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On A Separation Of A Stopping Game Problem For Standard Brownian Motion

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ABSTRACT. The stopping game problem for standard Brownian motion is separated into two ordinary optimal stopping problems under appropriate conditions. We show that the zero-sum equilibrium point is directly composed of two problems of a maximization and a minimization for stopping times.

1. Introduction

The optimal stopping problem for a time-homogeneous one dimensional stochastic process, especially for Brownian motion, can be shown to have a concrete solution([1],[3] etc.). The optimal stopping time of the problem is the first hitting time of a connected region if the reward does not depend on time with a growth condition. It is the control-limit type policy in the decision theory. For the minimization problem where the payoff is an increasing function, it should be continue when the state is smaller than a certain threshold value and stop otherwise.

The stopping game problem is a game variant of the optimal stopping problem. There are so many papers and books([2],[4],[5],[6],[12],[15] etc.) The objectives are mainly the existance of the optimal(equilibruim) policy or the characterization of the game value. Also the analitical aspects of the game value concerning with the variational inequalities, the penalty method of the partial differential equation, the relationship with the impuls stochastic control problem, and the extension for the general stochastic processes are discussed in these refereces.

In this note, our aim is to solve the game problem explicitly. Considering a zero-sum two person game on a standard Brownian motion, we will determine the two threshold values for this stopping game problem and show that the game problem will be separated into two usual optimal stopping problems.

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The motivation is due to the optimal stopping problem discussed by [8]. It is a problem for the positive axis with absorption at the origin. We consider that its explicit solution for the negative axis reversely. Then the composition of the whole axis constructs the solution in the stopping game problem. The existence of the value or its charactorizaton are discussed abstractly by many authors, however, a concrete solution for the stopping game problem perhaps needs to be given.

The one dimensional standard Brownian motion $\{x_t; t \geq 0\}$ is as follows:

$$dx_t = bdt + adw_t, \quad x_0 = x \tag{1.1}$$

where the drift b, the diffusion coefficient $a \neq 0$ are constants and $\{w_t\}$ is a Wiener process. It is seen that letting $a = 1, b = 0$ does not affect our result, however we adapts this case in order to clarify a behavior of the terms.

The formulation of the stopping game problem under the system, the state space is the real line and there are three kinds of payoff functions: $\varphi(x), \psi(x), \chi(x); -\infty < x < \infty$. Observing the process $\{x_t\}$, let τ and σ be the two stopping times such that player 1's objective is to select τ so as to minimize his expected payoff and player 2's is to select σ as to maximize it. Define the upper and lower value of the game as

$$\begin{aligned} \overline{w}(x) &= \inf_{0\leq\tau<\infty} \sup_{0\leq\sigma<\infty} E^x[R(\tau,\sigma)], \\ \underline{w}(x) &= \sup_{0\leq\sigma<\infty} \inf_{0\leq\tau<\infty} E^x[R(\tau,\sigma)] \end{aligned} \tag{1.2}$$

where

$$R(\tau,\sigma) = \begin{cases} e^{-\alpha\tau}\varphi(x_\tau), & \text{if } \tau < \sigma; \\ e^{-\alpha\sigma}\psi(x_\sigma), & \text{if } \tau > \sigma; \\ e^{-\alpha\tau}\chi(x_\tau), & \text{if } \tau = \sigma \end{cases}$$

and $\alpha > 0$ and E^x denotes the expectation corresponding to the initial point x.

For all finite stopping times the expectaion are assumed well defined. And to avoid the possibility of not stopping the game, the payoff to players is zero. That is, we assume that $\lim_{t,s\to\infty} R(t,s) = 0, almost surely$. Fundamental next assumption([10]) is important for the policy of the game problem.

ASSUMPTION 1. *Assume*

$$\varphi(x) < \chi(x) < \psi(x); \quad -\infty < x < \infty. \tag{1.3}$$

It is known that the both value of (1.2) are equal and it is called as the value of the game: $w(x) = \overline{w}(x) = \underline{w}(x)$. Also there exists a pair of stopping times (τ^*, σ^*) which is called an equilibrium pair, i.e. corresponding expectation is $w(x)$.

