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# A Multiobjective Fuzzy Stopping in a Stochastic and Fuzzy Environment

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**Abstract**—In a stochastic and fuzzy environment, a multiobjective fuzzy stopping problem is discussed. The randomness and fuzziness are evaluated by probabilistic expectations and linear ranking functions, respectively. Pareto optimal fuzzy stopping times are given under the assumption of regularity for stopping rules, by using $\lambda$-optimal stopping times. 

**Keywords**—Multiobjective optimal stopping, Fuzzy stochastic systems, Fuzzy stopping, Pareto optimal.

## 1. INTRODUCTION

This paper presents a multiobjective fuzzy stopping model of 'fuzzy stochastic systems' in co-operation with sequences of 'fuzzy random variables'. The fuzzy random variable, which is a fuzzy-number-valued extension of classical random variables, was studied by Puri and Ralescu [1] and has been discussed by many authors. It is one of the successful hybrid notions of randomness and fuzziness. On the other hand, stopping problems for a sequence of real-valued random variables had a long history and were studied extensively. Their applications are well known in various fields [2,3] and especially in the finance theory, recently. The optimal fuzzy stopping for fuzzy random variables is discussed by Yoshida *et al.* [4], and dynamic fuzzy systems without randomness are studied by Yoshida [5–7]. This paper analyzes a multiobjective stopping model for fuzzy stochastic systems, by extending the results of the classical stochastic systems [8,9].

We also discuss the optimization by 'fuzzy' stopping times. Fuzzy stopping times are introduced for dynamic fuzzy systems by Kurano *et al.* [10] and they are discussed by Yoshida *et al.* [11], and this paper applies the notion of fuzzy stopping times in a stochastic and fuzzy environment. In this paper, we evaluate the randomness and fuzziness regarding the stopped fuzzy stochastic systems by probabilistic expectations and linear ranking functions, respectively. We also give

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Pareto optimal fuzzy stopping times for the multiobjective model, by introducing the notion of $\lambda$-optimal stopping times.

In Section 2, the notations and definitions of fuzzy random variables are given. In Section 3, fuzzy stopping times are introduced. We formulate a multiobjective optimal stopping problem for fuzzy stochastic systems by fuzzy stopping times. In Section 4, we give Pareto optimal fuzzy stopping times for the problem under the assumption of regularity for stopping rules.

## 2. FUZZY RANDOM VARIABLES

Some mathematical notations of fuzzy random variables are given in this section. Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a $\sigma$- field and $P$ is a nonatomic probability measure. Let $\mathbb{R}$ be the set of all real numbers, let $\mathcal{B}$ denote the Borel $\sigma$-field of $\mathbb{R}$, and let $\mathcal{I}$ denote the set of all bounded closed subintervals of $\mathbb{R}$. A fuzzy number is denoted by its membership function $\tilde{a} : \mathbb{R} \mapsto [0,1]$ which is normal, upper-semicontinuous, fuzzy convex, and has a compact support. Refer to [12] for the theory of fuzzy sets. $\mathcal{R}$ denotes the set of all fuzzy numbers. The $\alpha$-cut of a fuzzy number $\tilde{a}(\in \mathcal{R})$ is given by

$$\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\} \ (\alpha \in (0,1]) \qquad \text{and} \qquad \tilde{a}_0 := \text{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\},$$

where cl denotes the closure of an interval. In this paper, we write the closed intervals by

$$[\tilde{a}]_\alpha := \left[[\tilde{a}]^-_\alpha, [\tilde{a}]^+_\alpha\right], \qquad \text{for } \alpha \in [0,1].$$

A map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a fuzzy random variable if

$$\left\{(\omega, x) \in \Omega \times \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\right\} \in \mathcal{M} \times \mathcal{B}, \qquad \text{for all } \alpha \in [0,1]. \tag{2.1}$$

Condition (2.1) is also written as

$$\left\{(\omega, x) \in \Omega \times \mathbb{R} \mid x \in \left[\tilde{X}(\omega)\right]_\alpha\right\} \in \mathcal{M} \times \mathcal{B}, \qquad \text{for all } \alpha \in [0,1], \tag{2.2}$$

where $[\tilde{X}(\omega)]_\alpha = [[\tilde{X}(\omega)]^-_\alpha, [\tilde{X}(\omega)]^+_\alpha] := \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ is the $\alpha$-cut of the fuzzy number $\tilde{X}(\omega)$ for $\omega \in \Omega$. We can find some equivalent conditions [13]; however, in this paper, we adopt a simple equivalent condition in the following lemma.

