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Egoroff's Theorem on Fuzzy Measure Spaces

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Abstract In this paper, the concepts of order-continuity and pseudometric generating property of a fuzzy measure will be introduced on a fuzzy σ-algebra of fuzzy sets, and the Egoroff's theorem for a sequence of measurable functions will be proved on a fuzzy σ-algebra.

Key words fuzzy set fuzzy σ-algebra fuzzy measure pseudometric generating property fuzzy measurable function Egoroff's theorem

0 Introduction

The concepts of fuzzy measure and fuzzy integral, defined on a classical σ-algebra, were first proposed by Sugeno[6]. Some structural characteristics of fuzzy measures were introduced and discussed by Wang[5]. A generalization of fuzzy measure and fuzzy integral were established on fuzzy sets by Qiao[1], and the Lebesgue's theorem and Riesz's theorem for a sequence of measurable functions had been proved on a fuzzy σ-algebra of fuzzy set.

In this paper, We will show that Egoroff's theorem for a sequence of fuzzy measurable functions also holds on fuzzy σ-algebra by using the concepts of order-continuity and the pseudometric generating property of fuzzy measures.

1 Preliminary

Let X be a nonempty set and $F(X)$ be the class of all fuzzy subsets on X, i. e., $F(X) = \{\widetilde{A};\ \widetilde{A}: X \to [0,1]\}$.

Definition 1 Let $F^{*}(X) \subset F(X)$. $F^{*}(X)$ is called a fuzzy σ-algebra, if the following properties are satisfied:

(FA1) $\varnothing, X \in F^{*}(X)$ where $\varnothing(x)=0$ and $X(x)=1$ for any $x \in X$;

(FA2) $\{\widetilde{A}_n\} \subset F^{*}(X)$ implies $\bigcup_{n=1}^{\infty}\widetilde{A}_n \in F^{*}(X)$;

(FA3) $\widetilde{A} \in F^{*}(X)$ implies $\widetilde{A}^c \in F^{*}(X)$.

If $\widetilde{A}_n \subset \widetilde{A}_{n+1}$ (resp. $\widetilde{A}_{n+1} \subset \widetilde{A}_n$) for any $n \geqslant 1$, then we define that

$$(\lim_{n\to\infty}\widetilde{A}_n)(x) = \lim_{n\to\infty}\widetilde{A}_n(x), \forall\, x \in X.$$

Definition 2[1] A set function $\mu: F^*(X) \to [0,\infty]$ is said to be a fuzzy measure on fuzzy σ-algebra $F^*(X)$, if it satisfies the following conditions:

(FM1) $\mu(\varnothing)=0$;

(FM2) $\widetilde{A} \subset \widetilde{B} \Rightarrow \mu(\widetilde{A}) \leqslant \mu(\widetilde{B})$;

(FM3) $\widetilde{A}_1 \subset \widetilde{A}_2 \subset \cdots \Rightarrow \mu(\bigcup_{n=1}^{\infty} \widetilde{A}_n) = (\bar{\rho}) \lim_{n\to\infty} \mu(\widetilde{A}_n)$;

(FM4) $\widetilde{A}_1 \supset \widetilde{A}_2 \supset \cdots$, and $\mu(\widetilde{A}_1) < \infty \Rightarrow \mu(\bigcap_{n=1}^{\infty} \widetilde{A}_n) = \lim_{n\to\infty} \mu(\widetilde{A}_n)$.

We say that $(X, F^*(X))$ is a fuzzy measurable space and $(X, F^*(X), \mu)$ is a fuzzy measure space.

In the following part, we always suppose that $F^*(X)$ is a fuzzy σ-algebra and μ is a fuzzy measure defined on $F^*(X)$. Several characteristics for fuzzy measures and their relations are presented as follows:

Definition 3[1] μ is said to be null-additive if $\mu(\widetilde{A} \cup \widetilde{B}) = \mu(\widetilde{A})$ whenever $\widetilde{A}, \widetilde{B} \in F^*(X)$, and $\mu(\widetilde{B}) = 0$; autocontinuous from above if $\lim_{n\to\infty} \mu(\widetilde{A} \cup \widetilde{B}_n) = \mu(\widetilde{A})$ whenever $\widetilde{A} \in F^*(X)$ and $\widetilde{A}_n \in F_e^*$ with $\lim_{n\to\infty} \mu(\widetilde{B}_n) = 0$; order-continuous if $\lim_{n\to\infty} \mu(\widetilde{E}_n) = 0$ whenever $\widetilde{E}_n \searrow \varnothing$.

Definition 4 μ is said to have the pseudometric generating property, denoted by *p. g. p.*, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mu(\widetilde{E}) \vee \mu(\widetilde{F}) < \delta \Rightarrow \mu(\widetilde{E} \cup \widetilde{F}) < \varepsilon.$$

Proposition 1 If μ is autocontinuous from above, then μ has the pseudometric generating property.

Proof It is similar to the proof of proposition 4 in[3].

Remark 1 By propositions 1, we know that if a fuzzy measure is autocontinuous from above, then it has the pseudometric generating property, but the converse is not true.

