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Two Properties of Fuzzy Set Function Deduced from Fuzzy Number-Valued Fuzzy Integral

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Abstract In this paper, we shall discuss two properties of a fuzzy number-valued fuzzy set function deduced from a fuzzy number-valued fuzzy integral over the fuzzy sets. They are the property(s) and the property($p.g.p.$) ,which are concerned with the convergence of sequence of fuzzy measurable functions on fuzzy number-valued fuzzy measure space.

Key words fuzzy number-valued fuzzy measure fuzzy number-valued fuzzy integral property ($p.g.p.$) property(s)

0 Introduction

Zhang[1] introduced a fuzzy number-valued fuzzy measure (which is called as "a (z) fuzzy measure"hereafter)and a (z) fuzzy integral (this means a fuzzy number-valued fuzzy integral)on fuzzy sets. Zhang[2,3]discussed also some properties of a fuzzy number-valued fuzzy set function η deduced from a (z) fuzzy integral over fuzzy sets with respect to a (z) fuzzy measure μ. He proved that the fuzzy number-valued fuzzy set function η preserves some structural characteristics,such as F -additivity,null-additivity,autocontinuity and uniformly autocontinuity, of the original (z) fuzzy measure μ.

The concepts of the pseudometric generating property(it is abrebiated to "($p.g.p.$) "). and the property (s) of a (z) fuzzy measure are introduced by Li[4]. Both the property ($p.g.p.$) and the property (s) of a (z) fuzzy measure play an important role in the study of the convergence for the sequence of fuzzy measurable functions(see[4]). In this paper, we shall discuss, adding the previous characteristics, two properties of a fuzzy number-valued fuzzy set function deduced from a (z) fuzzy integral. It is proved that the property($p.g.p.$) and the property (s) of a (z) fuzzy measure μ conserves that of the fuzzy number-valued fuzzy set function η defined by a (z) fuzzy integral with respect to a (z) fuzzy measure μ.

2 Preliminary definition

In this paper, we shall assume that X is a nonempty set, F^* is the set of all fuzzy numbers,

$F(X) = \{\widetilde{A};\widetilde{A}:X \to [0,1]\}$ is the class of fuzzy subsets of X , $F^{\bullet}(X)$ is a fuzzy σ-algebra of subsets of $F(X)$. All fuzzy set $\widetilde{A}$ are supposed to belonge to $F^{\bullet}(X)$. All concepts and signs not defined in the paper may be found in [1~4].

Definition 1[1] Let $F_+^{\bullet} = \{\tilde{a};\tilde{a} \geqslant \tilde{0},\tilde{a} \in F^{\bullet}\}$. A fuzzy number-valued fuzzy set function $\widetilde{\mu}$: $F^{\bullet}(X) \to F_+^{\bullet}$ is said to be fuzzy number-valued fuzzy measure, called by (z) fuzzy measure, if it satisfies the following conditions:

(ZFM1) $\widetilde{\mu}(\varnothing) = \tilde{0}$;

(ZFM2) $\widetilde{A} \subset \widetilde{B} \Rightarrow \widetilde{\mu}(\widetilde{A}) \leqslant \widetilde{\mu}(\widetilde{B})$;

(ZFM3) $\widetilde{A}_1 \subset \widetilde{A}_2 \subset \cdots \Rightarrow \widetilde{\mu}(\bigcup_{n=1}^{\infty} \widetilde{A}_n) = (\tilde{\rho}) \lim_{n\to\infty} \widetilde{\mu}(\widetilde{A}_n)$;

(ZFM4) $\widetilde{A}_1 \supset \widetilde{A}_2 \supset \cdots$, and $\widetilde{\mu}(\widetilde{A}_1) \neq \tilde{\infty} \Rightarrow (\bigcap_{n=1}^{\infty} \widetilde{A}_n) = (\tilde{\rho}) \lim_{n\to\infty} \widetilde{\mu}(\widetilde{A}_n)$.