LEMMA 2.1. *(See [14, Theorems 2.1 and 2.2].) For a map $\tilde{X} : \Omega \mapsto \mathcal{R}$, the following (i) and (ii) are equivalent.*

(i) *$\tilde{X}$ is a fuzzy random variable.*
(ii) *The maps $\omega \mapsto [\tilde{X}(\omega)]^-_\alpha$ and $\omega \mapsto [\tilde{X}(\omega)]^+_\alpha$ are measurable for all $\alpha \in [0,1]$.*

Now we introduce expectations of fuzzy random variables for the description of stopping models for fuzzy stochastic systems. A fuzzy random variable $\tilde{X}$ is called integrably bounded if $\omega \mapsto [\tilde{X}(\omega)]^-_\alpha$ and $\omega \mapsto [\tilde{X}(\omega)]^+_\alpha$ are integrable for all $\alpha \in [0,1]$. Let $\tilde{X}$ be an integrably bounded fuzzy random variable. The expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number [15, Lemma 3]

$$E\left(\tilde{X}\right)(x) := \sup_{\alpha \in [0,1]} \min\left\{\alpha, 1_{[E(\tilde{X})]_\alpha}(x)\right\}, \qquad \text{for } x \in \mathbb{R}, \tag{2.3}$$

where $1_D$ is the classical indicator function of a set $D$ and we put closed intervals

$$\left[E\left(\tilde{X}\right)\right]_\alpha := \left[\int_\Omega \left[\tilde{X}(\omega)\right]^-_\alpha \mathrm{d}P(\omega), \int_\Omega \left[\tilde{X}(\omega)\right]^+_\alpha \mathrm{d}P(\omega)\right], \qquad \alpha \in [0,1].$$

## 3. FUZZY STOPPING IN A MULTIOBJECTIVE MODEL

Let $k$ be a positive integer. In this section, we formulate a multiobjective optimal 'fuzzy' stopping problem for $k$ fuzzy stochastic systems. Let $\{1,2,\dots,k\}$ denote the set of $k$ objects which are described by fuzzy stochastic systems with the time space $\mathbb{N} := \{0,1,2,\dots\}$. For an object $i = 1,2,\dots,k$, let $\{\tilde{X}_n^i\}_{n=0}^{\infty}$ be a sequence of fuzzy random variables such that, for $n = 0,1,2,\dots$,

$$E\left(\max_{1\le i\le k}\sup_{n\ge 0}\left[\tilde{X}_n^i(\cdot)\right]_0^+\right) < \infty \qquad \text{and} \qquad E\left(\min_{1\le i\le k}\left[\tilde{X}_n^i(\cdot)\right]_0^-\right) > -\infty,$$

where the interval $[[\tilde{X}_n^i(\omega)]_0^-, [\tilde{X}_n^i(\omega)]_0^+]$ is the 0-cut of the fuzzy number $\tilde{X}_n^i(\omega)$. Let $\mathcal{M}_n$, $n = 0,1,2,\dots$, denote the smallest $\sigma$-field on $\Omega$ generated by all random variables $[\tilde{X}_n^i(\omega)]_\alpha^-$ and $[\tilde{X}_n^i(\omega)]_\alpha^+$ ($i = 1,2,\dots,k$; $m = 0,1,2,\dots,n$; $\alpha \in [0,1]$), and $\mathcal{M}_\infty$ denote the smallest $\sigma$-field containing $\bigcup_{n=0}^{\infty}\mathcal{M}_n$. Then we call $(\{\tilde{X}_n^i\}_{n=0}^{\infty}, \{\mathcal{M}_n\}_{n=0}^{\infty})$ the fuzzy stochastic system for an object $i$. A map $\tau : \Omega \mapsto \mathbb{N}\cup\{\infty\}$ is called a stopping time if it satisfies

$$\{\omega \mid \tau(\omega) = n\} \in \mathcal{M}_n, \qquad \text{for all } n = 0,1,2,\dots. \tag{3.1}$$

Then we have the following lemma which is trivial from the definitions.

