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A Linear Structure and Convexity for Relations in Dynamic Fuzzy Systems

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Abstract—Introducing a linear structure for a fuzzy relation, we could prove the linearity of the fuzzy integral with respect to this relation. It is known that the fuzzy integral does not have the linearity, so this becomes an obstacle in developing the fuzzy dynamic system. To avoid this, we will impose some restrictive assumptions for a fuzzy relation defined on the positive orthant of an n-dimensional Euclidean space. Also, fuzzy Markov decision processes with deterministic stationary policies are formulated and its characterization is given by a recursive equation in the fuzzy state.

Keywords—Fuzzy integral, Fuzzy relation, Markov decision process.

1. INTRODUCTION

Let denote $\mathcal{F}(\mathbb{R}^n_+)$ be the set of all fuzzy sets $\tilde{s}$ on $\mathbb{R}^n_+$, being upper semicontinuous with a compact support and $\sup_{x\in\mathbb{R}^n_+} \tilde{s}(x) = 1$ where $\mathbb{R}^n_+$ is the positive orthant of an n-dimensional Euclidean space. We will follow the usual notation in the fuzzy thoery. For any fuzzy set $\tilde{s}$ on $\mathbb{R}^n_+$ and $\alpha \in [0,1]$, its α-cut is defined by

$$\tilde{s}_\alpha := \{x \in \mathbb{R}^n_+ \mid \tilde{s}(x) \geq \alpha\}\ (1 \geq \alpha > 0) \quad \text{and} \quad \tilde{s}_0 := c\ell\{x \in \mathbb{R}^n_+ \mid \tilde{s}(x) > 0\}$$

where $c\ell$ means the closure of a set. We call $\tilde{s}_0$ a support of $\tilde{s}$. For fuzzy sets $\tilde{s}, \tilde{r}$ and a scalar λ,

$$(\tilde{s} + \tilde{r})(x) := \sup_{y+z=x;\ y,z\in\mathbb{R}^n_+} \{\tilde{s}(y) \wedge \tilde{r}(z)\}$$

and

$$(\lambda\tilde{s})(x) := \begin{cases} \tilde{s}\left(\dfrac{x}{\lambda}\right), & \text{if } \lambda > 0, \\ I_{\{0\}}(x), & \text{if } \lambda = 0, \end{cases} \qquad x \in \mathbb{R}^n_+,$$

where $\lambda \wedge \mu = \min\{\lambda, \mu\}$ for $\lambda, \mu \in \mathbb{R}^1_+$, and $I_A(\cdot)$ is the indicator function for any ordinary subset A of $\mathbb{R}^n_+$.

Then, the corresponding α-cut representations are as follows (see [1]):

$$(\tilde{s} + \tilde{r})_\alpha = \tilde{s}_\alpha + \tilde{r}_\alpha, \qquad \text{for } \tilde{s},\ \tilde{r} \text{ and } \alpha \in [0,1],$$

$$(\lambda\tilde{s})_\alpha = \lambda\tilde{s}_\alpha, \qquad \text{for } \tilde{s},\ \lambda \in \mathbb{R}^1_+, \text{ and } \alpha \in [0,1],$$

where $A+B = \{x+y \mid x \in A \text{ and } y \in B\}$ $(A, B \subset \mathbb{R}^n_+)$ and $\lambda A = \{\lambda x \mid x \in A\}$ $(A \subset \mathbb{R}^n_+, \lambda \in \mathbb{R}^1_+)$.

This research is a joint work with M. Kurano, J. Nakagami, Y. Yoshida of Chiba University, Japan. The intensive discussions with the colleague have improved the results.

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A fuzzy relation, in this paper, is defined by $\tilde{q} : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to [0,1]$, which satisfies the next two conditions.

ASSUMPTION 1.

(i) *It is continuous on* $\mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus \{(0,0)\}$ *and*
(ii) $\tilde{q}(\cdot, y) \in \mathcal{F}(\mathbb{R}^n_+)$, *for* $y \in \mathbb{R}^n_+$.

Note that the continuity breaks at $\{(0,0)\}$ because of the linearity introduced in the next section. The fuzzy relation $\tilde{q}$ defines a rule of transition in dynamic fuzzy system. Also, it reduces the fuzzy measure $\mu_{\tilde{q}}$, and then the fuzzy integral can be defined.

It is shown that, for any fuzzy set $\tilde{p} \in \mathcal{F}(\mathbb{R}^n_+)$, the fuzzy integral with respect to relation $\tilde{q}(x, \cdot)$ becomes

$$\begin{aligned} ⨍ \tilde{p}\, d\tilde{q}(x,\cdot) &= ⨍ \tilde{p}(y)\, \mu_{\tilde{q}(x,\cdot)}\, dy \\ &= \sup_{0 \le \alpha \le 1} \left\{\alpha \wedge \mu_{\tilde{q}(x,\cdot)}\left(\tilde{p}_\alpha\right)\right\} \\ &= \sup_{y \in \mathbb{R}^n_+} \left\{\tilde{q}(x,y) \wedge \tilde{p}(y)\right\}, \qquad x \in \mathbb{R}^n_+, \end{aligned}$$

where $\mu_{\tilde{q}(x,\cdot)}(D) = \sup_{y \in D} \tilde{q}(x,y)$ and $\tilde{p}_\alpha = \{y \in \mathbb{R}^n_+ \mid \tilde{p}(y) \ge \alpha\}$, $\alpha \in [0,1]$.

