On Multi-Object Fuzzy Stopping for Sequences of Fuzzy Random Variables

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Abstract: In a stochastic and fuzzy environment, a multi-objective fuzzy stopping problemis discussed. The randomness and fuzziness are evaluated by expectations and scalarization functions respectively. Pareto optimal fuzzy stopping times are given under the assumption of regularityfor stopping rules, by using λ-optimal stopping times.

Keyword: Multi-objective optimal stopping; Fuzzy stochastic systems; Fuzzy stopping; Pareto optimal.

1. Introduction

This paper deals with a multi-objective fuzzy stopping model for 'fuzzy stochastic systems' introduced by sequences of fuzzy random variables. The 'fuzzy random variable', which is a fuzzy-number-valued extension of classical random variables, was first studied by Puri and Ralescu [7] and has been discussed bymany authors. It is one of the successful hybrid notions of randomness and fuzziness. On the other hand, stopping problems for a sequence of real-valued random variables were studied by many authors, and their applications are well-known in various fields (Chow et al. [2], Shiryayev [9]). The optimal fuzzy stopping for fuzzy random variables is discussed by Yoshida etal. [15], and also optimal stopping models for fuzzy systems without randomness are studied by Yoshida [12, 13, 14]. This paper analyze a multi-objective stopping model for fuzzy stochastic systems, by extending the results of the classical stochastic systems (Aubin [1], Ohtsubo [6]).

In this paper, we also discuss the optimization by 'fuzzy' stopping times. Fuzzy stopping times are introduced for dynamic fuzzy systems by Kurano et al. [5] and they are discussed by Yoshida et al. [11], and this paper applies the notion of fuzzy stopping times in a stochastic and fuzzy environment. In this paper, we evaluate the randomness and fuzziness regarding the stopped fuzzy stochastic systems by probablistic expectations and scalarization functions respectively. And we give Pareto optimal stopping times for the multi-objective model, by introducing the notion of λ-optimal stopping times.

In Section 2, the notations and definitions of fuzzy random variables are given. In Section 3, fuzzy stopping times are introduced. We formulate a multi-objective optimal stopping problem for fuzzy stochastic systems by fuzzy stopping times and we give Pareto optimal fuzzy stopping times for the problem under the assumption of regularity for stopping rules. Finally, in Section 4, a numerical example is given to illustrate our idea.

2. Fuzzy random variables

Some mathematical notations of fuzzy random variables are given in this section. Let $(\Omega, \mathcal{M}, P)$ be a non-atomic probability space, where $\mathcal{M}$is a σ-field and P is a probability measure. Let $\mathbf{R}$ be the set of all real numbers, let $\mathcal{B}$ denote the Borel σ-field of $\mathbf{R}$ and let $\mathcal{I}$ denote the set of all bounded closed sub-intervals of $\mathbf{R}$. A fuzzy number is denoted by its membership function $\tilde{a} : \mathbf{R} \mapsto [0,1]$ which is normal, upper-semicontinuous, fuzzy convex and has a compact support. Refer to Zadeh [16] for the theory of fuzzy sets. $\mathcal{R}$ denotes the set of all fuzzy numbers. The α-cut of a fuzzy number $\tilde{a}(\in \mathcal{R})$ is given by

$$\tilde{a}_\alpha := \{x \in \mathbf{R} \mid \tilde{a}(x) \geq \alpha\} \ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{a}_0 := \mathrm{cl}\{x \in \mathbf{R} \mid \tilde{a}(x) > 0\},$$

where cl denotes the closure of an interval. In this paper, we write the closed intervals by

$$[\tilde{a}]_\alpha := [[\tilde{a}]_\alpha^-, [\tilde{a}]_\alpha^+] \quad \text{for } \alpha \in [0,1].$$

A map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a fuzzy random variable if

$$\{(\omega, x) \in \Omega \times \mathbf{R} \mid \tilde{X}(\omega)(x) \geq \alpha\} \in \mathcal{M} \times \mathcal{B} \quad \text{for all } \alpha \in [0,1]. \tag{2.1}$$

The condition (2.1) is also written as

$$\{(\omega, x) \in \Omega \times \mathbf{R} \mid x \in [\tilde{X}(\omega)]_\alpha\} \in \mathcal{M} \times \mathcal{B} \quad \text{for all } \alpha \in [0,1], \tag{2.2}$$

where $[\tilde{X}(\omega)]_\alpha = [[\tilde{X}(\omega)]_\alpha^-, [\tilde{X}(\omega)]_\alpha^+] := \{x \in \mathbf{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ is the α-cut of the fuzzy number $\tilde{X}(\omega)$ for $\omega \in \Omega$. We can find some equivalent conditions ([8]), however, in this paper, we adopt a simple equivalent condition in the followinglemma.