LEMMA 3.1. *Let $i = 1,2,\dots,k$ be an object and let $\tau$ be a finite stopping time. We define*

$$\tilde{X}_\tau^i(\omega) := \tilde{X}_n^i(\omega), \qquad \omega \in \{\omega \mid \tau(\omega) = n\}, \quad \text{for } n = 0,1,2,\dots. \tag{3.2}$$

*Then, $\tilde{X}_\tau^i$ is a fuzzy random variable.*

Now, for an object $i$, we consider the estimation of the fuzzy stochastic system stopped at a finite stopping time $\tau$, by the evaluation of the fuzzy random variable $\tilde{X}_\tau^i$. Let a map $g : \mathcal{I} \mapsto \mathbb{R}$ satisfy the following three conditions (L.i)–(L.iii):

(L.i) $g([a,b] + [c,d]) = g([a,b]) + g([c,d])$ for $[a,b],[c,d] \in \mathcal{I}$.
(L.ii) $g(\lambda[a,b]) = \lambda g([a,b])$ for $[a,b] \in \mathcal{I}$, $\lambda \ge 0$.
(L.iii) $a \le g([a,b]) \le b$ for $[a,b] \in \mathcal{I}$.

This function is called a 'linear ranking function' which is used for the evaluation of fuzzy numbers [16]. Conditions (L.i) and (L.ii) mean linearity and (L.iii) means the regularity about the estimation of $\alpha$-cuts of fuzzy numbers. Then, $g$ preserves the order of intervals corresponding to the 'fuzzy max order'

$$g([a,b]) \le g([c,d]),$$

for $[a,b],[c,d] \in \mathcal{I}$ such that $[a,b] \preceq [c,d]$, which means that $a \le c$ and $b \le d$. Then, the following lemma can be checked easily.

LEMMA 3.2. *For a map $g : \mathcal{I} \mapsto \mathbb{R}$, the following statements (i)–(iii) are equivalent.*

(i) *$g$ is a linear ranking function.*
(ii) *$g$ satisfies*

$$g(\lambda_1[0,1] + \lambda_2) = \lambda_1 g([0,1]) + \lambda_2, \qquad \text{for } \lambda_1 \ge 0, \quad \lambda_2 \in \mathbb{R}.$$

(iii) *$g$ satisfies*

$$g([a,b]) = a(1-k) + bk, \qquad \text{for } [a,b] \in \mathcal{I},$$

*where $k := g([0,1]) \in [0,1]$.*

From Lemma 3.2(iii), there exist $p$, $q$ ($p, q \geq 0$, $p + q = 1$) such that

$$g([a,b]) = pa + qb \tag{3.3}$$

for an interval $[a, b] \in \mathcal{I}$. Also, we note that if we define

$$g_{(p,q)}(\tilde{u}) := \int_0^1 \left(p\tilde{u}_\alpha^- + q\tilde{u}_\alpha^-\right) \, d\alpha, \tag{3.4}$$

then $\tilde{u} \preceq \tilde{v}$ in the fuzzy max order implies $g_{(p,q)}(\tilde{u}) \leq g_{(p,q)}(\tilde{v})$. However, the reverse does not hold in general. From (3.2), for $\omega \in \Omega$, the $\alpha$-cut of the fuzzy number $\tilde{X}_\tau^i(\omega)$ must be a closed interval $[\tilde{X}_\tau^i(\omega)]_\alpha$. Therefore, from definition (2.3), the expectation is given by the closed interval

$$E\left(\left[\tilde{X}_\tau^i(\cdot)\right]_\alpha\right). \tag{3.5}$$

Using the above linear ranking function $g$, we put

$$g\left(E\left(\left[\tilde{X}_\tau^i(\cdot)\right]_\alpha\right)\right). \tag{3.6}$$

Therefore, the evaluation of the fuzzy random variable $\tilde{X}_\tau^i$ is represented by the following integral:

$$\int_0^1 g\left(E\left(\left[\tilde{X}_\tau^i(\cdot)\right]_\alpha\right)\right) \, d\alpha. \tag{3.7}$$

LEMMA 3.3. *For an object $i = 1, 2, \ldots, k$ and a finite stopping time $\tau$, it holds that*

$$\int_0^1 g\left(E\left(\left[\tilde{X}_\tau^i(\cdot)\right]_\alpha\right)\right) d\alpha = \int_0^1 E\left(g\left(\left[\tilde{X}_\tau^i(\cdot)\right]_\alpha\right)\right) d\alpha = E\left(\int_0^1 g\left(\left[\tilde{X}_\tau^i(\cdot)\right]_\alpha\right) d\alpha\right). \tag{3.8}$$