In this paper, the linearity of the fuzzy integral with respect to this relation is discussed in Section 2. In Section 3, fuzzy Markov decision processes with deterministic stationary policies are formulated and its characterization is given by a recursive equation in the fuzzy state. The discussion in Section 3 does not assume the linearity of the integral.

2. RELATIONS FOR THE DYNAMIC FUZZY SYSTEMS

Assumptions for relations of the dynamic fuzzy system are imposed as follows.

ASSUMPTION 2.

(i) **(Convexity)** $\tilde{q}(\cdot, y)$ *is convex for each* y, *that is,* $\tilde{q}_\alpha(y)$ *is a convex set for each* α, y.
(ii) **(Linearity)** $\tilde{q}(\cdot, ay + bz) = a\tilde{q}(\cdot, y) + b\tilde{q}(\cdot, z)$ *for each* y, z *and* $a, b \ge 0$.

LEMMA 1. *Let A be a convex set. For $a, b \ge 0$, it holds that*

$$aA + bA = (a+b)A.$$

PROOF. It is clear. The proof is omitted.

Note that since we are considering the integral by the fuzzy relation, the fuzzy integral is not the usual scalar one, but the fuzzy set. Therefore, the scalar product and sum appearing in the next theorem must be understood in the sense of the fuzzy operation.

LEMMA 2. *Assumption* 2(ii) *is equivalent to*

$$\tilde{q}_\alpha(ay + bz) = a\tilde{q}_\alpha(y) + b\tilde{q}_\alpha(z), \qquad \text{for } \alpha \ge 0.$$

PROOF. Since $\tilde{q}_\alpha(y) = \{x \in \mathbb{R}^n_+ \mid \tilde{q}(x,y) \ge \alpha\}$, it is easily obtained.

THEOREM 1. *Under these assumptions, the linearity of the fuzzy integral with respect to the relation holds, that is, for $\tilde{s}, \tilde{r} \in \mathcal{F}(\mathbb{R}^n_+)$ and $a, b \ge 0$,*

$$⨍ (a\tilde{s} + b\tilde{r})\, d\tilde{q}(x,\cdot) = a ⨍ \tilde{s}\, d\tilde{q}(x,\cdot) + b ⨍ \tilde{r}\, d\tilde{q}(x,\cdot), \qquad x \in \mathbb{R}^n_+.$$

PROOF. Using two lemmas, Assumption 2 implies this linearity.

3. RECURSIVE EQUATION OF FUZZY MDP

The previous assumption is somewhat too restrictive, so we try to discuss the dynamic decision process without this assumption. Our main tool is to use α-cuts. Firstly, using Zadeh's extention principle [2], we will define the total fuzzy reward in the deterministic Markov decision process(MDP) and their associated rewards of stationary policy are characterized by the fixed point of a mapping.

In this section, we will formulate a fuzzy Markov decision process which transition follows a deterministic law. The policy is restricted to a stationary deterministic one.

A rough sketch on the classical MDP is described by the set

$$(X, A, q, r).$$

We consider, for simplicity, the case such that X is a compact state space, A is a compact action space, a deterministic transition function $q : X \times A \to X$ and a bounded reward function $r : X \times A \to [0, M]$, where M is a positive constant. A stationary policy $f^{(\infty)} = (f, f, \dots)$ is a sequence of function f from X to A and β denotes a discount factor. Hereafter, assume that each of q, r, f is continuous.

Under these assumptions, the aim of this section is to formulate a fuzzy Markov decision process from the classical one.

Let us denote that $\mathcal{F}(Y)$ equals the set of all upper semicontinuous fuzzy sets on Borel set Y. Using the above notion, define the fuzzy state $\tilde{s}$ as in the set of all fuzzy state space $\mathcal{F}(X)$. Similarly the fuzzy action $\tilde{a}$ as in the set of all fuzzy action space $\mathcal{F}(A)$.

To consider the recursive equation in the fuzzy MDP, we fix the policy $f^{(\infty)} = (f, f, \dots)$ so

$$q_f(x) := q(x, f(x)) \qquad \text{and} \qquad r_f(x) := r(x, f(x))$$

could be used for simplicity.

For a fuzzy state $\tilde{p} \in \mathcal{F}(X)$ and fixed policy $f^{(\infty)} = (f, f, \dots)$, let define a fuzzy transition by

$$\tilde{q}_f(\tilde{p})(y) := \sup\{\tilde{p}(x);\ q_f(x) = y,\ x \in X\}, \qquad y \in X$$

and a fuzzy reward by

$$\tilde{r}_f(\tilde{p})(z) := \sup\{\tilde{p}(x);\ x \in X,\ r_f(x) = z\}, \qquad z \in [0, M],$$

respectively.