Lemma 2.1 (Wang and Zhang [10, Theorems 2.1 and 2.2]). *For a map $\tilde{X} : \Omega \mapsto \mathcal{R}$, the following (i) and (ii) are equivalent:*

(i) *$\tilde{X}$ is a fuzzy random variable.*

(ii) *The maps $\omega \mapsto [\tilde{X}(\omega)]_\alpha^-$ and $\omega \mapsto [\tilde{X}(\omega)]_\alpha^+$ are measurable for all $\alpha \in [0,1]$.*

Now we introduce expectations of fuzzy random variables for the description of stopping models for fuzzy stochastic systems. A fuzzy random variable $\tilde{X}$ is called integrably bounded if $\omega \mapsto [\tilde{X}(\omega)]_\alpha^-$ and $\omega \mapsto [\tilde{X}(\omega)]_\alpha^+$ are integrable for all $\alpha \in [0,1]$. Let $\tilde{X}$ be an integrably bounded fuzzy random variable. We put closed intervals

$$[E(\tilde{X})]_\alpha := \left[\int_\Omega [\tilde{X}(\omega)]_\alpha^- \,\mathrm{d}P(\omega), \int_\Omega [\tilde{X}(\omega)]_\alpha^+ \,\mathrm{d}P(\omega)\right], \quad \alpha \in [0,1]. \tag{2.3}$$

Since the map $\alpha \mapsto [E(\tilde{X})]_\alpha$ is left-continuous by the monotone convergence theorem, the expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number ([4, Lemma 3]):

$$E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min \left\{\alpha, 1_{[E(\tilde{X})]_\alpha}(x)\right\} \quad \text{for } x \in \mathbf{R}, \tag{2.4}$$

where 1_D is the classical indicator function of a set D.

3. A multi-objective fuzzy stopping problem

Let k be a positive integer. In this section, we formulate a multi-objective optimal'fuzzy' stopping problemfor k fuzzy stochastic systems and we give Pareto optimal solutions for the problem. Let $\{1, 2, \cdots, k\}$ denote the set of k objects which are described by fuzzy stochastic systems with the time space $\mathbf{N} := \{0, 1, 2, \cdots\}$. For an object $i = 1, 2, \cdots, k$, let $\{\tilde{X}_n^i\}_{n=0}^{\infty}$ be a sequence of fuzzy random variables such that such that

$$E\left(\max_{1\leq i\leq k}\sup_{n\geq 0}[\tilde{X}_n^i(\omega)]_0^+]\right) < \infty \quad \text{and} \quad E\left(\min_{1\leq i\leq k}[\tilde{X}_n^i(\omega)]_0^-]\right) > -\infty$$

for $n = 0, 1, 2, \cdots$, where the interval $[[\tilde{X}_n^i(\omega)]_0^-, [\tilde{X}_n^i(\omega)]_0^+]$ is the 0-cut of the fuzzy number $\tilde{X}_n^i(\omega)$. For $n = 0, 1, 2, \cdots$, $\mathcal{M}_n$ denotes the smallest σ-field on Ω generated by all random variables $[\tilde{X}_n^i(\omega)]_\alpha^-$ and $[\tilde{X}_n^i(\omega)]_\alpha^+$ $(i = 1, 2, \cdots, k; m = 0, 1, 2, \cdots, n; \alpha \in [0, 1])$, and $\mathcal{M}_\infty$ denotes the smallest σ-field containing $\bigcup_{n=0}^{\infty}\mathcal{M}_n$. Then we call $\left(\{\tilde{X}_n^i\}_{n=0}^{\infty}, \{\mathcal{M}_n\}_{n=0}^{\infty}\right)$ the fuzzy stochastic system for an object i. A map $\tau : \Omega \mapsto \mathbf{N} \cup \{\infty\}$ is called a stopping time if it satisfies

$$\{\omega \mid \tau(\omega) = n\} \in \mathcal{M}_n \quad \text{for all } n = 0, 1, 2, \cdots. \tag{3.1}$$

Then we have the following lemma which is trivial from the definitions.

Lemma 3.1. *Let $i = 1, 2, \cdots, k$ be an object and let τ be a finite stopping time. We define*

$$\tilde{X}_\tau^i(\omega) := \tilde{X}_n^i(\omega), \quad \omega \in \{\tau = n\} \quad \text{for } n = 0, 1, 2, \cdots. \tag{3.2}$$

Then, $\tilde{X}_\tau^i$ is a fuzzy random variable.

Now, for an object i, we consider the estimation of the fuzzy stochastic system stopped at a finite stopping time τ, by the evaluation of the fuzzy random variable $\tilde{X}_\tau^i$. Let $g : \mathcal{I} \mapsto \mathbf{R}$ be a σ-additively homogeneous map, that is, g satisfies the following (3.3) and (3.4):

$$g\left(\sum_{n=0}^{\infty} c_n\right) = \sum_{n=0}^{\infty} g(c_n) \tag{3.3}$$

for bounded closed intervals $\{c_n\}_{n=0}^{\infty} \subset \mathcal{I}$ such that $\sum_{n=0}^{\infty} c_n \in \mathcal{I}$, and