PROOF. Properties (3.3) and (3.4) of $g$ imply immediately, for each $\alpha$, $g(E([\tilde{X}_\tau^i(\cdot)]_\alpha)) = E(g([\tilde{X}_\tau^i(\cdot)]_\alpha))$. Therefore,

$$\int_0^1 g\left(E\left(\left[\tilde{X}_\tau^i(\cdot)\right]_\alpha\right)\right) d\alpha = \int_0^1 E\left(g\left(\left[\tilde{X}_\tau^i(\cdot)\right]_\alpha\right)\right) d\alpha.$$

Also, by Fubini's theorem, we have

$$\int_0^1 E\left(g\left(\left[\tilde{X}_\tau^i(\cdot)\right]_\alpha\right)\right) d\alpha = E\left(\int_0^1 g\left(\left[\tilde{X}_\tau^i(\cdot)\right]_\alpha\right) d\alpha\right).$$

These complete the proof of this lemma. ∎

In the following definition, we modify fuzzy stopping times introduced by Kurano *et al.* [10] in order to apply them to fuzzy random variables.

DEFINITION 3.1. *A map $\tilde{\tau} : \mathbb{N} \times \Omega \mapsto [0,1]$ is called a fuzzy stopping time if it satisfies the following (i)–(iii).*

(i) *For each $n = 0, 1, 2, \ldots$, the map $\omega \mapsto \tilde{\tau}(n,\omega)$ is $\mathcal{M}_n$-measurable.*
(ii) *For almost all $\omega \in \Omega$, the map $n \mapsto \tilde{\tau}(n,\omega)$ is nonincreasing.*
(iii) *For almost all $\omega \in \Omega$, there exists an integer $m$ such that $\tilde{\tau}(n,\omega) = 0$ for all $n \geq m$.*

Regarding the grade of membership of fuzzy stopping times, '$\tilde{\tau}(n,\omega) = 0$' means 'to stop at time $n$' and '$\tilde{\tau}(n,\omega) = 1$' means 'to continue at time $n$', respectively. And the intermediate value '$0 < \tilde{\tau}(n,\omega) < 1$' is a notion of 'fuzzy stopping'. It is easy to check the following lemma regarding construction of fuzzy stopping times [10].

LEMMA 3.4.

(i) *Let $\tilde{\tau}$ be a fuzzy stopping time. Define a map $\tilde{\tau}_\alpha : \Omega \mapsto \mathbb{N}$ by*

$$\tilde{\tau}_\alpha(\omega) := \inf\{n \mid \tilde{\tau}(n,\omega) < \alpha\}, \qquad \omega \in \Omega, \quad \text{for } \alpha \in (0,1], \tag{3.9}$$

*where the infimum of the empty set is understood to be $+\infty$. Then, we have*

(a) $\{\omega \mid \tilde{\tau}_\alpha(\omega) \le n\} \in \mathcal{M}_n$ *for* $n = 0,1,2,\dots$;

(b) $\tilde{\tau}_\alpha(\omega) \le \tilde{\tau}_{\alpha'}(\omega)$ *a.a.* $\omega \in \Omega$ *if* $\alpha \ge \alpha'$;

(c) $\lim_{\alpha' \uparrow \alpha} \tilde{\tau}_{\alpha'}(\omega) = \tilde{\tau}_\alpha(\omega)$ *a.a.* $\omega \in \Omega$ *if* $\alpha > 0$;

(d) $\tilde{\tau}_0(\omega) := \lim_{\alpha \downarrow 0} \tilde{\tau}_\alpha(\omega) < \infty$ *a.a.* $\omega \in \Omega$.

(ii) *Let $\{\tilde{\tau}_\alpha\}_{\alpha\in[0,1]}$ be maps $\tilde{\tau}_\alpha : \Omega \mapsto \mathbb{N}$ satisfying the above (a), (b), and (d). Define a map $\tilde{\tau} : \mathbb{N} \times \Omega \mapsto [0,1]$ by*

$$\tilde{\tau}(n,\omega) := \sup_{\alpha\in[0,1]} \min\left\{\alpha, 1_{\{\omega \mid \tilde{\tau}_\alpha(\omega) > n\}}(\omega)\right\}, \qquad \text{for } n = 0,1,2,\dots \quad \text{and} \quad \omega \in \Omega. \tag{3.10}$$

