A Stopping Problem for Fuzzy Stochastic System with a Linear Ranking Function

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Abstract

We consider a stopping problem based on fuzzy stochastic system which is defined by sequences of fuzzy random variables. By a fuzzy-number-valued extension of classical random variables, a fuzzy stopping time for these sequences is discussed. Here we will induce a linear ranking function on α-cut of fuzzy numbers and formulate the fuzzy reward of a stopping rule as an optimization problem. Then an optimal fuzzy stopping time can be analyzed. Our aim is to characterize the optimal fuzzy stopping time of fuzzy stochastic system and describe the optimality equation treated as Dynamic programming. An example of the Markov model is presented.

Fuzzy stopping problem; dynamic fuzzy system; fuzzy decision; One-step Look Ahead policy.

1 Introduction

Zaheh's extension principle for mapping f:

$$\widetilde{r}(b) = \sup_{b=f(a)} \widetilde{s}(a).$$

Fix ω throughout. Then, random variables are mapping

$$X_\cdot(\omega) : \mathbb{R} \mapsto \mathbb{R}\,.$$

Also stopping times are

$$\tau(\omega) \in \mathbb{R}\,.$$

We can apply Zaheh's extension principle as follows

$$\text{Fuzzification of values at stopped time}(x) = \sup_{n : x = X_n(\omega)} \text{Fuzzification of stopping times}(n)$$

Let a fuzzy stopping time $\tilde{\tau}(n, \omega)$ by

$$\tilde{\tau} : \mathbb{R} \mapsto [0, 1].$$

Then we obtain

$$\text{Fuzzification of values at stopped time}(x) = \sup_{n : x = X_n(\omega)} \tilde{\tau}(n, \omega).$$

Let a fuzzification of the mapping $X_\cdot(\omega) : \mathbb{R} \mapsto \mathbb{R}$ by a fuzzy random variable $\widetilde{X_n}(\omega)(x)$:

$$\widetilde{X_n} : \Omega \mapsto \mathcal{F}(\mathbb{R}).$$

Then it follows

$$\tilde{\psi}(x) = \text{Fuzzification of values at stopped time}(x) = \sup_n T(\tilde{\tau}(n, \omega), \widetilde{X_n}(\omega)(x)),$$

where T is a t-norm and means the intersection of fuzzy sets by extension principle.

If T is a product, then

$$\tilde{\psi}(x) = \sup_n \{\tilde{\tau}(n, \omega) \widetilde{X_n}(\omega)(x)\}.$$

If T is a min, then

$$\tilde{\psi}(x) = \sup_n \min\{\tilde{\tau}(n, \omega), \widetilde{X_n}(\omega)(x)\}.$$

It's another extension.

Let E, E_1, E_2 be convex compact subsets of some Banach space. Throughout the paper, we will denote a fuzzy set and a fuzzy relation by their membership functions. For the theory of fuzzy sets, refer to Zadeh [?], Novák [?] and Dubois and Prade [?]. A fuzzy set $\widetilde{u} : E \to [0, 1]$ is called convex if

$$\widetilde{u}(\lambda x + (1 - \lambda) y) \geq \widetilde{u}(x) \wedge \widetilde{u}(y), \quad x, y \in E,\ \lambda \in [0, 1],$$

where $a \wedge b := \min\{a, b\}$. Also, a fuzzy relation $\widetilde{h} : E_1 \times E_2 \to [0, 1]$ is called convex if

$$\widetilde{h}(\lambda x_1 + (1 - \lambda) x_2, \lambda y_1 + (1 - \lambda) y_2) \geq \widetilde{h}(x_1, y_1) \wedge \widetilde{h}(x_2, y_2)$$

for $x_1, x_2 \in E_1$, $y_1, y_2 \in E_2$ and $\lambda \in [0,1]$. The α-cut ($\alpha \in [0,1]$) of the fuzzy set $\widetilde{u}$ is defined by

$$\widetilde{u}_\alpha := \{x \in E \mid \widetilde{u}(x) \geq \alpha\} \ (\alpha > 0) \quad \text{and} \quad \widetilde{u}_0 := \text{cl}\ \{x \in E \mid \widetilde{u}(x) > 0\},$$

where 'cl' denotes the closure of a set.

Let $\mathcal{F}(E)$ be the set of all convex fuzzy sets, $\widetilde{u}$, on E whose membership functions are upper semi-continuous and have compact supports and the normality condition : $\sup_{x\in E} \widetilde{u}(x) = 1$. We denote by $\mathcal{C}(E)$ the collection of all compact convex subsets of E and by ρ_E the Hausdorff metric on $\mathcal{C}(E)$. Clearly, $\widetilde{u} \in \mathcal{F}(E)$ means $\widetilde{u}_\alpha \in \mathcal{C}(E)$ for all $\alpha \in [0,1]$. Let $\mathbb{R}$ be the set of all real numbers. We see, from the definition, that $\mathcal{C}(\mathbb{R})$ and $\mathcal{F}(\mathbb{R})$ are the set of all bounded closed intervals in $\mathbb{R}$ and all upper semi-continuous and convex fuzzy numbers on $\mathbb{R}$ with compact supports, respectively.