Example 1 Let $X_1 = \{-1, -2, \cdots\}$, $X_2 = \{0, 1, 2, \cdots\}$, $X = X_1 \cup X_2$, and $F^*(X) = P(X)$. Put

$$m(E) = \begin{cases} \sum_{i\in E} \frac{1}{2^{|i|+1}}, & \text{if } E \neq \varnothing, \\ 0, & \text{if } E = \varnothing, \end{cases}$$

and

$$\mu(E) = m(E \cap X_1)(1 + m(E \cap X_2)).$$

Then μ is a fuzzy measure and has pseudometric generating property, but it is not autocontinuous from above.

3 Convergence for a sequence of measurable functions

Let $f: X \to R^1$ and $\alpha \in R$. Put $F_\alpha = \{x; f(x) \geqslant \alpha\}$ and

$$\chi_{F_\alpha}(x) = \begin{cases} 1, & \text{if } x \in F_\alpha, \\ 0, & \text{if } x \notin F_\alpha. \end{cases}$$

Definition 5 f is said to be a measurable function if $\chi_{F_\alpha} \in F^*(X)$ for any $\alpha \in R$.

Let M^* denote the class of all fuzzy measurable function on fuzzy measure space $(X, F^*(X), \mu)$. Unless stated otherwise, all fuzzy set $\widetilde{A}$ are supposed to belong to $F^*(X)$ and all real functions we consider are assumed to be measurable function.

Definition 6 We say that (1) f_n converges to f everywhere (resp. uniformly) on $\widetilde{A}$, denote it by $f_n \xrightarrow{e.} f$ (resp. $f_n \xrightarrow{u.} f$) on $\widetilde{A}$ or $f_n \xrightarrow[\widetilde{A}]{e.} f$ (resp. $f_n \xrightarrow[\widetilde{A}]{u.} f$), if there exists a subset $D \subset X$ with $\chi_D \in F^*(X)$ such that f_n converges to f (resp. uniformly) on D and $\widetilde{A} \subset \chi_D$;

(2) f_n converges to f almost everywhere on $\widetilde{A}$, denote it by $f_n \xrightarrow{a.e} f$ on $\widetilde{A}$ or $f_n \xrightarrow[\widetilde{A}]{a.e.} f$, if there exists a $D \subset X$ with $\chi_D \in F^*(X)$ and $\mu(\chi_D) = \widetilde{0}$ such that f_n converges to f everywhere on $\widetilde{A} \cap \chi_D^c$;

(3) f_n converges to f almost unformly on $\widetilde{A}$, denote it by $f_n \xrightarrow{a.u.} f$ on $\widetilde{A}$ or $f_n \xrightarrow[\widetilde{A}]{a.u.} f$, if for any $\delta > 0$, there exists $\widetilde{E} \in F^*(X)$ with $\mu(\widetilde{E}) < \delta$, such that f_n converges to f uniformly on $\widetilde{A} \cap \widetilde{E}^c$;

Lemma 1 Let μ have the pseudometric generating property. If $\lim_{n\to\infty} \mu(\widetilde{E}_n) = 0$, then there exists a sequence $\{\delta_r\}_r$ of R_+ and a subsequence $\{\widetilde{E}_{n(i)}\}_i \subset \{\widetilde{E}_n\}_n$ with $\delta_r \searrow 0$ such that

$$\mu\left(\bigcup_{i=r+1}^{\infty} \widetilde{E}_{n(i)}\right) < \delta_r, \forall\, r \geqslant 1.$$

Theorem 1 (Egoroff's theorem) Let $\{f_n\}_n \subset M^*$, $f \in M^*$, $\widetilde{A} \in F^*(X)$ and $\widetilde{A} \cap \widetilde{A}^c = \varnothing$. If μ is order-continuous and has the pseudometric generating property, then

$$f_n \xrightarrow[\widetilde{A}]{a.e.} f \Rightarrow f_n \xrightarrow[\widetilde{A}]{a.u.} f.$$

Proof Since $f_n \xrightarrow[\widetilde{A}]{a.e.} f$, there exists a subset $D \subset X$ with $\chi_D \in F^*(X)$ and $\mu(\chi_D) = 0$ such that $f_n \xrightarrow[\widetilde{A} \cap \chi_D^c]{e.} f$. Put $\widetilde{B} = \widetilde{A} \cap \chi_D^c$. Then, $f_n \xrightarrow[\widetilde{B}]{e.} f$. By Definition 6, there exists $H \subset X$ with $\chi_H \in F^*(X)$ such that f_n converges to f on H and $\widetilde{B} \subset \chi_H$.