$$g(\mu c) = \mu g(c) \tag{3.4}$$

for bounded closed intervals $c \in \mathcal{I}$ and real numbers $\mu \geq 0$, where the operation on closed intervals is defined ordinary as $\sum_{n=0}^{\infty} c_n := \mathrm{cl}\{\sum_{n=0}^{\infty} x_n \mid x_n \in c_n, n = 0, 1, 2, \cdots\}$ and $\mu c := \{\mu x \mid x \in c\}$. Weighting functions, which satisfy (3.3) and (3.4), are used for the evaluation of fuzzy numbers (Fortemps and Roubens [3]). ¿From (3.2), for $\omega \in \Omega$, the α-cut of the fuzzy number $\tilde{X}_\tau^i(\omega)$ must be a closed interval $[\tilde{X}_\tau^i(\omega)]_\alpha$. Therefore, from the definition (2.3), the expectation is given by the closed interval

$$E([\tilde{X}_\tau^i(\cdot)]_\alpha). \tag{3.5}$$

Using the above scalarization function g, we put

$$g(E([\tilde{X}^i_\tau(\cdot)]_\alpha)). \tag{3.6}$$

Therefore, the evaluation of the fuzzy random variable $\tilde{X}^i_\tau$ is represented by the following integral:

$$\int_0^1 g(E([\tilde{X}^i_\tau(\cdot)]_\alpha))\,\mathrm{d}\alpha. \tag{3.7}$$

Lemma 3.2. *For an object $i = 1, 2, \cdots, k$ and a finite stopping time τ, it holds that*

$$\int_0^1 g(E([\tilde{X}^i_\tau(\cdot)]_\alpha))\,d\alpha = \int_0^1 E(g([\tilde{X}^i_\tau(\cdot)]_\alpha))\,d\alpha = E\left(\int_0^1 g([\tilde{X}^i_\tau(\cdot)]_\alpha)\,d\alpha\right). \tag{3.8}$$

Proof. The properties (3.3) and (3.4) of g imply

$$g(E([\tilde{X}^i_\tau(\cdot)]_\alpha)) = E(g([\tilde{X}^i_\tau(\cdot)]_\alpha)).$$

Therefore

$$\int_0^1 g(E([\tilde{X}^i_\tau(\cdot)]_\alpha))\,\mathrm{d}\alpha = \int_0^1 E(g([\tilde{X}^i_\tau(\cdot)]_\alpha))\,\mathrm{d}\alpha.$$

Also, by Fubini's theorem, we have

$$\int_0^1 E(g([\tilde{X}^i_\tau(\cdot)]_\alpha))\,\mathrm{d}\alpha = E\left(\int_0^1 g([\tilde{X}^i_\tau(\cdot)]_\alpha)\,\mathrm{d}\alpha\right).$$

These complete the proof of this lemma. □

In the following definition, we modify fuzzy stopping times introduced by Kurano et al. [5] in order to apply them to fuzzy random variables.

Definition 3.1. A map $\tilde{\tau} : \mathbf{N} \times \Omega \mapsto [0, 1]$ is called a fuzzy stopping time if it satisfies the following (i) – (iii):

(i) For each $n = 0, 1, 2, \cdots$, the map $\omega \mapsto \tilde{\tau}(n, \omega)$ is $\mathcal{M}_n$-measurable.

(ii) For almost all $\omega \in \Omega$, the map $n \mapsto \tilde{\tau}(n, \omega)$ is non-increasing.

(iii) For almost all $\omega \in \Omega$, there exists an integer m such that $\tilde{\tau}(n, \omega) = 0$ for all $n \geq m$.

Regarding the grade of membership of fuzzy stopping times, '$\tilde{\tau}(n, \omega) = 0$' means 'to stop at time n' and '$\tilde{\tau}(n, \omega) = 1$'means 'to continue at time n' respectively. And the intermediate value '$0 < \tilde{\tau}(n, \omega) < 1$' is a notion of 'fuzzy stopping'. It is easy to check the following lemma regarding construction of fuzzy stoppingtimes ([5]).

Lemma 3.3.

(i) *Let $\tilde{\tau}$ be a fuzzy stopping time. Define a map $\tilde{\tau}_\alpha : \Omega \mapsto \mathbf{N}$ by*

$$\tilde{\tau}_\alpha(\omega) := \inf\{n \mid \tilde{\tau}(n,\omega) < \alpha\}, \quad \omega \in \Omega \quad \textit{for } \alpha \in (0,1], \tag{3.9}$$

where the infimum of the empty set is understood to be $+\infty$. Then, we have:

(a) $\{\tilde{\tau}_\alpha \le n\} \in \mathcal{M}_n$ *for* $n = 0, 1, 2, \cdots$;

(b) $\tilde{\tau}_\alpha(\omega) \le \tilde{\tau}_{\alpha'}(\omega)$ a.a. $\omega \in \Omega$ *if* $\alpha \ge \alpha'$;

(c) $\lim_{\alpha' \uparrow \alpha} \tilde{\tau}_{\alpha'}(\omega) = \tilde{\tau}_\alpha(\omega)$ a.a. $\omega \in \Omega$ *if* $\alpha > 0$;

(d) $\tilde{\tau}_0(\omega) := \lim_{\alpha \downarrow 0} \tilde{\tau}_\alpha(\omega) < \infty$ a.a. $\omega \in \Omega$.