The addition and the scalar multiplication on $\mathcal{F}(\mathbb{R})$ are defined as follows: For $\widetilde{m}, \widetilde{n} \in \mathcal{F}(\mathbb{R})$ and $\lambda \geq 0$,

$$(\widetilde{m} + \widetilde{n})(x) := \sup_{x_1, x_2 \in \mathbb{R}:\ x_1 + x_2 = x} \{\widetilde{m}(x_1) \wedge \widetilde{n}(x_2)\} \quad (x \in \mathbb{R}) \tag{1.1}$$

and

$$(\lambda \widetilde{m})(x) := \begin{cases} \widetilde{m}(x/\lambda) & \text{if } \lambda > 0 \\ I_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbb{R}). \tag{1.2}$$

Hence $(\widetilde{m} + \widetilde{n})_\alpha = \widetilde{m}_\alpha + \widetilde{n}_\alpha$ and $(\lambda \widetilde{m})_\alpha = \lambda \widetilde{m}_\alpha$ ($\alpha \in [0,1]$) holds where $A + B := \{x + y \mid x \in A, y \in B\}$, $\lambda A := \{\lambda x \mid x \in A\}$, $A + \emptyset = \emptyset + A := A$ and $\lambda \emptyset := \emptyset$ for any non-empty closed intervals A, B in $\mathbb{R}$. We use the following lemma.

Lemma 1.1 *For any $\widetilde{u} \in \mathcal{F}(E_1)$ and $\widetilde{p} \in \mathcal{F}(E_1 \times E_2)$ satisfying $\widetilde{p}(x, \cdot) \in \mathcal{F}(E_2)$ for $x \in E_1$, it holds that $\sup_{x \in E_1} \{\widetilde{u}(x) \wedge \widetilde{p}(x, \cdot)\} \in \mathcal{F}(E_2)$.*

We consider the dynamic fuzzy system([8, ?]) with fuzzy rewards in order to consider the a fuzzy stopping problem.

Definition 1 *The dynamic fuzzy system is defined by three elements $(S, \widetilde{q}, \widetilde{r})$ as follows:*

(i) *The state space S is a convex compact subset of some Banach space and is a element of $\mathcal{F}(S)$. Since the system is in a fuzzy environment, so that a state of the system is called a fuzzy state.*

(ii) *The law of motion $\widetilde{q} : S \times S \mapsto [0,1]$ for the system is time-invariant and, is assumed that $\widetilde{q} \in \mathcal{F}(S \times S)$ and $\widetilde{q}(x, \cdot) \in \mathcal{F}(S)$ for all $x \in S$.*

(iii) *The fuzzy reward $\widetilde{r} : S \times \mathbb{R} \mapsto [0,1]$ is assumed that $\widetilde{r} \in \mathcal{F}(S \times \mathbb{R})$ and $\widetilde{r}(x, \cdot) \in \mathcal{F}(\mathbb{R})$ for all $x \in S$..*

If the system is in a fuzzy state $\widetilde{s} \in \mathcal{F}(S)$, a fuzzy reward $R(\widetilde{s})$ is earned and the state is moved to a new fuzzy state $Q(\widetilde{s})$, where $Q : \mathcal{F}(S) \to \mathcal{F}(S)$ and $R : \mathcal{F}(S) \to \mathcal{F}(\mathbb{R})$ are defined by

$$Q(\widetilde{s})(y) := \sup_{x \in S}\{\widetilde{s}(x) \wedge \widetilde{q}(x, y)\} \quad (y \in S) \tag{1.3}$$

and

$$R(\widetilde{s})(z) := \sup_{x \in S}\{\widetilde{s}(x) \wedge \widetilde{r}(x, z)\} \quad (z \in \mathbb{R}). \tag{1.4}$$

Note that by Lemma 1.1 these maps Q and R are well-defined.

For the dynamic fuzzy system $(S, \widetilde{q}, \widetilde{r})$, if we give an initial fuzzy state $\widetilde{s} \in \mathcal{F}(S)$, we can define a sequence of fuzzy rewards $\{R(\widetilde{s}_t)\}_{t=1}^{\infty}$, where a sequence of fuzzy states $\{\widetilde{s}_t\}_{t=1}^{\infty}$ is defined by

$$\widetilde{s}_1 := \widetilde{s} \quad \text{and} \quad \widetilde{s}_{t+1} := Q(\widetilde{s}_t) \quad (t \geq 1). \tag{1.5}$$

In the following section, a fuzzy stopping problem for $\{R(\widetilde{s}_t)\}_{t=1}^{\infty}$ is formulated.

