}$, the expectation $E\widetilde{X} \in \widetilde{\mathbb{R}}$ is defined by

$$E\widetilde{X}(x) = \sup_{\substack{X \in \mathscr{X} \\ EX = x}} \widetilde{\mu}(\widetilde{X})(X), \tag{1.2}$$

where $\widetilde{\mu}(\widetilde{X})$ is a fuzzy set of $\mathscr{X}$ and defined by

$$\widetilde{\mu}(\widetilde{X})(X) = \inf_{\omega \in \Omega} \widetilde{X}(\omega)(X(\omega)) \quad \text{for all } X \in \mathscr{X}. \tag{1.3}$$

Obviously, we have

$$E(\widetilde{X})_\alpha = [\int \widetilde{X}^-_\alpha(\omega) dP(\omega), \int \widetilde{X}^+_\alpha(\omega) dP(\omega)], \quad (\alpha \in [0, 1]). \tag{1.4}$$

Note that a fuzzy set $\widetilde{\mu}(\widetilde{X})$ on $\mathscr{X}$ is called fuzzy random variable induced by $\widetilde{X}$ (cf. [1]). Regarding the another (equivalent) definition of fuzzy random variables, we refer to

H. Kwakernaak [6] and Puri and Ralescu [7]. In this paper, the definition of fuzzy random variables from a perceptive stand point by Baswell and Taylor [1] is adopted for modeling a fuzzy perceptive stopping problem.

2. Stopped fuzzy perception rewards

Let $\mathscr{X}^n$ be the set of all n-dimensional row vectors whose elements are in $\mathscr{X}$, i.e.,

$$\mathscr{X}^n = \{\boldsymbol{X} = (X_1, X_2, \ldots, X_n) \mid X_t \in \mathscr{X}, t = 1, 2, \ldots, n\}.$$

The stopping time $\sigma : \Omega \to \mathbb{N}_n := \{1, 2, \ldots, n\}$ is said to be corresponding to $\boldsymbol{X} = (X_1, X_2, \ldots, X_n) \in \mathscr{X}^n$ if $\{\sigma = k\} \in \mathcal{B}(\boldsymbol{X}_k)$ $(k = 1, 2, \ldots, n)$ where $\boldsymbol{X}_k = (X_1, X_2, \ldots, X_k)$ and $\mathcal{B}(\boldsymbol{X}_k)$ is the σ-field on Ω generated by the random vector $\boldsymbol{X}_k$. The set of such stopping times will be denoted by $\Sigma\{\boldsymbol{X}\}$.

The map δ on $\mathscr{X}^n$ with $\delta(\boldsymbol{X}) \in \Sigma\{\boldsymbol{X}\}$ for all $\boldsymbol{X} \in \mathscr{X}^n$ is called a stopping time function. A stopping time function δ is called monotone if for any $\boldsymbol{X} = (X_1, X_2, \ldots, X_n), \boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n) \in \mathscr{X}^n$ with $\boldsymbol{X} \le \boldsymbol{Y}$, i.e., $X_t \le Y_t$ $(t = 1, 2, \ldots, n)$ P-$a.s.$, it holds that $E\boldsymbol{X}_\delta \le E\boldsymbol{Y}_\delta$, where $\boldsymbol{X}_\delta := X_{\delta(\boldsymbol{X})}$ and $\boldsymbol{Y}_\delta := Y_{\delta(\boldsymbol{Y})}$.

For any $\boldsymbol{X} = (X_1, X_2, \ldots, X_n), \boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n) \in \mathscr{X}^n$ and $\beta \in [0, 1]$, let $\boldsymbol{Z} := \beta\boldsymbol{X} + (1-\beta)\boldsymbol{Y} = (\beta X_1 + (1-\beta)Y_1, \beta X_2 + (1-\beta)Y_2, \ldots, \beta X_n + (1-\beta)Y_n) \in \mathscr{X}^n$. Then δ is called convex if $E\boldsymbol{Z}_\delta \le \beta E\boldsymbol{X}_\delta + (1-\beta)E\boldsymbol{Y}_\delta$ for all $\beta \in [0, 1]$, where $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_n)$ and $\boldsymbol{Z}_\delta := Z_{\delta(\boldsymbol{Z})}$. The set of all monotone and convex stopping time functions will be denoted by Δ.

