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ASYMPTOTIC RESULTS FOR THE BEST-CHOICE PROBLEM WITH A RANDOM NUMBER OF OBJECTS

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Abstract

This paper considers the best-choice problem with a random number of objects having a known distribution. The optimality equation of the problem reduces to an integral equation by a scaling limit. The equation is explicitly solved under conditions on the distribution, which relate to the condition for an OLA policy to be optimal in Markov decision processes. This technique is then applied to three different versions of the problem and an exact value for the asymptotic optimal strategy is found.

OPTIMAL STOPPING PROBLEM; BEST CHOICE; RANDOM NUMBER OF OBJECTS; SCALING LIMIT

1. Introduction

An optimal stopping problem is related to a Markov decision process with two actions: stop and continue. The equation for $v(i)$, the expected reward under an optimal policy when starting from state i, is given by

$$(1.1) \qquad v(i) = \max\left\{r(i),\ -c(i) + \sum p(i,j)v(j)\right\}, \qquad i \in \{1, 2, \cdots\}$$

where $r(i)$ is an immediate reward, $c(i)$ is a paying cost and $p(i,j)$ is a transition probability on the state space, $\{1, 2, \cdots\}$. The best-choice problem, variously called the secretary problem, Googol, dowry problem, etc. in Chow et al. (1971), in Gilbert and Mosteller (1966) and elsewhere, is an optimal stopping problem based on relative ranks for objects arriving in a random fashion; the objective is to find the stopping rule that maximizes the probability of attaining the best object of the sequence.

To consider the problem as a Markov decision process, suppose that the model is in state i iff the ith object to be examined is better than all its predecessors (the relatively best object) and the two actions are to accept this

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object, or reject it and wait for the successors. The immediate reward $r(i)$ is a probability that the object accepted in state i is absolutely the best one. The transition probability $p(i,j)$ is a conditional probability that the relatively next best object to appear will be the jth object in the sequence, given that the ith object in the sequence was relatively best.

The Markov chain formulation is considered, for example, by Dynkin and Yuskevich (1969) and so its details are omitted. The practical situation for the well-known problem of one choice among n objects then becomes: the state space is a set of integers $\{1,2,\cdots,n\}$, the reward $r(i)=i/n$ and the transition probability $p(i,j)=i/((j-1)j)$ for $1\leqq i<j\leqq n$, $p(i,j)=0$, otherwise. Hence (1.1) implies

$$(1.2)\quad v(i)=\max\left\{i/n,\ i\sum_{j=i+1}^{n} v(j)/((j-1)j)\right\},\qquad i=1,2,\cdots,n-1,\quad v(n)=1.$$

By solving this equation, one obtains the optimal value, i.e., the maximal probability of attaining the best object, and the optimal strategy, i.e., how to accept or reject an object.

Although the solution can be obtained easily in this case, let us consider the following alternative method. We investigate the conditional optimal value when the decision-maker rejects all objects until and including the ith relatively best, instead of the optimal value. Denote by $w(i;n)$ the second term on the right-hand side of (1.2). Since this term corresponds to the rejection and $v(i)$ is the optimal value, $w(i;n)$ will be the conditional optimal value. That is, let

$$w(i;n)=w(i)=i\sum_{j=i+1}^{n} v(j)/((j-1)j),\qquad i=1,2,\cdots,n-1$$

and $w(n)=0$. Then clearly $w(i)-w(i+1)=(v(i+1)-w(i+1))/(i+1)$ and so

$$(1.3)\quad w(i)-w(i+1)=((i+1)/n-w(i+1))^{+}/(i+1),\qquad i=1,2,\cdots,n-1$$

where $a^{+}=\max(a,0)$.

Following Mucci (1973) and Lorenzen (1981), we consider a scaling limit of (1.3), $f(x)=\lim w(i;n)$ as i and n tend to ∞ subject to $i/n=x$. This leads to the differential equation:

$$(1.4)\qquad df(x)/dx=-x^{-1}(x-f(x))^{+},\qquad 0<x<1$$

with boundary condition $f(1)=0$. Immediately we obtain $f(x)=-x\log(x)$ on $\{e^{-1}\leqq x\leqq 1\}$, $f(x)=e^{-1}$ on $\{0<x\leqq e^{-1}\}$. From this solution, we can determine the optimal value and the stopping island named after Presman and Sonin (1972). A relatively best object is accepted iff the time of occurrence of this object belongs to the stopping set. If $k,k+1,\cdots,m$ belong to this set, then the interval $[k,m]$ is a stopping island. The optimal value equals $v^{*}=$

$\lim_{n\to\infty} v(1;n)=\lim_{n\to\infty}\max\{1/n,w(1;n)\}=f(0+)=e^{-1}$ and the stopping island is the interval $[\alpha^{*},1]$ where $\alpha^{*}=\inf\{x;x\geqq f(x)\}=e^{-1}$.