2 Fuzzy stochastic system

For the sake of brevity, denote $\mathcal{F} = \mathcal{F}(S)$. The metric ρ on $\mathcal{F}$ is given as $\rho(\widetilde{u}, \widetilde{v}) = \sup_{\alpha \in [0,1]} \rho_S(\widetilde{u}_\alpha, \widetilde{v}_\alpha)$ for $\widetilde{u}, \widetilde{v} \in \mathcal{F}$ (see Nanda [?]). Let $\mathcal{B}(\mathcal{F})$ be the set of Borel measurable subsets of $\mathcal{F}$ with respect to ρ. Putting by $\Omega_t := \mathcal{F}^t$ the $t(\geq 1)$ times product of $\mathcal{F}$ and by $\mathcal{B}_t := \mathcal{B}(\mathcal{F}^t)$ the set of Borel measurable subsets of $\mathcal{F}^t$ with a metric ρ^t on $\mathcal{F}^t$ defined by

$$\rho^t(\{\widetilde{s}_l\}, \{\widetilde{s}'_l\}) := \sum_{l=1}^{t} 2^{-(l-1)} \rho(\widetilde{s}_l, \widetilde{s}'_l). \tag{2.1}$$

We can interpret $\{\widetilde{s}_t\}_{t=1}^{\infty} \in \Omega_\infty$, where $\{\widetilde{s}_t\}_{t=1}^{\infty}$ is defined by (1.5) with any given initial fuzzy state $\widetilde{s}_1 = \widetilde{s} \in \mathcal{F}$. Here, applying the idea of fuzzy termination time in Kacprzyk [?, ?, ?], we will define a fuzzy stopping time. Let $\mathbb{R}$ be the set of all natural numbers.

Definition 2 *A fuzzy stopping time is a fuzzy relation $\tilde{\sigma} : \Omega_\infty \times \mathbb{R} \to [0, 1]$ such that*

(i) *for each $t \geq 1$, $\tilde{\sigma}(\cdot, t)$ is $\mathcal{B}_t$-measurable, and*

(ii) *for each $\overline{\omega} \in \Omega_\infty$, $\tilde{\sigma}(\overline{\omega}, \cdot)$ is non-increasing and there exists $t_{\overline{\omega}} \in \mathbb{R}$ with $\tilde{\sigma}(\overline{\omega}, t) = 0$ for all $t \geq t_{\overline{\omega}}$.*

In the grade of membership of stopping times, '0' and '1' represent 'stop' and 'continue' respectively. That is, the lower the value, the higher the grade of "stop". We denote by Σ the set of all fuzzy stopping times.

Lemma 2.1 *Let any $\tilde{\sigma} \in \Sigma$. Define a map $\tilde{\sigma}_\alpha : \Omega_\infty \to \mathbb{R}$ by*

$$\tilde{\sigma}_\alpha(\overline{\omega}) = \min\{t \geq 1 \mid \tilde{\sigma}(\overline{\omega}, t) < \alpha\} \quad (\overline{\omega} \in \Omega_\infty) \quad \text{for } \alpha \in (0, 1]. \tag{2.2}$$

Then, we have:

(i) $\{\tilde{\sigma}_\alpha \leq t\} \in \mathcal{B}_t \quad (t \geq 1)$;

(ii) $\tilde{\sigma}_\alpha(\overline{\omega}) \leq \tilde{\sigma}_{\alpha'}(\overline{\omega}) \quad (\overline{\omega} \in \Omega_\infty)$ *if* $\alpha \geq \alpha'$;

(ii) $\lim_{\alpha' \uparrow \alpha} \tilde{\sigma}_{\alpha'}(\overline{\omega}) = \tilde{\sigma}_\alpha(\overline{\omega}) \quad (\overline{\omega} \in \Omega_\infty)$ *if* $\alpha > 0$.

*P*roof. (i) is from $\{\tilde{\sigma}_\alpha > t\} = \{\overline{\omega} \in \Omega_\infty \mid \tilde{\sigma}(\overline{\omega}, t) \geq \alpha\} \in \mathcal{B}_t$. (ii) and (iii) follow immediately from the definition. qed.

In order to treat an optimal fuzzy stopping problem, we specify a function $G(\widetilde{s}, \tilde{\sigma})$ with a linear ranking function g, which measures the system's performance when a fuzzy stopping time $\tilde{\sigma} \in \Sigma$ and an initial fuzzy state $\widetilde{s} \in \mathcal{F}$ were adapted. It seems to be natural that the scalarization of the total fuzzy reward should be incorporated for these kind of optimization. Refer to Fortemps and Roubens [7], Wang and Kerre [?, ?] and Kurano et al [9] for a ranking method and an ordering of fuzzy sets.