Let $\widetilde{\boldsymbol{X}} = (\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n)$ be a sequence of fuzzy perception functions. For any $\delta \in \Delta$, the δ-stopped fuzzy perception reward $\widetilde{\boldsymbol{X}}_\delta$ is defined by,

$$\widetilde{\boldsymbol{X}}_\delta(\omega)(x) := \sup_{\substack{\boldsymbol{X}_\delta(\omega) = x \\ \boldsymbol{X} = (X_1, \ldots, X_n) \in \mathscr{X}^n}} \{\widetilde{X}_1(\omega)(X_1(\omega)) \wedge \ldots \wedge \widetilde{X}_n(\omega)(X_n(\omega))\}. \tag{2.1}$$

Note that $\widetilde{\boldsymbol{X}}_\delta(\omega)(x)$ may be a fuzzy set on $\mathbb{R}$ but not necessarily a fuzzy perception function.

Similarly as (1.2), we define the expected value of $\widetilde{\boldsymbol{X}}_\delta(\omega)(x)$ by

$$E\widetilde{\boldsymbol{X}}_\delta(x) := \sup_{\substack{E(X) = x \\ X \in \mathscr{X}}} \inf_{\omega \in \Omega} \{\widetilde{\boldsymbol{X}}_\delta(\omega)(X(\omega))\}. \tag{2.2}$$

Let, for each $\alpha \in [0, 1]$, $\widetilde{\boldsymbol{X}}_\alpha^- := (\widetilde{X}_{1,\alpha}^-, \ldots, \widetilde{X}_{n,\alpha}^-) \in \mathscr{X}^n$ and $\widetilde{\boldsymbol{X}}_\alpha^+ := (\widetilde{X}_{1,\alpha}^+, \ldots, \widetilde{X}_{n,\alpha}^+) \in \mathscr{X}^n$, where the α-cut of $\widetilde{X}_k$ is described by $\widetilde{X}_{k,\alpha} = [\widetilde{X}_{k,\alpha}^-, \widetilde{X}_{k,\alpha}^+]$.

Then, we have the following.

Theorem 2.1 *For any $\delta \in \Delta$, it holds that*

(i) $E\widetilde{\boldsymbol{X}}_\delta \in \widetilde{\mathbb{R}}$ *and*

(ii) $(E\widetilde{\boldsymbol{X}}_\delta)_\alpha = [E((\widetilde{\boldsymbol{X}}_\alpha^-)_\delta), E((\widetilde{\boldsymbol{X}}_\alpha^+)_\delta)]$ *for* $\alpha \in [0, 1]$.

For the proof of Theorem 2.1, we need the several preliminary lemmas.

Here, we put, for each $\alpha \in [0,1]$,

$$Z^\alpha(\beta) := \beta \widetilde{\boldsymbol{X}}_\alpha^+ + (1-\beta)\widetilde{\boldsymbol{X}}_\alpha^- \quad (\beta \in [0,1]). \tag{2.3}$$

Lemma 2.1 *For any $\delta \in \Delta$, $E(Z^\alpha(\beta)_\delta)$ is continuous with respect to $\beta \in [0,1]$.*

Proof. For any β, β' with $0 \le \beta < \beta' < 1$,

$$Z^\alpha(\beta') = \frac{\beta'-\beta}{1-\beta}\widetilde{\boldsymbol{X}}_\alpha^+ + (1 - \frac{\beta'-\beta}{1-\beta})Z^\alpha(\beta).$$

So, from the monotonicity and convexity of $\delta \in \Delta$, we have for $0 \le \beta < \beta' < 1$,

$$\begin{aligned} E(Z^\alpha(\beta)_\delta) &\le E(Z^\alpha(\beta')_\delta) \\ &\le \frac{\beta'-\beta}{1-\beta}E((\widetilde{\boldsymbol{X}}_\alpha^+)_\delta) + (1-\frac{\beta'-\beta}{1-\beta})E(Z^\alpha(\beta)_\delta), \end{aligned}$$

which implies that $\lim_{\beta' \downarrow \beta} E(Z^\alpha(\beta')_\delta) = E(Z^\alpha(\beta)_\delta)$.