Put

$$E_n^{(m)} = \bigcap_{i=n}^{+\infty} \{x \in X; |f_i(x) - f(x)| < \frac{1}{m}\}, \forall\, m \geqslant 1.$$

Then, for each fixed $m \geqslant 1$, $E_n^{(m)}$ is increasing on n and, therfore, $\chi_{E_n^{(m)}}$ is increasing on n. Since $f_n \xrightarrow[\widetilde{B}]{e.} f$, we have $\lim_{n\to\infty} \chi_{E_n^{(m)}} \supset \widetilde{B}$ and, hence, $\bigcup_{n=1}^{\infty} \chi_{E_n^{(m)}} \supset \widetilde{B}$. Therefore, we have

$$\bigcap_{n=1}^{\infty} (\widetilde{B} \cap \chi_{E_n^{(m)}}^c) = \widetilde{B} \cap \left(\bigcup_{n=1}^{\infty} \chi_{E_n^{(m)}}\right)^c = \varnothing.$$

Thus, we get $\lim_{n\to\infty} \widetilde{B} \cap \chi_{E_n^{(m)}}^c = \varnothing$. From the order-continuouity of μ, we have $\lim_{n\to\infty} \mu(\widetilde{B} \cap \chi_{E_n^{(m)}}^c) = 0$ and, hence, there exists a subsequence $\{\widetilde{B} \cap \chi_{E_{n_m}^{(m)}}^c\}_{m=1}^{\infty}$ of $\{\widetilde{B} \cap \chi_{E_n^{(m)}}^c\}_{m,n=1}^{\infty}$ such that

$$\mu(\widetilde{B} \cap \chi_{E_{n_m}^{(m)}}^c) < \frac{1}{m}.$$

for any $m \geqslant 1$. Thus,

$$\lim_{m\to\infty}\mu(\widetilde{B}\cap\chi_{E^{(m)}_{n_m}})=0.$$

Therefore, by Lemma 1, there exists a sequence $\{\delta_r\}_{r=1}^{\infty}$ of R_+ and a subsequence $\{\widetilde{B}\cap\chi_{E^{(m_i)}_{n_{m_i}}}\}_i$ of $\{\widetilde{B}\cap\chi_{E^{(m)}_{n_m}}\}_m$ such that $\delta_r\searrow 0$ and

$$\mu[\bigcup_{i=r+1}^{\infty}(\widetilde{B}\cap\chi_{E^{(m_i)}_{n_{m_i}}})]<\delta_r,\quad r\geqslant 1.$$

For any $\delta>0$, since μ has the pseudometric generating property, there exists $\sigma>0$ such that

$$\mu(\widetilde{E})\vee\mu(\widetilde{F})<\sigma\Rightarrow\mu(\widetilde{E}\cup\widetilde{F})<\delta.$$

for $\sigma>0$ above, we can find $r_0\geqslant 1$ such that $\delta_{r_0}<\sigma$. If we take

$$\widetilde{E}=\bigcup_{i=r_0+1}^{\infty}(\widetilde{B}\cap\chi_{E^{(m_i)}_{n_{m_i}}}),$$

then $\widetilde{E}\in F^*(X)$ and $\mu(\widetilde{E})<\sigma$. Note that $\mu(\chi_D)=0<\sigma$, therefore $\mu(\chi_D\cup\widetilde{E})<\delta$.

To prove that $\{f_n\}$ converge to f almost uniformly on $\widetilde{A}$, we need only to prove that $\{f_n\}$ converge to f uniformly on $\widetilde{A}\cap(\chi_D\cup\widetilde{E})^c$. Since $\widetilde{A}\cap\widetilde{A}^c=\varnothing$, we have $\widetilde{B}\cap\widetilde{B}^c=\varnothing$. Therefore

$$\widetilde{A}\cap(\chi_D\cup\widetilde{E})^c=\widetilde{A}\cap\chi_D^c\cap\widetilde{E}^c\subset\chi_{\bigcap_{r_0+1}^{\infty}E^{(m_i)}_{n_{m_i}}}.$$

Now we come to show that $\{f_n\}$ converges to f uniformly on $\bigcap_{i=r_0+1}^{\infty}E^{(m_i)}_{n_{m_i}}$. For any $\varepsilon>0$, we take $i_0>r_0$ such that $m_{i_0+1}>\frac{1}{\varepsilon}$. Then, $x\in\bigcap_{i=r_0+1}^{\infty}E^{(m_i)}_{n_{m_i}}$ implies $x\in E^{(m_i)}_{n_{m_i}}$ as $i\geqslant r_0+1$. Thus,

$$x\in\bigcap_{i=n_{m_{i_0}+1}}^{\infty}\{x;|f_i(x)-f(x)|<\frac{1}{m_{i_0+1}}\}.$$

This means that $|f_i(x)-f(x)|<\frac{1}{m_{i_0+1}}<\varepsilon$ as $i\geqslant n_{m_{i_0}+1}$, so that f_n converges to f uniformly on $\bigcap_{i=r_0+1}^{\infty}E^{m_i}_{n_{m_i}}$. The proof of the theorem is now complete.

From Definition 2 and proposition 1, we can immediately obtain the following corollary.

Corollary 1 Let $\{f_n\}_n\subset M^*$, $f\in M^*$, $\widetilde{A}\in F^*(X)$, $\widetilde{A}\cap\widetilde{A}^c=\varnothing$ and $\widetilde{\mu}(\widetilde{A})<\infty$. If μ is autocontinuous from above, then

$$f_n\xrightarrow[\widetilde{A}]{\text{a.e.}}f\Rightarrow f_n\xrightarrow[\widetilde{A}]{\text{a.u.}}f.$$

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