We define $\omega_\infty(\cdot) : \mathcal{F} \to \Omega_\infty$ by

$$\omega_\infty(\widetilde{s}) := \{\widetilde{s}_t\}_{t=1}^\infty, \tag{2.3}$$

and $\{\widetilde{s}_t\}_{t=1}^\infty$ is defined by (1.5) with $\widetilde{s}_1 = \widetilde{s}$. Let $g : \mathcal{C}(\mathbb{R}) \to \mathbb{R}$ be a continuous and monotone function. Using this, the description of the scalarization of the total fuzzy reward will be completed by

$$G(\widetilde{s}, \tilde{\sigma}) := \int_0^1 g(\varphi(\widetilde{s}, \tilde{\sigma})_\alpha)\, d\alpha \tag{2.4}$$

where $\tilde{\sigma}_\alpha := \tilde{\sigma}_\alpha(\omega_\infty(\widetilde{s}))$ and $\varphi(\widetilde{s}, \tilde{\sigma})_\alpha := \sum_{t=1}^{\tilde{\sigma}_\alpha - 1} R(\widetilde{s}_t)_\alpha$ provided $\sum_1^0 := \{0\}$. Note that since $\varphi(\widetilde{s}, \tilde{\sigma})_\alpha \in \mathcal{C}(\mathbb{R})$ and the map $\alpha \mapsto g(\varphi(\widetilde{s}, \tilde{\sigma})_\alpha)$ is left-continuous on $(0, 1]$, the right-hand integral of (2.4) is well-defined. Now, our objective of the problem is to maximize (2.4) over all fuzzy stopping times $\tilde{\sigma} \in \Sigma$ for each initial fuzzy state $\widetilde{s} \in \mathcal{F}$.

Definition 3 *For $\widetilde{s} \in \mathcal{F}$, a fuzzy stopping time $\tilde{\sigma}^*$ is called $\widetilde{s}$-optimal if $G(\widetilde{s}, \tilde{\sigma}) \leq G(\widetilde{s}, \tilde{\sigma}^*)$ for all $\tilde{\sigma} \in \Sigma$. If $\tilde{\sigma}^*$ is $\widetilde{s}$-optimal for all $\widetilde{s} \in \Sigma$, $\tilde{\sigma}^*$ is called optimal.*

First, we establish several notations that will be used in the sequel. Associated with the fuzzy relations $\widetilde{q}$ and $\widetilde{r}$, the corresponding maps $Q_\alpha : \mathcal{C}(S) \to \mathcal{C}(S)$ and $R_\alpha : \mathcal{C}(S) \to \mathcal{C}(\mathbb{R})$ $(\alpha \in [0, 1])$ are defined, respectively, as follows: For $D \in \mathcal{C}(S)$,

$$Q_\alpha(D) := \begin{cases} \{y \in S \mid \widetilde{q}(x, y) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha > 0 \\ \mathrm{cl}\{y \in S \mid \widetilde{q}(x, y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0, \end{cases} \tag{2.5}$$

and

$$R_\alpha(D) := \begin{cases} \{z \in R \mid \widetilde{r}(x,z) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha > 0 \\ \mathrm{cl}\{z \in R \mid \widetilde{r}(x,z) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0. \end{cases} \tag{2.6}$$

By $\widetilde{q} \in \mathcal{F}(S \times S)$ and $\widetilde{r} \in \mathcal{F}(S \times R)$, these maps Q_α and R_α ($\alpha \in [0,1]$) are well-defined. The iterates Q_α^t ($t \geq 0$) are defined by setting $Q_\alpha^0 := I$(identity) and iteratively,

$$Q_\alpha^{t+1} := Q_\alpha Q_\alpha^t \quad (t \geq 0).$$

In the following lemma, which is easily verified by the idea in the proof of Kurano et al. [8, Lemma 1], the α-cuts of $Q(\widetilde{s})$ and $R(\widetilde{s})$ defined by (1.3) and (1.4) are specified using the maps Q_α and R_α.

Lemma 2.2 *([8, ?]). For any $\alpha \in [0,1]$ and $\widetilde{s} \in \mathcal{F}$, we have:*

(i) $Q(\widetilde{s})_\alpha = Q_\alpha(\widetilde{s}_\alpha)$;

(ii) $R(\widetilde{s})_\alpha = R_\alpha(\widetilde{s}_\alpha)$;

(iii) $\widetilde{s}_{t,\alpha} = Q_\alpha^{t-1}(\widetilde{s}_\alpha)$ $(t \geq 1)$,

where $\widetilde{s}_{t,\alpha} := (\widetilde{s}_t)_\alpha$ and $\{\widetilde{s}_t\}_{t=1}^\infty$ is defined by (1.5) with $\widetilde{s}_1 = \widetilde{s}$.

Here we need the following assumption which is assumed to hold henceforth.