(ii) *Let $\{\tilde{\tau}_\alpha\}_{\alpha \in [0,1]}$ be maps $\tilde{\tau}_\alpha : \Omega \mapsto \mathbf{N}$ satisfying the above (a) (b) and (d). Define a map $\tilde{\tau} : \mathbf{N} \times \Omega \mapsto [0,1]$ by*

$$\tilde{\tau}(n,\omega) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{\{\tilde{\tau}_\alpha > n\}}(\omega)\} \quad \textit{for } n = 0, 1, 2, \cdots \textit{ and } \omega \in \Omega. \tag{3.10}$$

Then $\tilde{\tau}$ is a fuzzy stopping time.

Fuzzy stopping times are always finite from Definition 3.1(iii). Now we consider the estimation of the fuzzy stochastic system stopped at a 'fuzzy' stopping time $\tilde{\tau}$ regarding the i-th object. Let $i = 1, 2, \cdots, k$ be an object and let $\tilde{\tau}$ be a fuzzy stopping time. ¿From Lemma 3.1, we have $[\tilde{X}^i_{\tilde{\tau}_\alpha}(\omega)]_\alpha := [\tilde{X}^i_n(\omega)]_\alpha$ for $\omega \in \{\tilde{\tau}_\alpha = n\}$, where $\tilde{\tau}_\alpha(\omega)$ are 'classical' stopping times given by (3.9). By Lemma 3.2, we define a random variable

$$G^i_{\tilde{\tau}}(\omega) := \int_0^1 g([\tilde{X}^i_{\tilde{\tau}_\alpha}(\omega)]_\alpha)\,\mathrm{d}\alpha, \quad \omega \in \Omega. \tag{3.11}$$

Note that (3.11) is well-defined since the function $\alpha \mapsto g([\tilde{X}^i_{\tilde{\tau}_\alpha}(\omega)]_\alpha)$ is left-continuous on $(0,1]$. Therefore the expectation $E(G^i_{\tilde{\tau}})$ is the evaluation (3.7) of the fuzzy random variable $\tilde{X}_{\tilde{\tau}}$. By Fubini's theorem, we have

$$E(G^i_{\tilde{\tau}}) := E\left(\int_0^1 g([\tilde{X}^i_{\tilde{\tau}_\alpha}(\cdot)]_\alpha)\,\mathrm{d}\alpha\right) = \int_0^1 E(g([\tilde{X}^i_{\tilde{\tau}_\alpha}(\cdot)]_\alpha))\,\mathrm{d}\alpha \tag{3.12}$$

for fuzzy stopping times $\tilde{\tau}$. Then, Pareto optimal solutions for the multi-objective stopping model are characterized as follows.

Definition 3.2. A fuzzy stopping time $\tilde{\tau}^*$ is called Pareto optimal if there exists no fuzzy stopping time $\tilde{\tau}$ such that

$$E(G^i_{\tilde{\tau}}) \ge E(G^i_{\tilde{\tau}^*}) \quad \text{for all objects } i = 1, 2, \cdots, k$$

and

$$E(G^i_{\tilde{\tau}}) > E(G^i_{\tilde{\tau}^*}) \quad \text{for some object } i = 1, 2, \cdots, k.$$

We introduce the following λ-optimal stopping times in order to obtain Pareto optimal stopping times. Real numbers $\{\lambda^i\}_{i=1}^k$ are called weights of objects if they satisfy

$$\sum_{i=1}^k \lambda^i = 1 \quad \text{and} \quad \lambda^i \geq 0 \quad (i = 1, 2, \cdots, k). \tag{3.13}$$

For a set of weights $\lambda := \{\lambda^i\}_{i=1}^k$, we define a fuzzy stochastic system $\{\tilde{X}_n^\lambda\}_{n=0}^\infty$, which is $\{\mathcal{M}_n\}_{n=0}^\infty$-adapted, by

$$\tilde{X}_n^\lambda(\omega)(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{[\tilde{X}_n^\lambda(\omega)]_\alpha}(x)\}, \quad \omega \in \Omega,\ x \in \mathbf{R},$$

where the α-cuts $[\tilde{X}_n^\lambda(\omega)]_\alpha$ are closed intervals given by

$$[\tilde{X}_n^\lambda(\omega)]_\alpha = \left[\sum_{i=1}^k \lambda^i [\tilde{X}_n^i(\omega)]_\alpha^-, \sum_{i=1}^k \lambda^i [\tilde{X}_n^i(\omega)]_\alpha^+\right], \quad \omega \in \Omega,\ n = 0, 1, 2, \cdots.$$

For fuzzy stopping times $\tilde{\tau}$, we define a random variable

$$G_{\tilde{\tau}}^\lambda(\omega) := \int_0^1 g([\tilde{X}_{\tilde{\tau}_\alpha}^\lambda(\omega)]_\alpha)\, \mathrm{d}\alpha \quad \text{for } \omega \in \Omega.$$

Similarly to the proof of Lemma 3.2, we can easily check that its expectation isreduced to

$$E(G_{\tilde{\tau}}^\lambda) = \sum_{i=1}^k \lambda^i E(G_{\tilde{\tau}}^i). \tag{3.14}$$

Now we give the definition of λ-optimal stopping times as follows.