Similarly, we have for $0 \le \beta'' < \beta < 1$,

$$E(Z^\alpha(\beta)_\delta) \le \frac{\beta-\beta''}{1-\beta''}E((\widetilde{\boldsymbol{X}}_\alpha^+)_\delta) + (1-\frac{\beta-\beta''}{1-\beta''})E(Z^\alpha(\beta'')_\delta).$$

Thus, it holds that

$$\begin{aligned} 0 &\le E(Z^\alpha(\beta)_\delta) - E(Z^\alpha(\beta'')_\delta) \le \frac{\beta-\beta''}{1-\beta''}(E((\widetilde{\boldsymbol{X}}_\alpha^+)_\delta) - E(Z^\alpha(\beta'')_\delta)) \\ &\le \frac{\beta-\beta''}{1-\beta''}(E((\widetilde{\boldsymbol{X}}_\alpha^+)_\delta) - E((\widetilde{\boldsymbol{X}}_\alpha^-)_\delta)). \end{aligned}$$

Thus we get $\lim_{\beta'' \uparrow \beta} E(Z^\alpha(\beta'')_\delta) = E(Z^\alpha(\beta)_\delta)$. □

The following lemma follows easily from (2.1) and (2.2).

Lemma 2.2 *For any $\delta \in \Delta$ and $\alpha \in [0,1]$, it holds that*

$$\begin{aligned} (E\widetilde{\boldsymbol{X}}_\delta)_\alpha &= \{E\boldsymbol{X}_\delta \mid \boldsymbol{X} = (X_1, X_2, \ldots, X_n) \in \mathscr{X}^n, \\ &\qquad X_t(\omega) \in [\widetilde{X}_{t,\alpha}^-(\omega), \widetilde{X}_{t,\alpha}^+(\omega)] \quad \textit{for} \ \ t = 1,2,\ldots,n\}. \end{aligned}$$

The proof of Theorem 2.1. Since (ii) means (i), it suffices to show that (ii) holds. By Lemma 2.2 and monotonicity of δ, the inclusion $\subset$ of (ii) is immediate. Also, the inclusion $\supset$ follows from the observation that $Z^\alpha(1) = \widetilde{\boldsymbol{X}}_\alpha^+$, $Z^\alpha(0) = \widetilde{\boldsymbol{X}}_\alpha^-$ and Lemma 2.1. □

By Theorem 2.1, we observe that $E\widetilde{\boldsymbol{X}}_\delta \in \widetilde{\mathbb{R}}$ for all $\delta \in \Delta$. Here we can specify the perceptive fuzzy stopping problem investigated in the next section: The problem is to maximize $E\widetilde{\boldsymbol{X}}_\delta$ for all $\delta \in \Delta$ with respect to the fuzzy max order $\preccurlyeq$ on $\widetilde{\mathbb{R}}$.

3. Perceptive optimization and recursive equations

In this section, for any given sequence of fuzzy perception functions $\widetilde{\boldsymbol{X}} = (\widetilde{X}_1, \widetilde{X}_2, \dots, \widetilde{X}_n)$, we find the optimal stopping time function δ^* and characterize the optimal fuzzy perception value $E\widetilde{\boldsymbol{X}}_{\delta^*}$.

For each sequence of random variables $\boldsymbol{X} = (X_1, X_2, \dots, X_n) \in \mathscr{X}^n$, we denote by $\delta^*(\boldsymbol{X})$ the optimal stopping time for $\boldsymbol{X}$ (cf. [2]), which is thought as a stopping time function.

Lemma 3.1 $\delta^* \in \Delta$.

Proof. For $\boldsymbol{X} = (X_1, X_2, \dots, X_n) \in \mathscr{X}^n$, involving $\boldsymbol{X}$, we define the sequence $\{\gamma_k^n = \gamma_k^n(\boldsymbol{X})\}_{k=1}^n$ by

$$\gamma_n^n(\boldsymbol{X}) = X_n, \quad \gamma_k^n(\boldsymbol{X}) = \max\{X_k, E[\gamma_{k+1}^n \mid \mathcal{B}(\boldsymbol{X}_k)]\} \ \ (k = n-1, \dots, 1), \tag{3.1}$$

where $\boldsymbol{X}_k = (X_1, X_2, \dots, X_k)$. Then, by the usual theory of optimal stopping problems (cf. [2]), we have $E(\boldsymbol{X}_{\delta^*}) = E\gamma_1^n(\boldsymbol{X})$.

Let $\boldsymbol{X} = (X_1, X_2, \dots, X_n), \boldsymbol{Y} = (Y_1, Y_2, \dots, Y_n) \in \mathscr{X}^n$ with $X_t \le Y_t$ $(t = 1, 2, \dots, n)$ P-a.s.. Then, by induction on k, we can easily prove that $\gamma_k^n(\boldsymbol{X}) \le \gamma_k^n(\boldsymbol{Y})$ for $k = n, n-1, \dots, 1$. Thus, we get

$$E(\boldsymbol{X}_{\delta^*}) = E(\gamma_1^n(\boldsymbol{X})) \le E(\gamma_1^n(\boldsymbol{Y})) = E(\boldsymbol{Y}_{\delta^*}),$$

which shows the monotonicity of δ^*.