*Then $\tilde{\tau}$ is a fuzzy stopping time.*

Fuzzy stopping times are always finite from Definition 3.1(iii). Now, by using Lemma 3.4 and the linear ranking function $g$, we consider the estimation of the fuzzy stochastic system stopped at a 'fuzzy' stopping time $\tilde{\tau}$ regarding the $i^{\text{th}}$ object. Let $i = 1,2,\dots,k$ be an object and let $\tilde{\tau}$ be a fuzzy stopping time $\tilde{\tau}$. From Lemma 3.1, we have $[\tilde{X}^i_{\tilde{\tau}_\alpha}(\omega)]_\alpha := [\tilde{X}^i_n(\omega)]_\alpha$ for $\omega \in \{\omega \mid \tilde{\tau}_\alpha(\omega) = n\}$, where $\tilde{\tau}_\alpha(\omega)$ are 'classical' stopping times given by (3.9). By Lemma 3.3, we define a random variable

$$G^i_{\tilde{\tau}}(\omega) := \int_0^1 g\left(\left[\tilde{X}^i_{\tilde{\tau}_\alpha}(\omega)\right]_\alpha\right) \mathrm{d}\alpha, \qquad \omega \in \Omega. \tag{3.11}$$

Note that (3.11) is well defined since the function $\alpha \mapsto g([\tilde{X}^i_{\tilde{\tau}_\alpha}(\omega)]_\alpha)$ is left-continuous on $(0,1]$. Therefore, the expectation $E(G^i_{\tilde{\tau}})$ is the evaluation (3.7) of the fuzzy random variable $\tilde{X}_{\tilde{\tau}}$. By Fubini's theorem, we have

$$E\left(G^i_{\tilde{\tau}}\right) := E\left(\int_0^1 g\left(\left[\tilde{X}^i_{\tilde{\tau}_\alpha}(\cdot)\right]_\alpha\right) \mathrm{d}\alpha\right) = \int_0^1 E\left(g\left(\left[\tilde{X}^i_{\tilde{\tau}_\alpha}(\cdot)\right]_\alpha\right)\right) \mathrm{d}\alpha, \tag{3.12}$$

for fuzzy stopping times $\tilde{\tau}$. Then, Pareto optimal solutions for the multiobjective stopping model are characterized as follows.

DEFINITION 3.2. *A fuzzy stopping time $\tilde{\tau}^*$ is called Pareto optimal if there exists no fuzzy stopping time $\tilde{\tau}$ such that*

$$E\left(G^i_{\tilde{\tau}}\right) \ge E\left(G^i_{\tilde{\tau}^*}\right), \qquad \text{for all objects } i = 1,2,\dots,k,$$

*and*

$$E\left(G^i_{\tilde{\tau}}\right) > E\left(G^i_{\tilde{\tau}^*}\right), \qquad \text{for some object } i = 1,2,\dots,k.$$

## 4. PARETO OPTIMAL FUZZY STOPPING TIMES

In this section, we give Pareto optimal solutions for the problem in Section 3. We introduce the following $\lambda$-optimal stopping times in order to obtain Pareto optimal stopping times. Real numbers $\{\lambda^i\}_{i=1}^k$ are called weights of objects if they satisfy

$$\sum_{i=1}^k \lambda^i = 1 \quad \text{and} \quad \lambda^i \ge 0, \qquad i = 1,2,\dots,k. \tag{4.1}$$

For a set of weights $\lambda := \{\lambda^i\}_{i=1}^k$, we define a fuzzy stochastic system $\{\tilde{X}_n^\lambda\}_{n=0}^\infty$, which is $\{\mathcal{M}_n\}_{n=0}^\infty$-adapted, by

$$\tilde{X}_n^\lambda(\omega)(x) := \sup_{\alpha\in[0,1]} \min\left\{\alpha, 1_{[\tilde{X}_n^\lambda(\omega)]_\alpha}(x)\right\}, \qquad \omega \in \Omega, \quad x \in \mathbb{R},$$

where the $\alpha$-cuts $[\tilde{X}_n^\lambda(\omega)]_\alpha$ are closed intervals given by

$$\begin{aligned}\left[\tilde{X}_n^\lambda(\omega)\right]_\alpha &= \sum_{i=1}^k \lambda^i \left[\tilde{X}_n^i(\omega)\right]_\alpha \\ &= \sum_{i=1}^k \lambda^i \left[\left[\tilde{X}_n^i(\omega)\right]_\alpha^-, \left[\tilde{X}_n^i(\omega)\right]_\alpha^+\right] \\ &= \left[\sum_{i=1}^k \lambda^i \left[\tilde{X}_n^i(\omega)\right]_\alpha^-, \sum_{i=1}^k \lambda^i \left[\tilde{X}_n^i(\omega)\right]_\alpha^+\right], \qquad \omega \in \Omega.\end{aligned}$$