Definition 3.3. Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects. Then a fuzzy stopping time $\tilde{\tau}^*$ is called λ-optimal if

$$E(G_{\tilde{\tau}^*}^\lambda) \geq E(G_{\tilde{\tau}}^\lambda)$$

for all fuzzy stopping times $\tilde{\tau}$.

Theorem 3.1. *Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects such that*

$$\sum_{i=1}^k \lambda^i = 1 \quad \textit{and} \quad \lambda^i > 0 \quad (i = 1, 2, \cdots, k). \tag{3.15}$$

Then a λ-optimal fuzzy stopping time $\tilde{\tau}^*$ is Pareto optimal.

Proof. Let $\tilde{\tau}^*$ be a finite λ-optimal fuzzy stopping time. If $\tilde{\tau}^*$ is not Pareto optimal, then there exists a fuzzy stopping time $\tilde{\tau}$ such that

$$E(G_{\tilde{\tau}}^i) \geq E(G_{\tilde{\tau}^*}^i) \quad \text{for all objects } i = 1, 2, \cdots, k$$

and

$$E(G_{\tilde{\tau}}^i) > E(G_{\tilde{\tau}^*}^i) \quad \text{for some object } i = 1, 2, \cdots, k.$$

Then from (3.14) we have

$$E(G^{\lambda}_{\tilde{\tau}}) = \sum_{i=1}^{k} \lambda^i E(G^i_{\tilde{\tau}}) > \sum_{i=1}^{k} \lambda^i E(G^i_{\tilde{\tau}^*}) = E(G^{\lambda}_{\tilde{\tau}^*}).$$

This contradicts the λ-optimality of $\tilde{\tau}^*$, and so we obtain this theorem. □

Finally, in order to construct λ-optimal fuzzy stopping times, we introduce the following (λ, α)-optimal fuzzy stopping times.

Definition 3.4. Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects andlet $\alpha \in [0,1]$. A fuzzy stopping time $\tilde{\tau}^*$ is called (λ, α)-optimal if

$$E(g([\tilde{X}^{\lambda}_{\tilde{\tau}^*_\alpha}(\cdot)]_\alpha)) \geq E(g([\tilde{X}^{\lambda}_{\tilde{\tau}_\alpha}(\cdot)]_\alpha))$$

for all fuzzy stopping times $\tilde{\tau}$.

In order to characterize (λ, α)-optimal stopping times, we let

$$\gamma^{\lambda}_{n,\alpha} := \operatorname*{ess\,sup}_{\tau:\ \text{stopping times},\, \tau \geq n} E(g([\tilde{X}^{\lambda}_{\tau}(\cdot)]_\alpha) | \mathcal{M}_n) \quad \text{for } n = 0, 1, 2, \cdots, \tag{3.16}$$

where the definition of the essential supremum is referred to [2, Chapter 1-6]. Define a stopping time $\sigma^{\lambda}_{\alpha} : \Omega \mapsto \mathbf{N}$ by

$$\sigma^{\lambda}_{\alpha}(\omega) := \inf \left\{ n \mid g([\tilde{X}^{\lambda}_n(\omega)]_\alpha) = \gamma^{\lambda}_{n,\alpha}(\omega) \right\} \tag{3.17}$$

for $\omega \in \Omega$ and $\alpha \in [0,1]$, where the infimum of the empty set is understood to be $+\infty$. Then the following lemma can be checked easily by Chow et al. [2, Theorem 4.1].

Lemma 3.4. *Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects. Suppose*

$$P(\sigma^{\lambda}_{\alpha} < \infty) = 1 \quad \textit{for all } \alpha \in [0,1]. \tag{3.18}$$

Then, for $\alpha \in [0,1]$, the following (i) and (ii) hold:

(i) $\gamma^{\lambda}_{n,\alpha}(\omega) = \max\{g([\tilde{X}^{\lambda}_n(\omega)]_\alpha), \gamma^{\lambda}_{n+1,\alpha}(\omega)\}$ a.a. $\omega \in \Omega$ *for* $n = 0, 1, 2, \cdots$;

(ii) $\sigma^{\lambda}_{\alpha}$ *is (λ, α)-optimal and* $E(\gamma^{\lambda}_{0,\alpha}) = E(g([\tilde{X}^{\lambda}_{\sigma^{\lambda}_{\alpha}}(\cdot)]_\alpha))$.

In order to construct an optimal fuzzy stopping time from the (λ, α)-optimal stopping times $\{\sigma^{\lambda}_{\alpha}\}_{\alpha \in [0,1]}$, we need the following regularity condition.

Assumption A (Regularity). The map $\alpha \mapsto \sigma^{\lambda}_{\alpha}(\omega)$ is non-increasing for almost all $\omega \in \Omega$.

Under Assumption A, we can define a map $\tilde{\sigma}^{\lambda} : \mathbf{N} \times \Omega \mapsto [0,1]$ by

$$\tilde{\sigma}^{\lambda}(n, \omega) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{\{\sigma^{\lambda}_{\alpha} > n\}}(\omega)\} \quad \text{for } n = 0, 1, 2, \cdots \text{ and } \omega \in \Omega. \tag{3.19}$$

Put the α-cut (3.9) of $\tilde{\sigma}^\lambda(n,\omega)$ by $\tilde{\sigma}^\lambda_\alpha(\omega)$. Then $\tilde{\sigma}^\lambda_\alpha(\omega)$ and $\sigma^\lambda_\alpha(\omega)$ may not equal only at most countable many $\alpha \in (0,1]$, so we obtain the following result.