For $\boldsymbol{Z} = \beta\boldsymbol{X} + (1-\beta)\boldsymbol{Y}$ $(\beta \in [0,1])$, we have

$$\begin{aligned} E[Z_{\delta^*(\boldsymbol{Z})}] &= \beta E[X_{\delta^*(\boldsymbol{Z})}] + (1-\beta)E[Y_{\delta^*(\boldsymbol{Z})}] \\ &\le \beta E[X_{\delta^*(\boldsymbol{X})}] + (1-\beta)E[Y_{\delta^*(\boldsymbol{Y})}], \end{aligned}$$

where $\boldsymbol{Z} = (Z_1, Z_2, \dots, Z_n)$. This shows the convexity of δ^*. $\square$

By Lemma 3.1, we observe that δ^* is an optimal stopping time function.

For simplicity, we assume the sequence of perception functions $\widetilde{\boldsymbol{X}} = (\widetilde{X}_1, \widetilde{X}_2, \dots, \widetilde{X}_n)$ is independent with each $\widetilde{X}_t$ $(t = 1, 2, \dots, n)$. Then, in the following theorem it will be shown that the optimal fuzzy perception value $E\widetilde{\boldsymbol{X}}_{\delta^*}$ is given by the backward recursive equation:

$$\widetilde{\gamma}_n^n = E\widetilde{X}_n, \quad \widetilde{\gamma}_k^n = E\widetilde{\max}\{\widetilde{X}_k, \widetilde{\gamma}_{k+1}^n\} \ (k = n-1, \dots, 2, 1). \tag{3.2}$$

Let the α-cut of $\widetilde{\gamma}_k^n$ in (3.2) be

$$\widetilde{\gamma}_{k,\alpha}^n = [\widetilde{\gamma}_{k,\alpha}^{n,-}, \widetilde{\gamma}_{k,\alpha}^{n,+}] \quad (k = 1, 2, \dots, n).$$

Then, the α-cut expression of (3.2) is as follows:

$$\begin{aligned} &\widetilde{\gamma}_{n,\alpha}^{n,-} = E(\widetilde{X}_{n,\alpha}^-), \quad \widetilde{\gamma}_{n,\alpha}^{n,+} = E(\widetilde{X}_{n,\alpha}^+) \\ &\widetilde{\gamma}_k^{n,-} = E\max\{\widetilde{X}_{k,\alpha}^-, \widetilde{\gamma}_{(k+1),\alpha}^{n,-}\}, \quad \widetilde{\gamma}_k^{n,+} = E\max\{\widetilde{X}_{k,\alpha}^+, \widetilde{\gamma}_{(k+1),\alpha}^{n,+}\} \\ &\qquad (\alpha \in [0,1], \quad k = n-1, n-2, \dots, 1). \end{aligned} \tag{3.3}$$

Theorem 3.1 $E\widetilde{\boldsymbol{X}}_{\delta^*} = \widetilde{\gamma}_n^1$.

Proof. By (3.2) and (3.3), we have that, for $\alpha \in [0,1]$,

$$\begin{aligned}\widetilde{\gamma}_{k,\alpha}^n &= [E\max\{\widetilde{X}_{k,\alpha}^-, \gamma_{(k+1),\alpha}^{n,-}\}, E\max\{\widetilde{X}_{k,\alpha}^+, \gamma_{(k+1),\alpha}^{n,+}\}] \\ &= [E\max\{\widetilde{X}_{k,\alpha}^-, \widetilde{\gamma}_{(k+1)}^n(\widetilde{\boldsymbol{X}}_\alpha^-)\}, E\max\{\widetilde{X}_{k,\alpha}^+, \widetilde{\gamma}_{(k+1)}^n(\widetilde{\boldsymbol{X}}_\alpha^+)\}],\end{aligned}$$

where $\gamma_{(k+1)}^n(\widetilde{\boldsymbol{X}}_\alpha^-)$ and $\gamma_{(k+1)}^n(\widetilde{\boldsymbol{X}}_\alpha^+)$ are defined in (3.1). Applying Theorem 2.1, we get $(E\widetilde{\boldsymbol{X}}_{\delta^*})_\alpha = (\widetilde{\gamma}_n^1)_\alpha$. Thus, $E\widetilde{\boldsymbol{X}}_{\delta^*} = \widetilde{\gamma}_n^1$, as required. □

Example 1 (The finite horizon case)
As a numerical example, we will compute the optimal fuzzy perception value for the perception stopping problem described by simple triangular fuzzy numbers.