For fuzzy stopping times $\tilde{\tau}$, in the same way as (3.11) we define a random variable

$$G_{\tilde{\tau}}^\lambda(\omega) := \int_0^1 g\left(\left[\tilde{X}_{\tilde{\tau}_\alpha}^\lambda(\omega)\right]_\alpha\right) \mathrm{d}\alpha, \qquad \text{for } \omega \in \Omega.$$

Similarly to the proof of Lemma 3.3, we can easily check that its expectation is reduced to the weighted sum of the expectations for objects

$$E\left(G_{\tilde{\tau}}^\lambda\right) = \sum_{i=1}^k \lambda^i E\left(G_{\tilde{\tau}}^i\right). \tag{4.2}$$

Now we give the definition of $\lambda$-optimal stopping times as follows.

DEFINITION 4.1. *Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects. Then a fuzzy stopping time $\tilde{\tau}^*$ is called $\lambda$-optimal if*

$$E\left(G_{\tilde{\tau}^*}^\lambda\right) \geq E\left(G_{\tilde{\tau}}^\lambda\right),$$

*for all fuzzy stopping times $\tilde{\tau}$.*

THEOREM 4.1. *Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects such that*

$$\sum_{i=1}^k \lambda^i = 1 \quad \text{and} \quad \lambda^i > 0, \qquad i = 1, 2, \ldots, k. \tag{4.3}$$

*Then a $\lambda$-optimal fuzzy stopping time $\tilde{\tau}^*$ is Pareto optimal.*

PROOF. Let $\tilde{\tau}^*$ be a finite $\lambda$-optimal fuzzy stopping time. If $\tilde{\tau}^*$ is not Pareto optimal, then there exists a fuzzy stopping time $\tilde{\tau}$ such that

$$E\left(G_{\tilde{\tau}}^i\right) \geq E\left(G_{\tilde{\tau}^*}^i\right), \qquad \text{for all objects } i = 1, 2, \ldots, k,$$

and

$$E\left(G_{\tilde{\tau}}^i\right) > E\left(G_{\tilde{\tau}^*}^i\right), \qquad \text{for some object } i = 1, 2, \ldots, k.$$

Then from (4.2) we have

$$E\left(G_{\tilde{\tau}}^\lambda\right) = \sum_{i=1}^k \lambda^i E\left(G_{\tilde{\tau}}^i\right) > \sum_{i=1}^k \lambda^i E\left(G_{\tilde{\tau}^*}^i\right) = E\left(G_{\tilde{\tau}^*}^\lambda\right).$$

This contradicts the $\lambda$-optimality of $\tilde{\tau}^*$, and so we obtain this theorem. ∎

Finally, in order to construct $\lambda$-optimal fuzzy stopping times, we introduce the following $(\lambda, \alpha)$-optimal fuzzy stopping times.

DEFINITION 4.2. *Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects and let $\alpha \in [0,1]$. A fuzzy stopping time $\tilde{\tau}^*$ is called $(\lambda, \alpha)$-optimal if*

$$E\left(g\left(\left[\tilde{X}^\lambda_{\tilde{\tau}^*_\alpha}(\cdot)\right]_\alpha\right)\right) \geq E\left(g\left(\left[\tilde{X}^\lambda_{\tilde{\tau}_\alpha}(\cdot)\right]_\alpha\right)\right),$$

*for all fuzzy stopping times $\tilde{\tau}$.*

In order to characterize $(\lambda, \alpha)$-optimal stopping times, we let real random variables

$$\gamma^\lambda_{n,\alpha} := \operatorname*{ess\,sup}_{\tau:\text{stopping times}, \tau \geq n} E\left(g\left(\left[\tilde{X}^\lambda_\tau(\cdot)\right]_\alpha\right) \mid \mathcal{M}_n\right), \qquad \text{for } n = 0, 1, 2, \dots, \tag{4.4}$$

where the definition of the essential supremum is referred to [2, Chapters 1– 6]. Define a stopping time $\sigma^\lambda_\alpha : \Omega \mapsto \mathbb{N}$ by

$$\sigma^\lambda_\alpha(\omega) := \inf\left\{n \mid g\left(\left[\tilde{X}^\lambda_n(\omega)\right]_\alpha\right) = \gamma^\lambda_{n,\alpha}(\omega)\right\}, \tag{4.5}$$

for $\omega \in \Omega$ and $\alpha \in [0,1]$, where the infimum of the empty set is understood to be $+\infty$. Then the following lemma can be checked easily by Chow *et al.* [2, Theorem 4.1].