Assumption A (Lipschitz condition). There exists a constant $K > 0$ such that

$$\rho_S(Q_\alpha(D_1), Q_\alpha(D_2)) \leq K\, \rho_S(D_1, D_2) \tag{2.7}$$

for all $\alpha \in [0,1]$ and $D_1, D_2 \in \mathcal{C}(S)$.

Theorem 2.1 *Let a fuzzy stopping time $\tilde{\sigma} \in \Sigma$. Then, the map $\tilde{\sigma}'(\cdot,\cdot) : \mathcal{F} \times \mathbb{R} \mapsto [0,1]$ defined by $\tilde{\sigma}(\widetilde{s}, t) := \tilde{\sigma}(\omega_\infty(\widetilde{s}), t)$ $(\widetilde{s} \in \mathcal{F}, t \in \mathbb{R})$ has the following properties (i) and (ii):*

(i) *$\tilde{\sigma}'(\cdot, t)$ is $\mathcal{B}(\mathcal{F})$-measurable for each $t \geq 1$.*

(ii) *For each $\widetilde{s} \in \mathcal{F}$, $\tilde{\sigma}'(\widetilde{s}, \cdot)$ is non-increasing and there exists $t_{\widetilde{s}} \in \mathbb{R}$ such that $\tilde{\sigma}'(\widetilde{s}, t) = 0$ for all $t \geq t_{\widetilde{s}}$.*

Proof. For $t \geq 1$, we define a map $\omega_t : \mathcal{F} \mapsto \mathcal{F}^t$ by $\omega_t(\widetilde{s}) := \{\widetilde{s}_l\}_{l=1}^t$, where $\{\widetilde{s}_l\}_{l=1}^\infty$ is defined by (1.5) with $\widetilde{s}_1 = \widetilde{s}$. For (i), it suffices to prove that ω_T is continuous for each $t \geq 1$, together with the measurability of $\tilde{\sigma}$. We will show only the case of $t = 2$, since the case of $t \geq 3$ is proved from (2.1) in the same manner. For $\widetilde{s}, \widetilde{s}' \in \mathcal{F}$, we have

$$\rho^2(\omega_2(\widetilde{s}), \omega_2(\widetilde{s}')) \leq \rho(\widetilde{s}, \widetilde{s}') + 2^{-1}\rho(Q(\widetilde{s}), Q(\widetilde{s}')) \leq (1 + K/2)\rho(\widetilde{s}, \widetilde{s}'),$$

from Lemma 2.2 and Assumption A. This shows the continuity of $\omega_2(\cdot)$. Also, (ii) follows from the definition of a fuzzy stopping time. qed.

Observing the scalarization (2.4) and the objective function $G(\widetilde{s}, \tilde{\sigma})$ for the stopping problem, we can confine ourselves to the class of fuzzy stopping times $\tilde{\sigma}'(\cdot, \cdot) : \mathcal{F} \times \mathbb{R} \mapsto [0,1]$ satisfying (i) and (ii) in Theorem 2.1. The class of such fuzzy stopping times will be denoted by Σ'. The following theorem is useful in constructing an optimal fuzzy time which is done in Section ??.

Theorem 2.2 *Suppose that, for each $\alpha \in [0,1]$, there exists a $\mathcal{B}(\mathcal{C}(S))$-measurable map $\sigma_\alpha : \mathcal{C}(S) \mapsto \mathbb{R}$. Using this family $\{\sigma_\alpha\}_{\alpha \in [0,1]}$, define the map $\tilde{\sigma} : \mathcal{F} \times \mathbb{R} \mapsto [0,1]$ by*

$$\tilde{\sigma}(\widetilde{s}, t) = \sup_{\alpha \in [0,1]} \{\alpha \wedge 1_{\{t : \sigma_\alpha(\widetilde{s}_\alpha) > t\}}(t)\}, \quad \widetilde{s} \in \mathcal{F}, \ t \geq 1. \tag{2.8}$$

Then, if for each $\widetilde{s} \in \mathcal{F}$, $\sigma_\alpha(\widetilde{s}_\alpha)$ is non-increasing and left-continuous in $\alpha \in [0,1]$, it holds that

(i) $\tilde{\sigma} \in \Sigma'$, *and*

(ii) $\sigma_\alpha(\widetilde{s}_\alpha) = \min\{t \geq 1 \mid \tilde{\sigma}(\widetilde{s}, t) < \alpha\} \quad (\alpha \in (0,1])$.