The triangular fuzzy number (a,m,b) with $a > 0$ and $b > 0$ is given by

$$(a,m,b)(x) = \begin{cases} \max\{(x-m+a)/a, 0\} & \text{if } x \le m \\ \max\{(x-m-b)/b, 0\} & \text{if } x > m. \end{cases}$$

Obviously, the α-cut of (a,m,b) is

$$(a,m,b)_\alpha = [m - a(1-\alpha), m + b(1-\alpha)] \quad \alpha \in [0,1].$$

Let $\widetilde{\boldsymbol{X}} = (\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n)$ be independent and identically distributed sequence of fuzzy perception functions with $\widetilde{X}_t = (Y_t, X_t, Z_t)$ $(t = 1, 2, \ldots, n)$. (See Fig.1). We assume that $X_t \sim U[0,1]$ and $Y_t,\ Z_t \sim U[0,1/2]$ $(t = 1,2,\ldots,n)$, where $X \sim U[a,b]$ $(a < b)$ means that the distribution of X is a uniform distribution on $[a,b]$.

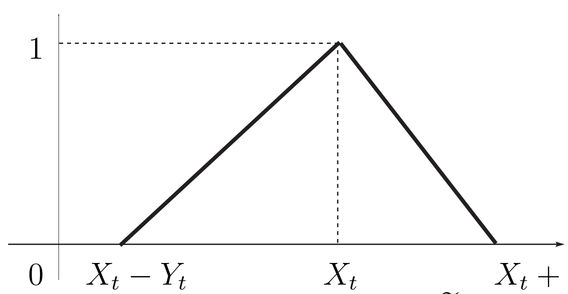


Figure 1. The fuzzy perception $\widetilde{X}_t = (Y_t, X_t, Z_t)$

The optimal fuzzy perception value $E\widetilde{\boldsymbol{X}}_{\delta^*} = \widetilde{\gamma}_1^n$ is computed recursively by (3.3), which is given as follows.

$$\begin{aligned}\widetilde{\gamma}_{n,\alpha}^{n,-} &= \tfrac{1+\alpha}{2}, \quad \widetilde{\gamma}_{n,\alpha}^{n,+} = \tfrac{3-\alpha}{2} \\ \widetilde{\gamma}_{k,\alpha}^{n,-} &= E\max\{X_k - (1-\alpha)Y_k, \widetilde{\gamma}_{(k+1),\alpha}^{n,-}\} \\ \widetilde{\gamma}_{k,\alpha}^{n,+} &= E\max\{X_k + (1-\alpha)Z_k, \widetilde{\gamma}_{(k+1),\alpha}^{n,+}\} \\ &\quad (\alpha \in [0,1],\ k = n-1, n-2, \ldots, 1).\end{aligned}$$

The graph of $\widetilde{\gamma}_1^n$ $(n = 1, 5, 20)$ evaluated by Maple 7 is shown in Fig. 2, and we observe that $\widetilde{\gamma}_1^{20}$ is concave on its left-side slope and convex on its right-side slope.

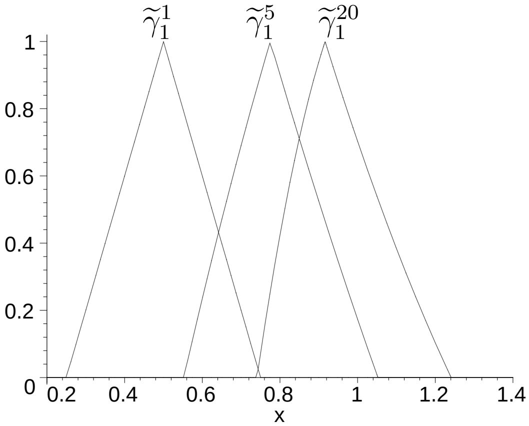


Figure 2. The graph of $\widetilde{\gamma}_1^n$ $(n = 1, 5, 20)$

4. The infinite horizon case

In this section, we consider the infinite horizon case for fuzzy stopping problems by the use of the same idea as in the finite horizon case treated in the preceding sections.