According to the zero-sum matrix game theory, if the equally minimax and maximin value is denoted by 'val', we could write down the value of the game as the following equation, which corresponds to the optimality equality in the dynamic programming theory:

$$\text{val}\begin{pmatrix} \chi - w & \varphi - w \\ \psi - w & \mathcal{A}w - \alpha w \end{pmatrix} = 0 \tag{1.4}$$

where the infinitesimal generator $\mathcal{A}$ of Brownian motion:

$$\mathcal{A}w = \frac{a^2}{2}w'' + bw'. \tag{1.5}$$

The assumption 2 implies that the game matrix (1.4) is determined, i.e. the pure strategy exists in this case and it makes the strategy simple. In other word, there exists a stopping time for each player and there is no needs to consider the extended stopping time([14]). And the case $\chi - w = 0$ never occure from the assumption. So it is sufficient to consider the following three cases:

$$w = \varphi, \qquad w = \psi \qquad \text{and} \qquad \mathcal{A}w - \alpha w = 0.$$

Each region corresponding to the above equality denotes the stop region for player 1, player 2 and the continuity region of both players respectively. If we impose some conditions, it is expected that these region will be divided into three intervals on the real line. In order to divide the interval one must therefore determine two threshold values. This type of the free boundary problem is called as two obstacles problem([8]). That is, we have reduced the problem into finding two values z_1, z_2 and a function $w = w(x); -\infty < x < \infty$ which satisfies

$$\begin{aligned} w(x) &= \varphi(x) && \text{for} \quad z_1 < x \\ w(x) &= \psi(x) && \text{for} \quad x < z_2 \\ \mathcal{A}w(x) - \alpha w(x) &= 0 && \text{for} \quad z_2 \le x \le z_1 \end{aligned} \tag{1.6}$$

under some conditions.

In the next section 2 we will make a separation of the solution $w = w(x)$ on the real line into two. So that the right-hand-side of the line corresponds to the minimization of player 1 and the left-hand-side to the maximization of player 2 respectively. Using the two problems, an explicit expession for the solution of (1.6) is obtained in the section 3.

2. Two optimal stopping problems

The next two problems are the ordinary optimal stopping problems where the reward is absorbed at some point, for example, zero on the real line. Therefore it is restricted to either in a positive part or a negative part. We are given a constant K, which means that the reward is absorbed at zero, and the two functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$, are the rewards of players.

(I) *Minimization of Player 1 with the reward φ, K :*

$$u(x) = u(x;K) = \inf_{0 \le \tau < \infty} E^x[R_\varphi(\tau, \sigma_0)] \tag{2.1i}$$

where

$$R_\varphi(\tau,\sigma)=\begin{cases} e^{-\alpha\tau}\varphi(x_\tau), & \text{if } \tau<\sigma; \\ Ke^{-\alpha\sigma}, & \text{if } \sigma\le\tau \end{cases}$$

and $\sigma_0=\inf\{t\ge 0; x_t\ge 0\}$.

(II) *Maximization of Player 2 with the reward* ψ, K :

$$v(x)=v(x;K)=\sup_{0\le\sigma<\infty} E^x[R_\psi(\tau_0,\sigma)] \tag{2.1ii}$$

where

$$R_\psi(\tau,\sigma)=\begin{cases} e^{-\alpha\sigma}\psi(x_\sigma), & \text{if } \sigma<\tau; \\ Ke^{-\alpha\tau}, & \text{if } \tau\le\sigma \end{cases}$$

and $\tau_0=\inf\{t\ge 0; x_t\le 0\}$.

As the example of (I) or (II), it is already discussed by [8] having a connection with stochastic controls and Gittens index in the bandit problem.

We impose a condition as (2.2) so that the state space $-\infty<x<\infty$ of the real line is separated into two. For simplicity, the separation point is assumed at the origin.