*P*roof. If $\sigma_\alpha(\widetilde{s}_\alpha)$ is non-increasing in $\alpha \in [0,1]$, the inequalities $\tilde{\sigma}(\widetilde{s}, t) \geq \tilde{\sigma}(\widetilde{s}, t+1)$ $(t \geq 1)$ follow from (2.8). Also, (2.8) implies that, for each $t \geq 1$ and $\alpha \in [0,1]$,

$$\{\widetilde{s} \in \mathcal{F} \mid \tilde{\sigma}(\widetilde{s}, t) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{\widetilde{s} \in \mathcal{F} \mid \sigma_{\alpha - 1/n}(\widetilde{s}_{\alpha - 1/n}) > t\}. \tag{2.9}$$

For a continuous map $\eta_\alpha : \mathcal{F} \mapsto \mathcal{C}(S)$ defined by $\eta_\alpha(\widetilde{s}) = \widetilde{s}_\alpha$ $(\widetilde{s} \in \mathcal{F})$, we have

$$\{\widetilde{s} \in \mathcal{F} \mid \sigma_\alpha(\widetilde{s}_\alpha) > t\} = \eta_\alpha^{-1}(\{D \in \mathcal{C}(S) \mid \sigma_\alpha(D) \geq t+1\}),$$

so that $\{\widetilde{s} \in \mathcal{F} \mid \tilde{\sigma}(\widetilde{s}, t) \geq \alpha\} \in \mathcal{B}(\mathcal{F})$ follows from (2.9) and $\mathcal{B}(\mathcal{C}(S))$-measurability of σ_α. The above facts imply $\tilde{\sigma} \in \Sigma'$. Also, (ii) holds obviously. qed.

3 Optimal fuzzy stopping time

In this section, we try to construct an optimal fuzzy stopping time, by applying an approach by α-cuts. Now, we define a non-fuzzy stopping problem specified by $\mathcal{C}(S)$, Q_α and R_α $(\alpha \in [0,1])$, associated with the fuzzy stopping problem considered in the preceding section. For each $\alpha \in [0,1]$ and any initial subset $c \in \mathcal{C}(S)$, a sequence $\{c_t\}_{t=1}^{\infty} \subset \mathcal{C}(S)$ is defined by

$$c_1 := c \quad \text{and} \quad c_{t+1} := Q_\alpha(c_t) \ \ (t \geq 1). \tag{3.1}$$

Let

$$\Sigma_1 := \{\sigma : \mathcal{C}(S) \mapsto \mathbb{R} \mid \{\sigma = t\} \in \mathcal{B}(\mathcal{C}(S)) \text{ for each } t \geq 1\}. \tag{3.2}$$

Using this sequence $\{c_t\}_{t=1}^{\infty}$ given by (4.1) with $c_1 := c$, let

$$\varphi^\alpha(c,t) := \sum_{l=1}^{t-1} R_\alpha(c_l) \quad \text{for } c \in \mathcal{C}(S). \tag{3.3}$$

Note that $\varphi^\alpha(c,\sigma(c)) = \sum_{l=1}^{\sigma(c)-1} R_\alpha(Q_\alpha^{t-1}(c)) \in \mathcal{C}(\mathbb{R})$ for all $\sigma \in \Sigma_1$. The non-fuzzy stopping problem considered here is to maximize $g(\varphi^\alpha(c,\sigma(c)))$ over all $\sigma \in \Sigma_1$, where g is the weighting function given in Section ??. A map $\tau_\alpha \in \Sigma_1$ is called an α-optimal stopping time if

$$g(\varphi^\alpha(c,\tau_\alpha(c))) \geq g(\varphi^\alpha(c,\sigma(c))) \quad \text{for all } \sigma \in \Sigma_1.$$

In order to characterize α-optimal stopping times, let

$$\gamma_t^\alpha(c) := \sup_{\sigma \in \Sigma_t} g(\varphi^\alpha(c,\sigma(c))) \quad \text{for } t \geq 1 \text{ and } c \in \mathcal{C}(S), \tag{3.4}$$

where $\Sigma_t := \{\sigma \vee t \mid \sigma \in \Sigma_1\}$ $(t \geq 1)$.

Assumption B (Closedness). For any $\alpha \in [0,1]$, if $(\varphi^\alpha(\widetilde{s}_\alpha, t), \widetilde{s}_{t,\alpha}) \in K^\alpha(g)$ for some t, then $(\varphi^\alpha(\widetilde{s}_\alpha, t'), \widetilde{s}_{t',\alpha}) \in K^\alpha(g)$ for all $t' > t$ where $K^\alpha(g) := \{(h,c) \in \mathcal{C}(\mathbb{R}) \times \mathcal{C}(S) \mid g(h) \geq g(h + R_\alpha(Q_\alpha(c)))\}$.

For $c \in \mathcal{C}(S)$, let

$$\tau_\alpha^*(c) := \min\{t \in \mathbb{R} \mid (\varphi^\alpha(c,t), c_t) \in K^\alpha(g)\}. \tag{3.5}$$

Then, the next lemma is given as deterministic versions of the results for stochastic stopping problems in Chow et al. [4] and Kadota et al. [?].