Let $\mathscr{X}^\infty$ be the set of all infinite dimensional row vectors whose elements are in $\mathscr{X}$, i.e.,

$$\mathscr{X}^\infty := \{\boldsymbol{X} = (X_1, X_2, \dots) \mid X_t \in \mathscr{X},\ t \geq 1\}.$$

The stopping time for the infinite horizon case $\sigma : \Omega \to \mathbb{N}_\infty = \{1, 2, \dots, \infty\}$ is said to be corresponding to $\boldsymbol{X} = (X_1, X_2, \dots) \in \mathscr{X}^\infty$ if $P(\sigma < \infty) = 1$ and $\{\sigma = k\} \in \mathcal{B}(\boldsymbol{X}_k)$ for $k \geq 1$, where $\boldsymbol{X}_k = (X_1, X_2, \dots, X_k)$.

For any $\boldsymbol{X} = (X_1, X_2, \dots) \in \mathscr{X}^\infty$, let $\mathcal{C}(\boldsymbol{X})$ be the set of all stopping times corresponding to $\boldsymbol{X}$ with $E(-\min\{0, -X_\sigma\}) < \infty$.

Let $\widetilde{\boldsymbol{X}} = (\widetilde{X}_1, \widetilde{X}_2, \dots)$ be a sequence of fuzzy perception functions and put

$$\mathcal{M}(\widetilde{\boldsymbol{X}}) := \{\boldsymbol{X} = (X_1, X_2, \dots) \in \mathscr{X}^\infty \mid \widetilde{X}_{t,0}^- \leq X_t \leq \widetilde{X}_{t,0}^+ \ P\text{-}a.s.\ (t \geq 1)\},$$

where $\widetilde{X}_{t,0} = [X_{t,0}^-, X_{t,0}^+]$ is the 0-cut of $\widetilde{X}_t$.

Clearly, $\mathcal{M}(\widetilde{\boldsymbol{X}})$ is convex, i.e., for any $\boldsymbol{X} = (X_1, X_2, \dots), \boldsymbol{Y} = (Y_1, Y_2, \dots) \in \mathcal{M}(\widetilde{\boldsymbol{X}})$, $\alpha\boldsymbol{X} + (1-\alpha)\boldsymbol{Y} = (\alpha X_1 + (1-\alpha)Y_1, \alpha X_2 + (1-\alpha)Y_2, \dots) \in \mathcal{M}(\widetilde{\boldsymbol{X}})$.

The map δ on $\mathcal{M}(\widetilde{\boldsymbol{X}})$ with $\delta(\boldsymbol{X}) \in \mathcal{C}(\boldsymbol{X})$ for all $\boldsymbol{X} \in \mathcal{M}(\widetilde{\boldsymbol{X}})$ is called a stopping time function for $\mathcal{M}(\widetilde{\boldsymbol{X}})$, whose monotonicity and convexity are defined similarly as in the finite horizon case.

The set of all monotone and convex stopping times on $\mathcal{M}(\widetilde{\boldsymbol{X}})$ will be denoted by $\Delta(\widetilde{\boldsymbol{X}})$.

For any $\delta \in \Delta(\widetilde{\boldsymbol{X}})$, the δ-stopped fuzzy perception reward $\widetilde{\boldsymbol{X}}_\delta$ is denoted similarly as (2.1) by

$$\widetilde{\boldsymbol{X}}_{\delta(\omega)}(x) := \sup_{\substack{\boldsymbol{X}_{\delta(\omega)} = x \\ \boldsymbol{X} \in \mathcal{M}(\widetilde{\boldsymbol{X}})}} \bigwedge_{t=1}^{\infty} \widetilde{X}_t(\omega)(X_t(\omega)), \tag{4.1}$$

where $\boldsymbol{X}_{\delta(\omega)} = X_\delta(\omega)$ and $\boldsymbol{X} = (X_1, X_2, \dots) \in \mathcal{M}(\widetilde{\boldsymbol{X}})$.

The expectation $E\widetilde{\boldsymbol{X}}_\delta$ is defined by (2.2) with $\widetilde{\boldsymbol{X}}_\delta$ defined by (4.1). Then, the same results as Theorem 2.1 hold for the expectation $E\widetilde{\boldsymbol{X}}_\delta$. That is, we have $E\widetilde{\boldsymbol{X}}_\delta \in \widetilde{\mathbb{R}}$.

The problem for the infinite horizon case is to maximize $E\widetilde{\boldsymbol{X}}_\delta$ for all $\delta \in \Delta(\widetilde{\boldsymbol{X}})$ with respect to the fuzzy max order $\preccurlyeq$ on $\widetilde{\mathbb{R}}$.

For simplicity, the following Assumption A is supposed to hold the henceforth.