Theorem 3.2. *Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects satisfying (3.15). Suppose (3.18) and Assumption A hold. Then $\tilde{\sigma}^\lambda$ is a λ-optimal fuzzy stopping time and it is also Pareto optimal.*

Proof. ¿From Assumption A and Lemma 3.3(ii), $\tilde{\sigma}^\lambda$ is a fuzzy stopping time. ¿From Lemma 3.4, for all fuzzy stopping times $\tilde{\tau}$ we obtain

$$E(G^\lambda_{\tilde{\tau}}) \le \int_0^1 \sup_\tau E\left(g([\tilde{X}^\lambda_\tau(\cdot)]_\alpha)\right) \mathrm{d}\alpha = \int_0^1 E(\gamma^\lambda_{0,\alpha}) \,\mathrm{d}\alpha = \int_0^1 E(g([\tilde{X}^\lambda_{\sigma^\lambda_\alpha}(\cdot)]_\alpha)) \,\mathrm{d}\alpha. \quad (3.20)$$

Since $\tilde{\sigma}^\lambda_\alpha(\omega) \neq \sigma^\lambda_\alpha(\omega)$ holds only at most countable $\alpha \in (0,1]$, we have

$$E\left(\int_0^1 g([\tilde{X}^\lambda_{\sigma^\lambda_\alpha}(\cdot)]_\alpha) \,\mathrm{d}\alpha\right) = E\left(\int_0^1 g([\tilde{X}^\lambda_{\tilde{\sigma}^\lambda_\alpha}(\cdot)]_\alpha) \,\mathrm{d}\alpha\right). \quad (3.21)$$

By (3.20), (3.21) and Fubini's theorem, we obtain

$$\begin{aligned} E(G^\lambda_{\tilde{\tau}}) &\le \int_0^1 E(g([\tilde{X}^\lambda_{\sigma^\lambda_\alpha}(\cdot)]_\alpha)) \,\mathrm{d}\alpha \\ &= E\left(\int_0^1 g([\tilde{X}^\lambda_{\sigma^\lambda_\alpha}(\cdot)]_\alpha) \,\mathrm{d}\alpha\right) \\ &= E\left(\int_0^1 g([\tilde{X}^\lambda_{\tilde{\sigma}^\lambda_\alpha}(\cdot)]_\alpha) \,\mathrm{d}\alpha\right) \\ &= \int_0^1 E(g([\tilde{X}^\lambda_{\tilde{\sigma}^\lambda_\alpha}(\cdot)]_\alpha)) \,\mathrm{d}\alpha \\ &= E(G^\lambda_{\tilde{\sigma}^\lambda}). \end{aligned}$$

Therefore $\tilde{\sigma}^\lambda$ is λ-optimal. We also obtain Pareto optimality of $\tilde{\sigma}^\lambda$ from Theorem 4.1. □

4. A numerical example

An example is given to illustrate our idea of the multi-objective optimal fuzzystopping problem in Section 3. In this example, k objects mean k assets in a financial market $\{B_n\}_{n=0}^\infty$ which is a sequence of real random variables. We assume that $\{B_n\}_{n=0}^\infty$ is a random walk:

$$B_n := \sum_{m=0}^n W_m \quad n = 0,1,2,\cdots, \quad (4.1)$$

where $\{W_n\}_{n=0}^\infty$ is a sequence of independent random variables such that

$$P(W_n = -1) = 1/2 \quad \text{and} \quad P(W_n = 1) = 1/2 \quad \text{for } n = 0,1,2,\cdots. \quad (4.2)$$

The price of each asset $i(=1,2,\cdots,k)$ is described by a fuzzy stochastic system $\{\tilde{X}_n^i\}_{n=0}^{\infty}$ as follows.

$$Y_n^i := p^i + r^i n + v^i B_n, \quad i = 1,2,\cdots,k. \tag{4.3}$$

where p^i, r^i and v^i are constants such that p^i is the initial price of an asset i in the market, r^i is the rate of growth of the asset price in the market, and v^i is the volatility of the asset i in the market. Let c^i and d^i be constants satisfying $0 < d^i < 3(c^i - r^i)$. We put

$$M_n^i := Y_n^i - c^i(n+1) \quad \text{for } n = 0,1,2,\cdots. \tag{4.4}$$

where $c^i(n+1)$ means the maintenance cost for asset i. Hence we take a sequence of fuzzy random variables $\{\tilde{X}_n^i\}_{n=0}^{\infty}$ by

$$\tilde{X}_n^i(\omega)(x) := \begin{cases} L((M_n^i(\omega) - x)/a_n^i) & \text{if } x \le M_n^i(\omega) \\ L((x - M_n^i(\omega))/a_n^i) & \text{if } x \ge M_n^i(\omega) \end{cases} \tag{4.5}$$