Definition 2[4] A fuzzy number-valued fuzzy set function $\widetilde{\mu}$ is said to have the property (s) if $(\tilde{\rho}) \lim_{n\to\infty} \widetilde{\mu}(\widetilde{E}_n) = \tilde{0}$, there exists a subsequence $\{\widetilde{E}_{n(i)}\}_i$ of $\{\widetilde{E}_n\}_n$ such that $\widetilde{\mu}(\overline{\lim_{i\to\infty}} \widetilde{E}_{n(i)}) = \tilde{0}$.

Definition 3[4] A fuzzy number-valued fuzzy set function $\widetilde{\mu}$ is said to have the pseudometric generating property, called by $(p.g.p.)$, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\widetilde{\mu}(\widetilde{E}) \vee \widetilde{\mu}(\widetilde{F}) \leqslant \tilde{\delta} \Rightarrow \widetilde{\mu}(\widetilde{E} \cup \widetilde{F}) \leqslant \tilde{\varepsilon}.$$

Let $f:X \to R^1$ and $\alpha \in R^1$. Put $F_\alpha = \{x;f(x) \geqslant \alpha\}$ and

$$\chi_{F_\alpha}(x) = \begin{cases} 1, & \text{if } x \in F_\alpha, \\ 0, & \text{if } x \notin F_\alpha. \end{cases}$$

Definition 4 f is said to be a fuzzy measurable function if $\chi_{F_\alpha} \in F^{\bullet}(X)$ for any $\alpha \in R$.

Let $M_+^{\bullet}$ denote the class of all non-negative fuzzy measurable function on (z) fuzzy measure space $(X,F^{\bullet}(X),\widetilde{\mu})$. Unless stated otherwise, all fuzzy set $\widetilde{A}$ are supposed to belong to $F^{\bullet}(X)$ and all real functions we consider are assumed to be non-negative fuzzy measurable function.

Definition 5[1] Let $(X,F^{\bullet}(X),\widetilde{\mu})$ be a (z) fuzzy measure space, $\widetilde{A} \in F^{\bullet}(X)$, $f \in M_+^{\bullet}$. The fuzzy number-valued fuzzy integral ((z) fuzzy integral in short) of f on $\widetilde{A}$ with respect to $\widetilde{\mu}$ is defined by

$$\int_{\widetilde{A}} f \mathrm{d}\widetilde{\mu} = \bigcup_{\lambda \in [0,1]} \lambda[\sup_{\alpha \in [0,\infty)} \alpha \wedge (\widetilde{\mu}(\widetilde{A} \cap \chi_{F_\alpha}))_\lambda^-, \sup_{\alpha \in [0,\infty)} \alpha \wedge ((\widetilde{\mu}(\widetilde{A} \cap \chi_{F_\alpha}))_\lambda^+].$$

where $F_\alpha = \{x:f(x) \geqslant \alpha\}, \alpha \in [0,\infty)$.

3 The inheritance of fuzzy set function

In this section, we discuss the inheritance of the fuzzy number-valued fuzzy set function induced by the (z) fuzzy integral.

Definition 6[2] For a given and fixed function $f \in M_+^{\bullet}$, we define $\tilde{\eta}_f:F^{\bullet}(X) \to F_+^{\bullet}$ by

$$\tilde{\eta}_f(\widetilde{A}) = \int_{\widetilde{A}} f \mathrm{d}\widetilde{\mu}.$$

for every $\widetilde{A} \in F^{\bullet}(X)$. $\tilde{\eta}_f$ is called the fuzzy number-valued fuzzy set function induced by f. In this section, $\tilde{\eta}$ is used instead of $\tilde{\eta}_f$ if there is no confusion.

Lemma 1[3] Let $\{\widetilde{A}_n\} \subset F^*(X)$ and $(\tilde{\rho})\ \lim\limits_{n\to\infty}\tilde{\eta}(\widetilde{A}_n) = \tilde{0}$. Then,

$$(\tilde{\rho})\ \lim_{n\to\infty}\tilde{\mu}(\widetilde{A}_n \cap \chi_{F_\beta}) = \tilde{0}$$

for any fixed $\beta > 0$.