Assumption A. For each $\boldsymbol{X} = (X_1, X_2, \dots) \in \mathcal{M}(\widetilde{\boldsymbol{X}})$, there exists an optimal stopping time $\delta^*(\boldsymbol{X}) \in \mathcal{C}(\boldsymbol{X})$.

Applying Theorem 4.5 in [2], the following Assumption A' is easily proved to be sufficient for Assumption A.

Assumption A'. For a given $\widetilde{\boldsymbol{X}} = (\widetilde{X}_1, \widetilde{X}_2, \dots)$, $E(\sup_{t \geq 1} \widetilde{X}^+_{t,0}) < \infty$ and $\lim_{t \to \infty} \widetilde{X}^+_{t,0} = -\infty$.

$\delta^*(\boldsymbol{X})$ can be thought as a stopping time function on $\mathcal{M}(\widetilde{\boldsymbol{X}})$. By a slight modification of the proof of Lemma 3.1, we yield that $\delta^* \in \Delta(\widetilde{\boldsymbol{X}})$. Obviously, δ^* is optimal, i.e.,

$$E(\widetilde{\boldsymbol{X}}_{\delta^*}) \succcurlyeq E(\widetilde{\boldsymbol{X}}_\delta) \quad \text{for all} \quad \delta \in \Delta(\widetilde{\boldsymbol{X}}).$$

Now, we will characterize the optimal fuzzy perception reward $E(\widetilde{\boldsymbol{X}}_{\delta^*})$ applying the usual non-fuzzy optimal stopping theory (cf. [2]).

Using the extreme points of α-cuts $\widetilde{\boldsymbol{X}}^-_{1,\alpha} = (\widetilde{X}^-_{1,\alpha}, \widetilde{X}^-_{2,\alpha}, \dots)$ and $\widetilde{\boldsymbol{X}}^+_{1,\alpha} = (\widetilde{X}^+_{1,\alpha}, \widetilde{X}^+_{2,\alpha}, \dots)$ with $\widetilde{X}_{t,\alpha} = [\widetilde{X}^-_{t,\alpha}, \widetilde{X}^+_{t,\alpha}]$ $(t \geq 1, \alpha \in [0,1])$, we define $\widetilde{\gamma}^-_t$ and $\widetilde{\gamma}^+_t$ $(t \geq 1)$ by

$$\widetilde{\gamma}^-_{t,\alpha} := \operatorname*{ess\,sup}_{\sigma \in \mathcal{C}_t(\widetilde{\boldsymbol{X}}^-_\alpha)} E(\widetilde{X}^-_{\sigma,\alpha} \mid \mathcal{B}(\widetilde{\boldsymbol{X}}^-_t)), \tag{4.2}$$

$$\widetilde{\gamma}^+_{t,\alpha} := \operatorname*{ess\,sup}_{\sigma \in \mathcal{C}_t(\widetilde{\boldsymbol{X}}^+_\alpha)} E(\widetilde{X}^+_{\sigma,\alpha} \mid \mathcal{B}(\widetilde{\boldsymbol{X}}^+_t)), \tag{4.3}$$

where, for any $\boldsymbol{X} = (X_1, X_2, \dots) \in \mathscr{X}^\infty$, $\mathcal{C}_t(\boldsymbol{X})$ is the set of $\sigma \in \mathcal{C}(\boldsymbol{X})$ with $\sigma \leq t$ and $\boldsymbol{X}_t = (X_1, X_2, \dots, X_t)$. Then, we have the following left- and right-hand optimality equations.

$$\widetilde{\gamma}^-_{t,\alpha} = \max\{\widetilde{X}^-_{t,\alpha}, E(\widetilde{\gamma}^-_{t+1,\alpha} \mid \mathcal{B}(\widetilde{\boldsymbol{X}}^-_t))\} \quad \text{and} \tag{4.4}$$

$$\widetilde{\gamma}^+_{t,\alpha} = \max\{\widetilde{X}^+_{t,\alpha}, E(\widetilde{\gamma}^+_{t+1,\alpha} \mid \mathcal{B}(\widetilde{\boldsymbol{X}}^+_t))\} \quad (t \leq 1). \tag{4.5}$$

Theorem 4.1 *Concerning with the α-cut of $E(\widetilde{\boldsymbol{X}}_{\delta^*})$, we have*

$$E(\widetilde{\boldsymbol{X}}_{\delta^*})_\alpha = [\widetilde{\gamma}^-_{1,\alpha}, \widetilde{\gamma}^+_{1,\alpha}] \quad (\alpha \in [0,1]).$$