LEMMA 4.1. *Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects. Suppose*

$$P\left(\sigma^\lambda_\alpha < \infty\right) = 1, \qquad \text{for all } \alpha \in [0,1]. \tag{4.6}$$

*Then, for $\alpha \in [0,1]$, the following (i) and (ii) hold:*

(i) $\gamma^\lambda_{n,\alpha}(\omega) = \max\{g([\tilde{X}^\lambda_n(\omega)]_\alpha), E(\gamma^\lambda_{n+1,\alpha} \mid \mathcal{M}_n)(\omega)\}$ a.a. $\omega \in \Omega$ *for* $n = 0, 1, 2, \dots$;
(ii) $\sigma^\lambda_\alpha$ *is $(\lambda, \alpha)$-optimal and* $E(\gamma^\lambda_{0,\alpha}) = E(g([\tilde{X}^\lambda_{\sigma^\lambda_\alpha}(\cdot)]_\alpha))$.

In order to construct an optimal fuzzy stopping time from the $(\lambda, \alpha)$- optimal stopping times $\{\sigma^\lambda_\alpha\}_{\alpha\in[0,1]}$, we need the following regularity condition.

ASSUMPTION A. *Regularity. The map $\alpha \mapsto \sigma^\lambda_\alpha(\omega)$ is nonincreasing for almost all $\omega \in \Omega$.*

Under Assumption A, we can define a map $\tilde{\sigma}^\lambda : \mathbb{N} \times \Omega \mapsto [0,1]$ by

$$\tilde{\sigma}^\lambda(n,\omega) := \sup_{\alpha\in[0,1]} \min\left\{\alpha, 1_{\{\omega \mid \sigma^\lambda_\alpha(\omega) > n\}}(\omega)\right\}, \qquad \text{for } n = 0, 1, 2, \dots \quad \text{and} \quad \omega \in \Omega. \tag{4.7}$$

Then, from Assumption A, $\sigma^\lambda_\alpha(\omega)$ satisfies (a), (b), and (d) of Lemma 3.4(i). So, $\tilde{\sigma}^\lambda(n,\omega)$ defined by (4.7) is a fuzzy stopping time from Lemma 3.4(ii). Put the $\alpha$-cut (3.9) of $\tilde{\sigma}^\lambda(n,\omega)$ by $\tilde{\sigma}^\lambda_\alpha(\omega)$. Then, $\tilde{\sigma}^\lambda_\alpha(\omega)$ has properties (a)–(d) of Lemma 3.4(i) and we also have $\tilde{\sigma}^\lambda(n,\omega) = \sup_{\alpha\in[0,1]} \min\{\alpha, 1_{\{\omega\mid\tilde{\sigma}^\lambda_\alpha(\omega)>n\}}(\omega)\}$. Thus, the map $\alpha \mapsto \tilde{\sigma}^\lambda_\alpha(\omega)$ is a left continuous version of the non-increasing map $\alpha \mapsto \sigma^\lambda_\alpha(\omega)$. Therefore, $\tilde{\sigma}^\lambda_\alpha(\omega)$ coinsides with $\sigma^\lambda_\alpha(\omega)$ except for at most countable many $\alpha \in (0,1]$, so we obtain the following result.

THEOREM 4.2. *Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects satisfying (4.3). Suppose (4.6) and Assumption A hold. Then $\tilde{\sigma}^\lambda$ is a $\lambda$-optimal fuzzy stopping time and it is also Pareto optimal.*

PROOF. From Assumption A and Lemma 3.4(ii), $\tilde{\sigma}^\lambda$ is a fuzzy stopping time and we obtain

$$E\left(G^\lambda_{\tilde{\tau}}\right) \leq \int_0^1 \sup_\tau E\left(g\left(\left[\tilde{X}^\lambda_\tau(\cdot)\right]_\alpha\right)\right) \, d\alpha,$$

for all fuzzy stopping times $\tilde{\tau}$ by Lemma 4.1. And also the last equals

$$\int_0^1 E\left(\gamma^\lambda_{0,\alpha}\right) \, d\alpha = \int_0^1 E\left(g\left(\left[\tilde{X}^\lambda_{\sigma^\lambda_\alpha}(\cdot)\right]_\alpha\right)\right) \, d\alpha.$$