Lemma 2 Let any $\alpha \in [0,\infty)$, we have

$$\int_{\widetilde{A}} f \mathrm{d}\tilde{\mu} \leqslant \tilde{\alpha} \Leftrightarrow \tilde{\mu}(\widetilde{A} \cap \chi_{F_\gamma}) \leqslant \tilde{\alpha}, \qquad \forall\, \gamma > \alpha \Leftrightarrow \tilde{\mu}(\widetilde{A} \cap \chi_{F_\alpha}) \leqslant \tilde{\alpha}.$$

Proof If $\tilde{\mu}(\widetilde{A} \cap \chi_{F_\gamma}) \leqslant \tilde{\alpha}$ for any $\gamma > \alpha$, then for any $\lambda \in (0,1]$,

$$(\tilde{\mu}(\widetilde{A} \cap \chi_{F_\alpha}))_\lambda^- \leqslant \alpha \text{ and } (\tilde{\mu}(\widetilde{A} \cap \chi_{F_\alpha}))_\lambda^+ \leqslant \alpha.$$

Therefore,

$$\begin{aligned}
\int_{\widetilde{A}} f \mathrm{d}\tilde{\mu} &= \bigcup_{\lambda\in[0,1]} \lambda[\sup_{s\in[0,\infty)} s \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^-,\ \sup_{s\in[0,\infty)} s \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^+] \\
&= \bigcup_{\lambda\in[0,1]} \lambda[\sup_{s\in[0,\alpha]} s \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^- \vee \sup_{s\in(\alpha,\infty)} s \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^-], \\
&\quad \sup_{s\in[0,\alpha]} s \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^+ \vee \sup_{s\in(\alpha,\infty)} s \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^+] \\
&\leqslant \bigcup_{\lambda\in[0,1]} \lambda[\alpha \vee \sup_{s\in(\alpha,\infty)} s \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^-, \alpha \vee \sup_{s\in(\alpha,\infty)} s \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^+] \\
&\leqslant \bigcup_{\lambda\in[0,1]} \lambda[\alpha \vee \sup_{s\in(\alpha,\infty)} (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^-, \alpha \vee \sup_{s\in(\alpha,\infty)} (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^+] \\
&\leqslant \bigcup_{\lambda\in[0,1]} \lambda[\alpha \vee \alpha, \alpha \vee \alpha] \\
&= \tilde{\alpha}.
\end{aligned}$$

Conversely, let $\int_{\widetilde{A}} f \mathrm{d}\tilde{\mu} \leqslant \tilde{\alpha}$, then for any $\lambda \in (0,1]$, we have

$$\sup_{s\in[0,\infty)} s \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^- \leqslant \alpha \text{ and } \sup_{s\in[0,\infty)} s \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^+ \leqslant \alpha.$$

Therefore,

$$\sup_{s\in(\alpha,\infty)} s \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^- \leqslant \alpha \text{ and } \sup_{s\in(\alpha,\infty)} s \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_s}))_\lambda^+ \leqslant \alpha.$$

Hence, for any $\gamma > \alpha$,

$$\gamma \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_\gamma}))_\lambda^- \leqslant \alpha \text{ and } \gamma \wedge (\tilde{\mu}(\widetilde{A} \cap \chi_{F_\gamma}))_\lambda^+ \leqslant \alpha.$$

Consequently, for any $\gamma > \alpha$, we have

$$(\tilde{\mu}(\widetilde{A} \cap \chi_{F_\gamma}))_\lambda^- \leqslant \alpha \text{ and } (\tilde{\mu}(\widetilde{A} \cap \chi_{F_\gamma}))_\lambda^+ \leqslant \alpha.$$

That is,

$$\tilde{\mu}(\widetilde{A} \cap \chi_{F_\gamma}) \leqslant \tilde{\alpha}, \qquad \forall\, \gamma > \alpha.$$

Thus, we have proved the first part of conclusion in this lemma. The second issue from the monotonicity of $\tilde{\mu}$.