If we adapt the fuzzy policy $f^{(\infty)} = (f, f, \dots)$,the fuzzy state process changes as time goes by,

$$\tilde{p}_0 := \tilde{p} \in \mathcal{F}(X), \quad \tilde{p}_{t+1} := \tilde{q}_f(\tilde{p}_t)$$

for $t = 0, 1, 2, \dots$.

Now for the initial fuzzy state $\tilde{p}$ and the stationary policy $f^{(\infty)} = (f, f, \dots)$, the discounted total fuzzy reward is defined formally as

$$\tilde{R}_f(\tilde{p}) := \sum_{t=0}^{\infty} \beta^t \tilde{r}_f(\tilde{p}_t).$$

This is a formal sum so that we must discuss its convergence.

Let $\mathcal{CS}[0, M]$ be a set of all closed subset in the interval $[0, M]$. And let ρ be Hausdroff metric on $\mathcal{CS}[0, M]$. Hence, the space $(\mathcal{CS}[0, M], \rho)$ becomes a compact metric space.

To discuss the convergence for fuzzy states, the α-cut is a useful tool as previously applied in [3].

DEFINITION 1. *For $\tilde{s}_n, \tilde{s} \in \mathcal{F}(X)$,*

$$\tilde{s}_n \to \tilde{s} \text{ as } n \to \infty,$$

if and only if

$$\sup\{\rho(\tilde{s}_{n,\alpha}, \tilde{s}_\alpha);\ 0 \le \alpha \le 1\} \to 0 \quad \text{as } n \to \infty,$$

where the subscript α means the corresponding α-cut.

THEOREM 2. *It holds that*

(a) *The fuzzy total reward $\tilde{R}_f(\tilde{p})$ with a stationary policy $f^{(\infty)} = (f, f, \dots)$ and the initial fuzzy state $\tilde{p}$ converges.*

(b) *For any $\alpha \ge 0$,*

$$\tilde{R}_f(\tilde{p})_\alpha = \sum_{t=0}^{\infty} \beta^t \tilde{r}_f(\tilde{p}_t)_\alpha$$

where $\tilde{p}_0 := \tilde{p} \in \mathcal{F}(X)$, $\tilde{p}_{t+1} := \tilde{q}_f(\tilde{p}_t)$, for $t = 0, 1, 2, \dots$ with the fuzzy transition $\tilde{q}_f$.

Let us consider the following function space

$$V := \left\{ v;\ \mathcal{CS}(X) \to \mathcal{CS}\left(\left[0, \frac{M}{1-\beta}\right]\right) \right\}.$$

That is, for any closed set $D \subset X$, if $v \in V$, then $v(D)$ is a closed subset of $[0, M/(1-\beta)]$. Let define an operator U_f associated with a policy $f^{(\infty)} = (f, f, \dots)$ from V to V such that

$$U_f v(D) := r_f(D) + \beta v(q_f(D)), \qquad \text{for } v \in V,\ D \in \mathcal{CS}(X)$$

where $q_f(D) := \{q_f(x); x \in D\}$.

LEMMA 3. *The operator U_f for each policy $f^{(\infty)} = (f, f, \dots)$ is a contractive mapping on V.*

Hence, by this lemma, there exists a fixed point and we denote it as $v_f^* \in V$.

THEOREM 3. *The fuzzy total reward $\tilde{R}_f(\tilde{p})$ with an initial fuzzy state $\tilde{p}$ and a fuzzy stationary policy $f^{(\infty)} = (f, f, \dots)$ is characterized by using α-set as*

$$\tilde{R}_f(\tilde{p})_\alpha = v_f^*(\tilde{p}_\alpha), \qquad 0 \le \alpha \le 1.$$

PROOF. From Theorem 2 and Lemma 3,

$$\begin{aligned} \tilde{R}_f(\tilde{p})_\alpha &= \sum_{t=0}^{\infty} \beta^t \tilde{r}_f(\tilde{p}_t)_\alpha \\ &= \sum_{t=0}^{\infty} \beta^t \tilde{r}_f(\tilde{p}_{t,\alpha}) \\ &= \sum_{t=0}^{\infty} \beta^t \tilde{r}_f(\tilde{q}_f^t(\tilde{p}_\alpha)) \\ &= \tilde{r}_f(\tilde{p}_\alpha) + \beta \tilde{R}_f(\tilde{q}_f(\tilde{p}_\alpha)). \end{aligned}$$

Since $\tilde{R}_f(\tilde{p})_\alpha$ depends on $\tilde{p}_\alpha$, it could be assigned as $v(\tilde{p}_\alpha) := \tilde{R}_f(\tilde{p})_\alpha$. Letting $D = \tilde{p}_\alpha$,

$$v(D) = U_f v(D).$$

That is, v is the fixed point of U_f and $v = v_f^*$. ∎

This report is the preliminary results about the fuzzy integral and Markov decision processes. It will be applied to define a fuzzy potential and the multistage decision processes in order to expand the notion of fuzzy.

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