Application Let $\widetilde{Y}, \widetilde{Y}_1, \widetilde{Y}_2, \dots$ be a sequence of fuzzy perception functions such that the pair of extreme points of α-cuts $(\widetilde{Y}^-_\alpha, \widetilde{Y}^+_\alpha), (\widetilde{Y}^-_{1,\alpha}, \widetilde{Y}^+_{1,\alpha}), (\widetilde{Y}^-_{2,\alpha}, \widetilde{Y}^+_{2,\alpha}), \dots$ are independent and identically distributed with $E(|\widetilde{Y}^-_\alpha|) < \infty$ and $E(|\widetilde{Y}^+_\alpha|) < \infty$ for any $\alpha \in [0,1]$. We define $\widetilde{\boldsymbol{X}} = (\widetilde{X}_1, \widetilde{X}_2, \dots)$ by $\widetilde{X}_t = \widetilde{Y}_t - t\widetilde{c}$ $(t \geq 1)$, where $\widetilde{c} \in \widetilde{\mathbb{R}}$ is a fuzzy observation cost with $\widetilde{c} \succ \widetilde{0} := \mathbf{1}_0$.

Proposition *There exists an optimal stopping time function δ^* for $\widetilde{\boldsymbol{X}}$, whose optimal fuzzy perception reward $E(\widetilde{\boldsymbol{X}}_{\delta^*})$ is given as a solution of the following fuzzy optimal relation.*

$$\widetilde{\gamma} = E\widetilde{\max}\{\widetilde{Y} - \widetilde{c}, \widetilde{\gamma} - \widetilde{c}\}, \quad (\widetilde{\gamma} \in \widetilde{\mathbb{R}}). \tag{4.6}$$

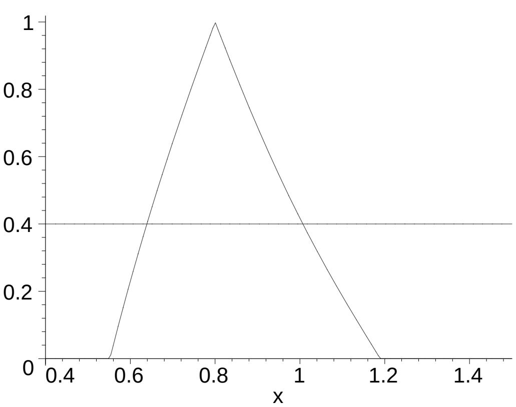
The α-cut expression of (4.6) is as follows.

$$\widetilde{c}^-_\alpha = E\max\{\widetilde{Y}^+_\alpha - \widetilde{\gamma}^+_\alpha, 0\} \tag{4.7}$$

$$\widetilde{c}^+_\alpha = E\max\{\widetilde{Y}^-_\alpha - \widetilde{\gamma}^-_\alpha, 0\} \quad \alpha \in [0,1].$$

Example 2 (The infinite horizon case)
As a numerical example, we will compute the solution $\widetilde{\gamma}$ of the fuzzy optimal relation (4.6) or (4.7) under the same assumption on Example 1 (The finite horizon case) described by simple triangular fuzzy numbers $(a, m, b)(x)$. Let $\widetilde{Y} = (U_2, U_1, U_3), \widetilde{Y}_t = (U_{2,t}, U_{1,t}, U_{3,t})$ $(t = 1, 2, \dots)$ where $U_1, U_{1,t} \sim U[0,1]$ and $U_2, U_3, U_{2,t} U_{3,t} \sim U[0, 1/2]$. Let $\widetilde{c} = (0.01, 0.02, 0.01)$ be a fuzzy observation cost.

The graph of $\widetilde{\gamma}$ and the value of $\widetilde{\gamma}^-_0$ and $\widetilde{\gamma}^+_0$ evaluated by Maple 7 are shown in Fig. 3. We also observe that $\widetilde{\gamma}$ is concave on its left-side slope and convex on its right-side slope.



$$\widetilde{\gamma}^-_1 = \widetilde{\gamma}^+_1 = 0.8$$
$$\widetilde{\gamma}^-_{0.4} = 0.6390697674 \qquad \widetilde{\gamma}^+_{0.4} = 1.006893910$$
$$\widetilde{\gamma}^-_0 = 0.5518936209 \qquad \widetilde{\gamma}^+_0 = 1.189297951$$

Figure 3. The graph of $\widetilde{\gamma}$ and the value of $\widetilde{\gamma}^-_0$ and $\widetilde{\gamma}^+_0$

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