Theorem 1 Let $\tilde{\eta}$ be fuzzy number-valued fuzzy set function induced by f. If $\tilde{\mu}$ has the property($p.g.p.$), then $\tilde{\eta}$ also has the property ($p.g.p.$).

Proof Since $\tilde{\mu}$ has the property ($p.g.p.$), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\tilde{\mu}(\widetilde{A}) \vee \tilde{\mu}(\widetilde{B}) \leqslant \tilde{\delta} \Rightarrow \tilde{\mu}(\widetilde{A} \cup \widetilde{B}) \leqslant \tilde{\varepsilon}.$$

Without loss of generality, suppose $\delta < \varepsilon$.

To prove that $\tilde{\eta}$ has the property ($p.g.p.$), we should show that

$$\bar{\eta}(\widetilde{A}) \vee \bar{\eta}(\widetilde{B}) \leqslant \delta \Rightarrow \bar{\eta}(\widetilde{A} \cup \widetilde{B}) \leqslant \tilde{\varepsilon}.$$

In fact, by using Lemma 2, we have

$$\bar{\eta}(\widetilde{A}) \vee \bar{\eta}(\widetilde{B}) \leqslant \delta \Rightarrow \tilde{\mu}(\widetilde{A} \cap \chi_{F_\varepsilon}) \vee \tilde{\mu}(\widetilde{B} \cap \chi_{F_\varepsilon}) \leqslant \delta.$$

Therefore,

$$\tilde{\mu}((\widetilde{A} \cup \widetilde{B}) \cap \chi_{F_\varepsilon}) \leqslant \tilde{\varepsilon}.$$

Thus, by using again Lemma 2, we obtain that

$$\int_{\widetilde{A} \cup \widetilde{B}} f \mathrm{d}\tilde{\mu} \leqslant \tilde{\varepsilon},$$

that is, $\bar{\eta}(\widetilde{A} \cup \widetilde{B}) \leqslant \tilde{\varepsilon}$. This show that $\bar{\eta}$ has the property $(p.g.p.)$.

Theorem 2 Let $\bar{\eta}$ be (z) fuzzy set function induced by f. If $\tilde{\mu}$ has the property (s), then $\bar{\eta}$ also has the property (s).

Proof Let $(\bar{\rho}) \lim\limits_{n\to\infty} \bar{\eta}(\widetilde{E}_n) = \tilde{0}$. By lemma 1, for $\beta = \frac{1}{2}$, we have

$$(\bar{\rho}) \lim_{n\to+\infty} \tilde{\mu}(\widetilde{E}_n \cap \chi_{F_{\frac{1}{2}}}) = \tilde{0}.$$

Since $\tilde{\mu}$ has the property (s), there exists a subsequence $\{\widetilde{A}^{(1)}_{n(k)}\}_k$ of $\{\widetilde{E}_n\}_n$ such that

$$(\tilde{\mu}) \overline{\lim_{k\to+\infty}} \widetilde{E}^{(1)}_{n(k)} \cap \chi_{F_{\frac{1}{2}}}) = \tilde{0}.$$

As $(\bar{\rho}) \lim\limits_{n\to\infty} \bar{\eta}(\widetilde{E}^{(1)}_{n(k)}) = \tilde{0}$ too, by Lemma 1, for $\beta = \frac{1}{2^2}$, we have

$$(\bar{\rho}) \lim_{k\to+\infty} \tilde{\mu}(\widetilde{E}^{(1)}_{n(k)} \cap \chi_{F_{\frac{1}{2^2}}}) = \tilde{0}.$$