for $n = 0,1,2,\cdots$, $\omega \in \Omega$ and $x \in \mathbf{R}$ (see Figure 1), where $\{a_n^i\}_{n=0}^{\infty}$ is a sequence given by $a_n^i := d^i(n+1)$ $(n = 0,1,2,\cdots)$ and the shape function is given by $L(x) := \max\{1-|x|, 0\}$ $(x \in \mathbf{R})$. The corresponding σ-field $\mathcal{M}_n$ is the smallest σ-field generated by the random variables $W_0, W_1, W_2, \cdots, W_n$. Then their α-cuts are

$$[\tilde{X}_n^i(\omega)]_\alpha = [M_n^i(\omega) - (1-\alpha)a_n^i, M_n^i(\omega) + (1-\alpha)a_n^i], \quad \omega \in \Omega$$

for $n = 0,1,2,\cdots$ and $\alpha \in [0,1]$.

Figure 1.

Now we take a weighting function by $g([x,y]) := (x+2y)/3$ for $x, y \in \mathbf{R}$ satisfying $x \le y$. Then g satisfies the properties (3.3) and (3.4), and we can easily check

$$g([\tilde{X}_n^i(\omega)]_\alpha) = M_n^i(\omega) + \frac{1-\alpha}{3} a_n^i, \quad \omega \in \Omega$$

for $\alpha \in [0,1]$, and so

$$G_n^i(\omega) = \int_0^1 g([\tilde{X}_n^i(\omega)]_\alpha)\,\mathrm{d}\alpha = M_n^i(\omega) + \frac{1}{6} a_n^i, \quad \omega \in \Omega.$$

Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for assets satisfying (3.15). It means a kind of portfolio for the assets. Then we have

$$G_n^\lambda(\omega) = \sum_{i=1}^k \lambda^i \int_0^1 g([\tilde{X}_n^i(\omega)]_\alpha)\,\mathrm{d}\alpha = \sum_{i=1}^k \lambda^i M_n^i(\omega) + \frac{1}{6}\sum_{i=1}^k \lambda^i a_n^i, \quad \omega \in \Omega.$$

For simplicity, we put

$$M_n^\lambda(\omega) := \sum_{i=1}^k \lambda^i M_n^i(\omega) \quad \text{and} \quad a_n^\lambda := \sum_{i=1}^k \lambda^i a_n^i.$$

Hence we check Assumption A. Let $\alpha, \alpha' \in [0,1]$ satisfy $\alpha' \le \alpha$ and let $\omega \in \Omega$. If $g(\tilde{X}_{n,\alpha'}^\lambda(\omega)) = \gamma_{n,\alpha'}^\lambda(\omega)$ forsome n, then we have

$$\begin{aligned} g([\tilde{X}_n^\lambda(\omega)]_\alpha) &= M_n^\lambda(\omega) + \tfrac{1-\alpha}{3} a_n^\lambda \\ &= M_n^\lambda(\omega) + \tfrac{1-\alpha'}{3} a_n^\lambda + \tfrac{\alpha'-\alpha}{3} a_n^\lambda \\ &= g(\tilde{X}_{n,\alpha'}^\lambda(\omega)) + \tfrac{\alpha'-\alpha}{3} a_n^\lambda \\ &= \gamma_{n,\alpha'}^\lambda(\omega) + \tfrac{\alpha'-\alpha}{3} a_n^\lambda \\ &\ge E(g(\tilde{X}_{\tau,\alpha'}^\lambda) \mid \mathcal{M}_n)(\omega) + \tfrac{\alpha'-\alpha}{3} E(a_\tau^\lambda \mid \mathcal{M}_n)(\omega) \\ &= E\left(g(\tilde{X}_{\tau,\alpha'}^\lambda) + \tfrac{\alpha'-\alpha}{3} a_\tau^\lambda \mid \mathcal{M}_n\right)(\omega) \\ &= E(g([\tilde{X}_\tau^\lambda(\cdot)]_\alpha) \mid \mathcal{M}_n)(\omega) \quad \text{a.a. } \omega \in \Omega \end{aligned}$$

for all stopping times τ such that $\tau \ge n$. It follows $g([\tilde{X}_n^\lambda(\omega)]_\alpha) = \gamma_{n,\alpha}^\lambda(\omega)$. Therefore we obtain $\tilde{\sigma}_\alpha^*(\omega) \le \tilde{\sigma}_{\alpha'}^*(\omega)$ for almost all $\omega \in \Omega$, and Assumption A is fulfilled. On the other hand we have

$$\begin{aligned} g([\tilde{X}_n^\lambda(\omega)]_\alpha) &= \sum_{i=1}^k \lambda^i g([\tilde{X}_n^\lambda(\omega)]_\alpha) \\ &= \sum_{i=1}^k \lambda^i \left(p^i - r^i\right) + \sum_{i=1}^k \lambda^i v^i B_n(\omega) - \sum_{i=1}^k \lambda^i \left(c^i - r^i - \frac{1-\alpha}{3} d^i\right)(n+1). \end{aligned}$$

Therefore the finiteness (3.18) is trivial from Chow et al. [2, Theorem 4.5] since

$$\sum_{i=1}^k \lambda^i \left(c^i - r^i - \frac{1-\alpha}{3} d^i\right) \ge \sum_{i=1}^k \lambda^i \left(c^i - r^i - \frac{1}{3} d^i\right) > 0.$$