Lemma 3.1 *(c.f. [4, Theorems 4.1 and 4.5] and [?]). Suppose Assumption B holds. Let $\alpha \in [0,1]$. The following (i) and (ii) hold:*

(i) $\gamma_t^\alpha(c) = \max\{g(\varphi^\alpha(c,t)), \gamma_{t+1}^\alpha(c)\}$ $\quad (t \geq 1, c \in \mathcal{C}(S))$.

(ii) *Suppose that $\lim_{t\to\infty} g(\varphi^\alpha(c,t)) = -\infty$ and $\sup_{t\geq 1} g(\varphi^\alpha(c,t)) < \infty$ for each $c \in \mathcal{C}(S)$. Then, τ_α^* is α-optimal and $\gamma_1^\alpha(\cdot) = g(\varphi^\alpha(\cdot, \tau_\alpha^*(\cdot)))$.*

Chow et al. [4] studied the general case in optimal stopping problems, and Kadota et al. [?] discussed the one-step look ahead optimal stopping times given by (3.5). For each $\alpha \in [0,1]$, applying the above lemma, we can find an α-optimal stopping time τ_α^* under conditions of Lemma 3.1(ii). Assuming the existence of α-optimal stopping times for each $\alpha \in [0,1]$, let $\{\tau_\alpha^*\}_{\alpha\in[0,1]}$ be the family of such stopping times. Here, we try to construct an optimal fuzzy stopping time from $\{\tau_\alpha^*\}_{\alpha\in[0,1]}$. For this purpose, a regularity condition is need to prove our main results Theorem 3.1.

Assumption C (Regularity). $\tau_\alpha^*(\widetilde{s}_\alpha)$ is non-increasing in $\alpha \in [0,1]$.

We can assume the left-continuity of the map $\alpha \mapsto \tau^*_\alpha(\widetilde{s}_\alpha)$, by considering $\lim_{\alpha'\uparrow\alpha} \tau^*_{\alpha'}(\widetilde{s}_{\alpha'})$ instead of $\tau^*_\alpha(\widetilde{s}_\alpha)$. Define a map $\tilde{\tau}^* : \mathcal{F} \times \mathbb{R} \mapsto [0,1]$ by

$$\tilde{\tau}^*(\widetilde{s}, t) := \sup_{\alpha\in[0,1]} \left\{ \alpha \wedge 1_{\{t:\tau^*_\alpha(\widetilde{s}_\alpha)>t\}}(t) \right\}. \tag{3.6}$$

for all $\widetilde{s} \in \mathcal{F}$ and $t \in \mathbb{R}$.

Theorem 3.1 *Suppose Assumptions B and C hold. Then, $\tilde{\tau}^*$ defined by (3.6) is an $\widetilde{s}$-optimal fuzzy stopping time.*

*P*roof. From Assumption C, $\tau^*_\alpha(\widetilde{s}_\alpha) \le \tau^*_{\alpha'}(\widetilde{s}_{\alpha'})$ if $\alpha \ge \alpha'$, so that $\tilde{\tau}^* \in \Sigma'$ follows from Theorem 2.2. For any $\widetilde{s} \in \mathcal{F}$ and $\tilde{\sigma} \in \Sigma'$, from Lemma 2.1 and 2.2 we have

$$\varphi(\widetilde{s}, \tilde{\sigma})_\alpha = \sum_{t=1}^{\tilde{\sigma}_\alpha(\widetilde{s}_\alpha)-1} R_\alpha(\widetilde{s}_{t,\alpha}) = \sum_{t=1}^{\tilde{\sigma}_\alpha(\widetilde{s}_\alpha)-1} R_\alpha(Q^{t-1}_\alpha(\widetilde{s}_\alpha)). \tag{3.7}$$

Since $\sigma_\alpha \in \Sigma_1$, the optimality of τ^*_α implies by (3.7) that, for all $\alpha \in [0,1]$,

$$g(\varphi(\widetilde{s}, \tilde{\sigma})_\alpha) = g(\varphi^\alpha(\widetilde{s}_\alpha, \sigma_\alpha(\widetilde{s}_\alpha))) \le g(\varphi^\alpha(\widetilde{s}_\alpha, \tau^*_\alpha(\widetilde{s}_\alpha))) = g(\varphi(\widetilde{s}, \tilde{\tau}^*)_\alpha).$$

Therefore, we have

$$G(\widetilde{s}, \tilde{\sigma}) = \int_0^1 g(\varphi(\widetilde{s}, \tilde{\sigma})_\alpha)\ d\alpha \le \int_0^1 g(\varphi(\widetilde{s}, \tilde{\tau}^*)_\alpha)\ d\alpha = G(\widetilde{s}, \tilde{\tau}^*).$$

This means that $\tilde{\tau}^*$ is $\widetilde{s}$-optimal, as required. qed.