Therefore, by the property (s) of $\tilde{\mu}$, there exists a subsequence $\{\widetilde{E}^{(2)}_{n(k)}\}_k$ of $\{\widetilde{E}^{(1)}_{n(k)}\}_k$ such that

$$(\tilde{\mu}) \overline{\lim_{k\to+\infty}} \widetilde{E}^{(2)}_{n(k)} \cap \chi_{F_{\frac{1}{2^2}}}) = \tilde{0}.$$

Repeating this procedure, we can obtain a sequence $\{\varepsilon_m\}_m$ of subsequences of $\{\widetilde{E}_n\}_n$, where $\varepsilon_m = \{\widetilde{E}^{(m)}_{n(k)}\}_k, m = 1, 2, \cdots$, such that

$$\{\widetilde{E}^{(m)}_{n(k)}\}_k \supset \{\widetilde{E}^{(m+1)}_{n(k)}\}_k, \quad \forall\, m \geqslant 1$$

and

$$\tilde{\mu}(\overline{\lim_{k\to+\infty}} \widetilde{E}^{(m)}_{n(k)} \cap \chi_{F_{\frac{1}{2^m}}}) = \tilde{0}.$$

Taking $\widetilde{E}_{n(m)} = \widetilde{E}^{(m)}_{n(m)}, m = 1, 2, \cdots$ Then, $\{\widetilde{E}_{n(m)}\}$ is a subsequence of $\{\widetilde{E}_n\}$ and satisfies

$$\bigcup_{m=k}^{\infty} \widetilde{E}_{n(m)} \subset \bigcup_{m=k}^{\infty} \widetilde{E}^{n(m)}_{n(m)},$$

$k = 1, 2, \cdots$. For any $\alpha > 0$, take $m_0 \geqslant 1$ such that $\frac{1}{2^{m_0}} < \alpha$. Then, we have

$$\begin{aligned}\bigcap_{k=1}^{+\infty} \bigcup_{m=k}^{+\infty} (\widetilde{E}_{n(m)} \cap \chi_{F_\alpha}) &= \bigcap_{k=m_0}^{+\infty} \bigcup_{m=k}^{+\infty} (\widetilde{E}_{n(m)} \cap \chi_{F_\alpha}) \\ &\subset \bigcap_{k=m_0}^{+\infty} \bigcup_{m=k}^{+\infty} (\widetilde{E}^{(k)}_{n(m)} \cap \chi_{F_\alpha}) \\ &\subset \bigcap_{k=m_0}^{+\infty} \bigcup_{m=k}^{+\infty} (\widetilde{E}^{(m_0)}_{n(m)} \cap \chi_{F_\alpha})\end{aligned}$$

$$=\bigcap_{k=1}^{+\infty}\bigcup_{m=k}^{+\infty}(\widetilde{E}_{n(m)}^{(m_0)}\cap \chi_{F_\alpha})$$

$$\subset\bigcap_{k=1}^{+\infty}\bigcup_{m=k}^{+\infty}(\widetilde{E}_{n(m)}^{(m_0)}\cap \chi_{F_{\frac{1}{2^{m_0}}}}).$$

Consequently,

$$\widetilde{\mu}(\overline{\lim_{m\to+\infty}}\widetilde{E}_{n(m)}\cap\chi_{F_\alpha})\leqslant\widetilde{\mu}(\overline{\lim_{k\to+\infty}}\widetilde{E}_{n(k)}^{(m_0)}\cap\chi_{F_{\frac{1}{2^{m_0}}}})=\widetilde{0},$$

Thus

$$\widetilde{\mu}(\overline{\lim_{m\to+\infty}}\widetilde{E}_{n(m)}\cap\chi_{F_\alpha})=\widetilde{0},\qquad \forall\,\alpha>0.$$

It follows that, by Definition 11, $\widetilde{\eta}(\overline{\lim_{m\to+\infty}}\widetilde{E}_{n(m)})=\widetilde{0}$. This shows that $\widetilde{\eta}$ has the property (s) .

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