Putting $p^\lambda := \sum_{i=1}^k \lambda^i \left(p^i - r^i\right)$, $v^\lambda := \sum_{i=1}^k \lambda^i v^i$ and $c_\alpha^\lambda := \sum_{i=1}^k \lambda^i \left(c^i - r^i - \frac{1-\alpha}{3} d^i\right)$, we have

$$g([\tilde{X}_n^\lambda(\omega)]_\alpha) = p^\lambda + v^\lambda B_n(\omega) - c_\alpha^\lambda (n+1). \tag{4.6}$$

Thus the finite (λ, α)-optimal stopping times for the problem are

$$\begin{aligned} \sigma_\alpha^*(\omega) &= \inf\left\{ n \mid g([\tilde{X}_n^\lambda(\omega)]_\alpha) = \gamma_{n,\alpha}^\lambda(\omega) \right\} \\ &= \inf\left\{ n \mid \operatorname*{ess\,sup}_{\tau \ge n+1} E\left(v^\lambda B_\tau - c_\alpha^\lambda \tau \mid \mathcal{M}_n\right)(\omega) \le v^\lambda B_n(\omega) - c_\alpha^\lambda n \right\} \\ &= \inf\left\{ n \mid \operatorname*{ess\,sup}_{\tau \ge n+1} E\left(v^\lambda \sum_{m=n+1}^{\tau} W_m(\omega) - c_\alpha^\lambda(\tau - n) \mid \mathcal{M}_n\right)(\omega) \le 0 \right\} \end{aligned}$$

for $\omega \in \Omega$, where $\gamma^{\lambda}_{n,\alpha}$ is given by (3.16). By Theorem 4.1, λ-optimal fuzzy stopping time, which is also one of Pareto optimal stopping times, is given by

$$\tilde{\sigma}^*(n,\omega) = \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{\{\sigma^*_\alpha > n\}}(\omega)\}, \tag{4.7}$$

for $n = 0, 1, 2, \cdots$ and $\omega \in \Omega$. We can easily check that the corresponding optimal expected value for the fuzzy stopping problem is

$$E(G^{\lambda}_{\tilde{\sigma}^*}) = \int_0^1 E(g([\tilde{X}^{\lambda}_{\sigma^*_\alpha}(\cdot)]_\alpha))\,\mathrm{d}\alpha = p^{\lambda} + v^{\lambda} \int_0^1 E(B_{\sigma^*_\alpha})\,\mathrm{d}\alpha - \int_0^1 c^{\lambda}_{\alpha} E(\sigma^*_\alpha + 1)\,\mathrm{d}\alpha \tag{4.8}$$

for a portfolio $\lambda := \{\lambda^i\}_{i=1}^k$ for the assets.

Finally, when letting d^i to zero especially in this example, we note that the fuzzy random variables $\tilde{X}^{\lambda}_n$ are reduced to the 'classical' random variables.

References

[1] J.P.Aubin, *Mathematical Methods of Game and Economic Theory* (North-Holland, Amsterdam, 1979).

[2] Y.S.Chow, H.Robbins and D.Siegmund, *The Theory of Optimal Stopping: Great Expectations* (Houghton Mifflin Company, New York, 1971).

[3] P.Fortemps and M.Roubens, Ranking and defuzzification methods based on area compensation, *Fuzzy Sets and Systems* **82** (1996) 319-330.

[4] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51** (1992) 83-88.

[5] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, An approach to stopping problems of a dynamic fuzzy system, preprint.

[6] Y.Ohtsubo, Multi-objective stopping problem for a monotone case, *Mem. Fac. Sci. Kochi Univ. Ser. A* **18** (1997) 99-104.

[7] M.L.Puri and D.A.Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.* **114** (1986) 409-422.

[8] M.L.Puri and D.A.Ralescu, Convergence theorem for fuzzy martingales, *J. Math. Anal. Appl.* **160** (1991) 107-122.

[9] A.N.Shiryayev, *Optimal Stopping Rules* (Springer, New York, 1979).

[10] G.Wang and Y.Zhang, The theory of fuzzy stochastic processes, *Fuzzy Sets and Systems* **51** (1992) 161-178.

[11] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A monotone fuzzy stopping time for dynamic fuzzy systems, *Bull. Infor. Cyber. Res. Ass. Stat. Sci., Kyushu University* **31** (1999) 91-99.

[12] Y.Yoshida, Markov chains with a transition possibility measure and fuzzy dynamic programming, *Fuzzy Sets and Systems* **66** (1994) 39-57.

[13] Y.Yoshida, An optimal stopping problem in dynamic fuzzy systems with fuzzy rewards, *Computers Math. Appl.* **32** (1996) 17-28.

[14] Y.Yoshida, Duality in dynamic fuzzy systems, *Fuzzy Sets and Systems* **95** (1998) 53-65.

[15] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, Optimal stopping problems of a stochastic and fuzzy system, submitted.

[16] L.A.Zadeh, Fuzzy sets, *Inform. and Control* **8** (1965) 338-353.