If the regularity does not hold for some $\widetilde{s} \in \mathcal{F}$, the $\widetilde{s}$-optimality of $\tilde{\tau}^*$ does not follow. But, $\tilde{\tau}^*$ defined by (3.6) is thought of as a good fuzzy stopping time.

4 Numerical example

In this section, an example is given to illustrate the theoretical results. Let $S := [0,1]$ and $0 < \beta < 0.98$. The fuzzy relations $\widetilde{q}$ and $\widetilde{r}$ are given by

$$\widetilde{q}(x,y) = (1 - 10^{-2}|y - \beta x|) \vee 0, \quad x, y \in [0,1]$$

and, for a given constant $\lambda > 10^{-2}(1-\beta)$,

$$\widetilde{r}(x,z) = \begin{cases} 1 & \text{if } x - z = \lambda \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } x \in [0,1],\ z \in \mathbb{R},$$

where λ means an observation cost. Then, each Q_α and R_α of (2.5) and (2.6) are calculated easily as follows: For $0 \le a \le b \le 1$,

$$Q_\alpha([a,b]) = [\beta a - (1-\alpha), \beta b + (1-\alpha)] \quad \text{and} \quad R_\alpha([a,b]) = [a - \lambda, b - \lambda].$$

Now, let the linear ranking function to be $g([a,b]) = b$ $(0 \leq a \leq b \leq 1)$. Easily we have that

$$g(\varphi^\alpha(c,t)) = g\left(\sum_{l=1}^{t-1} R_\alpha(c_l)\right) = \frac{(1-\beta^{t-1})b_\alpha}{1-\beta} - \lambda_\alpha(t-1)$$

and

$$\gamma_t^\alpha(c) = \sup_{\sigma\in\Sigma_t} g(\varphi^\alpha(c,\sigma(c))) = \sup_{n\geq t}\left\{\frac{(1-\beta^{n-1})b_\alpha}{1-\beta} - \lambda_\alpha(n-1)\right\},$$

where $b_\alpha := b - 10^{-2}(1-\alpha)/(1-\beta)$ and $\lambda_\alpha := \lambda - 10^{-2}(1-\alpha)/(1-\beta)$ for $\alpha \in [0,1]$. Then, applying Lemma 4.1, the α-optimal stopping time τ_α^* is given by

$$\begin{aligned}
\tau_\alpha^*([a,b]) &= \min\left\{t\geq 1 \mid \left(\varphi^\alpha([a,b],t), \beta^{t-1}[a,b]\right) \in K^\alpha(g)\right\} \\
&= \min\left\{t \geq 1 \left| \frac{(1-\beta^{t-1})b_\alpha}{1-\beta} - \lambda_\alpha(t-1) \geq \frac{(1-\beta^t)b_\alpha}{1-\beta} - \lambda_\alpha t\right.\right\}
\end{aligned}$$

for each $\alpha \in [0,1]$. Let $\widetilde{s}(x) = (1 - 4|2x-1|) \vee 0$ for $x \in [0.1]$. We see that $\widetilde{s}_\alpha = [(3+\alpha)/8, (5-\alpha)/8]$. Therefore

$$\tau_\alpha^*(\widetilde{s}_\alpha) = \left\lfloor \log\frac{\lambda_\alpha(1-\beta)}{(-b_\alpha)\log\beta}\Big/ \log\beta\right\rfloor + 1,$$

where $\lfloor\cdot\rfloor$ is the largest dominated integer. Since $\widetilde{s}$ is regular with respect to $\{\tau_\alpha^*\}_{\alpha\in[0,1]}$, Theorem 3.1 implies that the $\widetilde{s}$-optimal fuzzy stopping time $\tilde{\tau}^*$ is given by

$$\begin{aligned}
\tilde{\tau}^*(\widetilde{s},t) &= \sup\left\{\alpha\in[0,1] \left| \left\lfloor \log\frac{\lambda_\alpha(1-\beta)}{(-b_\alpha)\log\beta}\Big/ \log\beta\right\rfloor \geq t\right.\right\} \\
&= \left\{0 \vee \frac{8(1-\beta+\beta^t\log\beta) + 500\beta^t\log\beta + 800(1-\beta)\lambda}{8(1-\beta+\beta^t\log\beta)+100\beta^t\log\beta}\right\} \wedge 1.
\end{aligned}$$

The numerical values are given in Table 1.

t	1	2	3	4	5	6	7	8	9	$\cdots$
$\tilde{\tau}^*(\widetilde{s},t)$	0.938	0.812	0.681	0.546	0.405	0.260	0.108	0.000	0.000	$\cdots$

Table 1. $\widetilde{s}$-optimal fuzzy stopping time $\tilde{\tau}^*(\widetilde{s},\cdot)$ when $\lambda = 0.5$ and $\beta = 0.97$.

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