ASSUMPTION 2. *The function $\varphi(x),\psi(x)$ satisfy the next inequality in the each region:*

$$\begin{aligned} \mathcal{A}\varphi(x)-\alpha\varphi(x)>0 \quad & \textit{for} \quad x>0, \\ \mathcal{A}\psi(x)-\alpha\psi(x)<0 \quad & \textit{for} \quad x<0 \end{aligned} \tag{2.2}$$

provided $\varphi(x),\psi(x)$ are belong to the domain of the operator (1.5).

LEMMA 2.1.

(a) *The optimality equations of* (I),(II) *become*

$$\min\{\mathcal{A}u(x)-\alpha u(x),\ \varphi(x)-u(x)\}=0 \quad \textit{for} \quad x>0, \qquad u(0)=K \tag{2.3i}$$

and

$$\max\{\mathcal{A}v(x)-\alpha v(x),\ \psi(x)-v(x)\}=0 \quad \textit{for} \quad x<0, \qquad v(0)=K \tag{2.3ii}$$

respectively.

(b) *The stopping region for the minimization* (I) *is included in* $(0,\infty)$, *and the stop region for the maximization* (II) *is included in* $(-\infty,0)$.

PROOF. (a) The equation $(2.3i),(2.3ii)$ is well known in the theory of optimal stopping, and the additional condition is obtained because one gets the reward K at the origin. (b) By Dynkin's formula, consider Infinitesimal Look Ahead policy([11]). If one continues the observation in the region $\{x\,|\,\mathcal{A}\varphi(x)-\alpha\varphi(x)>0\}$, one's reward decreases in case of (I). The decision is better to stop. Hence the optimal stopping region is included by the region. But the system is not closed(absorbing) in this region so the proper sub-region becomes the optimal region for stopping. The region is also included by $(0,\infty)$ since (2.2) hold for $x>0$. For the case (II), it is shown by the oppsite side of inequality. □

The following discussion owes much to the condition that coefficients in (1.1) are constant. The extension to the non-constant case seems the solution complex and hard.

Hereafter, we will define the reals λ_1,λ_2 and the functions $C_1(x;f), C_2(x;f)$, $C(x;f)$ as follows:

(i) The real numbers $\lambda_1,\lambda_2(\lambda_1\geq\lambda_2)$ denote the solutions of the quadratic equation: $a^2\lambda^2+2b\lambda-2\alpha=0$, where a,b is the drift and the diffusion coefficient in (1.1). Note that $b^2+2a^2\alpha>0$ and there exists always two real roots.

(ii) For the smooth function $f=f(x), -\infty<x<\infty$,

$$
C_1(x;f)=\frac{\exp(-\lambda_1 x)}{\lambda_1-\lambda_2}\{f'(x)-\lambda_2 f(x)\},
$$

$$
C_2(x;f)=\frac{\exp(-\lambda_2 x)}{\lambda_1-\lambda_2}\{\lambda_1 f(x)-f'(x)\}, \tag{2.4}
$$

$$
C(x;f)=C_1(x;f)+C_2(x;f).
$$

Using the above notation, we can write down the optimal value of the minimization (I) and the maximization (II) explicitly.

LEMMA 2.2. *The optimal value of* (2.1*i*) *is as follows:*

$$
u(x;K)=\begin{cases} C_1(z_1;\varphi)e^{\lambda_1 x}+C_2(z_1;\varphi)e^{\lambda_2 x} & \text{for } 0\leq x\leq z_1 \\ \varphi(x) & \text{for } x\geq z_1 \end{cases} \tag{2.5i}
$$

where z_1 *depends on* K *and* $C(z_1;\varphi)=K$. *Similarly the optimal value of* (2.1*ii*) *is as follows:*

$$
v(x;K)=\begin{cases} C_1(z_2;\psi)e^{\lambda_1 x}+C_2(z_2;\psi)e^{\lambda_2 x} & \text{for } z_2\leq x\leq 0 \\ \psi(x) & \text{for } x\leq z_2 \end{cases} \tag{2.5ii}
$$

where z_2 *depends on* K *and* $C(z_2;\psi)=K$.