Since $\tilde{\sigma}_\alpha^\lambda(\omega) \neq \sigma_\alpha^\lambda(\omega)$ holds only at most countable $\alpha \in (0,1]$, we have

$$E\left(\int_0^1 g\left(\left[\tilde{X}^\lambda_{\sigma^\lambda_\alpha}(\cdot)\right]_\alpha\right) \mathrm{d}\alpha\right) = E\left(\int_0^1 g\left(\left[\tilde{X}^\lambda_{\tilde{\sigma}^\lambda_\alpha}(\cdot)\right]_\alpha\right) \mathrm{d}\alpha\right)$$

Thus, by Fubini's theorem, we can show

$$\begin{aligned} E\left(G^\lambda_{\tilde{\tau}}\right) &\leq \int_0^1 E\left(g\left(\left[\tilde{X}^\lambda_{\sigma^\lambda_\alpha}(\cdot)\right]_\alpha\right)\right) \mathrm{d}\alpha \\ &= E\left(\int_0^1 g\left(\left[\tilde{X}^\lambda_{\sigma^\lambda_\alpha}(\cdot)\right]_\alpha\right) \mathrm{d}\alpha\right) \\ &= E\left(\int_0^1 g\left(\left[\tilde{X}^\lambda_{\tilde{\sigma}^\lambda_\alpha}(\cdot)\right]_\alpha\right) \mathrm{d}\alpha\right) \\ &= \int_0^1 E\left(g\left(\left[\tilde{X}^\lambda_{\tilde{\sigma}^\lambda_\alpha}(\cdot)\right]_\alpha\right)\right) \mathrm{d}\alpha \\ &= E\left(G^\lambda_{\tilde{\sigma}^\lambda}\right). \end{aligned}$$

Therefore, $\tilde{\sigma}^\lambda$ is $\lambda$-optimal. We also obtain Pareto optimality of $\tilde{\sigma}^\lambda$ from Theorem 4.1 in [2]. ∎

## REFERENCES


1. M.L. Puri and D.A. Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.* **114**, 409–422, (1986).
2. Y.S. Chow, H. Robbins and D. Siegmund, *The Theory of Optimal Stopping: Great Expectations*, Houghton Mifflin Company, New York, (1971).
3. A.N. Shiryayev, *Optimal Stopping Rules*, Springer, New York, (1979).
4. Y. Yoshida, M. Yasuda, J. Nakagami and M. Kurano, Optimal stopping problems in a stochastic and fuzzy system, *J. Math. Analy. and Appl.* **246**, 135–149, (2000).
5. Y. Yoshida, Markov chains with a transition possibility measure and fuzzy dynamic programming, *Fuzzy Sets and Systems* **66**, 39–57, (1994).
6. Y. Yoshida, An optimal stopping problem in dynamic fuzzy systems with fuzzy rewards, *Computers Math. Applic.* **32** (10), 17–28, (1996).
7. Y. Yoshida, Duality in dynamic fuzzy systems, *Fuzzy Sets and Systems* **95**, 53–65, (1998).
8. J.P. Aubin, *Mathematical Methods of Game and Economic Theory*, North-Holland, Amsterdam, (1979).
9. Y. Ohtsubo, Multi-objective stopping problem for a monotone case, *Mem. Fac. Sci. Kochi Univ. Ser. A* **18**, 99–104, (1997).
10. M. Kurano, M. Yasuda, J. Nakagami and Y. Yoshida, An approach to stopping problems of a dynamic fuzzy system, (Preprint).
11. Y. Yoshida, M. Yasuda, J. Nakagami and M. Kurano, A monotone fuzzy stopping time for dynamic fuzzy systems, *Bull. Infor. Cyber. Res. Ass. Stat. Sci. Kyushu University* **31**, 91–99, (1999).
12. L.A. Zadeh, Fuzzy sets, *Inform. and Control* **8**, 338–353, (1965).
13. M.L. Puri and D.A. Ralescu, Convergence theorem for fuzzy martingales, *J. Math. Anal. Appl.* **160**, 107–122, (1991).
14. G. Wang and Y. Zhang, The theory of fuzzy stochastic processes, *Fuzzy Sets and Systems* **51**, 161–178, (1992).
15. M. Kurano, M. Yasuda, J. Nakagami and Y. Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51**, 83–88, (1992).
16. P. Fortemps and M. Roubens, Ranking and defuzzification methods based on area compensation, *Fuzzy Sets and Systems* **82**, 319–330, (1996).
17. J.C. Cox, S.A. Ross and M. Rubinstein, Option pricing: A simplified approach, *J. Financial Economics* **7**, 229–263, (1979).