PROOF. Since the standard Brownian motion is regular, it is known that the optimal value is differentiable. So the principle of smooth-fit([13])

$$
u(x)=\varphi(x)|_{x=z_1},\quad u'(x)=\varphi'(x)|_{x=z_1}
$$

holds. By solving the diffrential equation for u of $\mathcal{A}u(x)-\alpha u(x)=0$ under this boundary condition, we obtain (2.5*i*). Similarly (2.5*ii*) can be shown by $\mathcal{A}v(x)-\alpha v(x)=0$ and the condition

$$
v(x)=\psi(x)|_{x=z_2},\quad v'(x)=\psi'(x)|_{x=z_2}.
$$

□

3. Separation of the stopping game problem

In the previous section we have seen that the assumptions implies the simultaneous stopping dose not occur and so the two kind of the stopping problems could be considered for each player. To give an explicit solution of (1.6) in the game problem, it needs making an adjustment between them. More precisely, One must align the value of K for each of two optimal stopping problems so as to fit the game value. We consider the following simultaneous non-linear equation.

LEMMA 3.1. *If $\varphi = \varphi(x), \psi = \psi(x)$ satisfy Assumption 2, the equation for (z_1, z_2) with $z_1 > 0 > z_2$ such that*

$$C_1(z_1;\varphi) = C_1(z_2;\psi), \qquad C_2(z_1;\varphi) = C_2(z_2;\psi) \tag{3.1}$$

has at most one solution.

PROOF. Differentiating (2.4), we have that

$$C_1'(x;f) = \frac{2\exp(-\lambda_1 x)}{(\lambda_1-\lambda_2)a^2}\{\mathcal{A}f(x) - \alpha f(x)\},$$
$$C_2'(x;f) = \frac{-2\exp(-\lambda_2 x)}{(\lambda_1-\lambda_2)a^2}\{\mathcal{A}f(x) - \alpha f(x)\}$$

where $a \neq 0$ is the diffusion coefficient in (1.1). By Assumption 2 on φ and ψ, $C_1(x;\varphi)$ is strictly increasing in x and $C_1(x;\psi)$ is strictly decreasing in x. Therefore along the curve $\{(x,y); C_1(x;\varphi) - C_1(y;\psi) = 0\}$, if x increases then y decreases in the domain of $\{x > 0, y < 0\}$. Similarly $C_2(x;\varphi)$ is strictly decreasing in x and $C_2(x;\psi)$ is strictly increasing in x. So along the curve $\{(x,y); C_2(x;\varphi) - C_2(y;\psi) = 0\}$, if x increases then y increases also. Because of this monotonicity, it crosses at most once. Hence the equations have at most one solution. □

THEOREM 3.2. *If there exists the solution (z_1, z_2) of (3.1), then the game value of the stopping game problem, $w(x)$, is separated directly as two optimal values of $u(x;K)$ and $v(x;K)$. That is,*

$$w(x) = \begin{cases} u(x;K) & \text{for } x \geq 0 \\ v(x;K) & \text{for } x \leq 0 \end{cases} \tag{3.2}$$

where $K = C(z_1;\varphi) = C(z_2;\psi)$.

PROOF. By Assumption 1 and 2, the game value $w = w(x); -\infty < x < \infty$ equals

$$w(x) = \begin{cases} \varphi(x) & \text{for } x \geq z_1 \\ \psi(x) & \text{for } x \leq z_2 \end{cases}$$

in the stopping region for both players. For the continuation region,

$$\mathcal{A}w(x) - \alpha w(x) = 0;\ z_2 < x < z_1$$

with the smooth fitness at the boundary. If z_1 and z_2 satisfy (3.1), we can apply Lemma 2.2. Therefore, determining K at the origin with $w(0) = u(0, K) = v(0, K) = K$, (3.2) could be obtained. □

Note that the equilibruim pair (τ^*, σ^*) of the stopping game problem such that

$$w(x) = E^x[R(\tau^*, \sigma^*)]$$

is the first hitting time for the region $(z_1, \infty), (-\infty, z_2)$ respectively.

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