The aim of this paper is to apply this method to the best-choice problem with a random number of objects, and obtain some explicit solutions in the asymptotic form. Instead of the differential equation, an integral equation is considered so as to treat the case with a general distribution of the number of objects. But here we assume that the total number of objects is a bounded random variable with known distribution. Presman and Sonin (1972) considered this problem by an approximation method of the parameter associated with its distribution, rather than by using the scaling limit. For another problem of minimizing the expected rank of the individual selected, Gianini (1979) has used a differential equation method.

In Section 2 an integral equation with a general distribution of the number of objects is derived by adapting the above method. However, if the distribution is absolutely continuous, it reduces to a differential equation, the simplest one being (1.4). To find an optimal strategy, we determine the stopping island. A certain condition implies that the stopping set is a single island of which the lower bound can be found, and of which the upper bound is 1. This condition is fundamental to our discussion and contributes to obtaining a solution of the integral equation exactly. As an extension of the uniformly distributed case, we obtain an intermediate result between the non-random case and the Rasmussen and Robbins (1975) problem. Another intermediate case of a distribution, which is not absolutely continuous, is also considered. The next three sections are devoted to discussing three different variants of the best-choice problem.

In Section 3 the result of Smith (1975) involving a refusal probability is extended to that of a uniformly distributed number of objects with non-constant refusal. For the variation of the multiple choice permitting r offers, Gilbert and Mosteller (1966) had formulated and Tamaki (1979a) had obtained the result for $r=2$ in the uniform case. In Section 4, we give a further result of the optimal value of r in an iterative form for the same situation. For the multiple-choice problem, the aim is to select the best and the second-best objects, a problem solved by Nikolaev (1977) and Sakaguchi (1979). We consider this problem with a random number of objects and calculate results for the uniformly distributed case in Section 5.

In the rest of this section we set out notations and preliminaries. For integration with respect to the probability measure $d\Phi$ on the unit interval $[0,1]$: $V(A)=\int_A v(x)d\Phi(x)$ for all intervals A in $[0,1]$, we shall use the abbreviation

$$(1.5)\qquad dV(x)=v(x)d\Phi(x).$$

For any bounded function $u(x)$ the relation (1.5) obviously implies $u(x)dV(x)=$

$u(x)v(x)d\Phi(x)$ (p. 137, Feller (1966)). Using this shorthand notation, an integral equation of the form

$$f(y)-f(x)=\int_x^y a(t,f(t))d\Phi(t)+\int_x^y b(t,f(t))dt$$

for all $0<x<y<1$ is equivalent to

$$df(x)=a(x,f(x))d\Phi(x)+b(x,f(x))dx, \qquad 0<x<1. \tag{1.6}$$

Let $f(x)$ and $g(x)$ be two functions of bounded variation over $[0,1]$, right-continuous and with left-hand limits, then, by Fubini's theorem,

$$d(fg)(x)=f(x)dg(x)+g(x)df(x)-\{f(x)-f(x-)\}dg(x) \tag{1.7}$$

holds (p. 336, Brémaud (1981)). If $f(x)$ is continuous in $0<x<1$, then

$$d(fg)(x)=f(x)dg(x)+g(x)df(x)$$

follows immediately.

2. A scaling limit of the optimality equation

The probability model for the best-choice problem with a random number of objects has been considered by Presman and Sonin (1972). We therefore omit details of its construction here. To take a scaling limit, we restrict ourselves to the case where the number of objects is bounded.

Assumption 1. A random number of objects N is bounded with a probability 1, that is, there is a positive integer n such that

$$n=\inf\{k\geqq 1;\, P(N>k)=0\}. \tag{2.1}$$

The state space is a set of integers $\{1,2,\cdots,n\}$. State i in the model means that the ith object appearing is the relatively best one (better than all its predecessors). The meanings of the transition probability and reward are similar to those for the deterministic case introduced in the previous section, with some learning procedures included. Let us denote $p_i=P(N=i)$ and $\pi_i=\sum_{k=i}^n p_k$. The transition probability matrix $P=(p(i,j);\, 1\leqq i,\, j\leqq n)$ is defined by

$$\begin{aligned} p(i,j)&=i\pi_j/(j(j-1)\pi_i), \qquad 1\leqq i<j\leqq n,\\ p(i,n)&=\sum_{k=i+1}^n ip_k/(k\pi_i), \qquad 1\leqq i<n \quad\text{and}\quad p(n,n)=1. \end{aligned} \tag{2.2}$$

The expected reward $r(i)$ is

$$r(i)=r(i;n)=\sum_{k=i}^n ip_k/(k\pi_i), \tag{2.3}$$

and the cost is $c(i)=0$ for each i. From the general equation (1.1), the optimal value $v(i)=v(i;n)$ satisfies an optimality equation:

$$v(i)=\max\{r(i),Pv(i)\}, \qquad i=1,2,\cdots,n-1, \quad v(n)=1 \tag{2.4}$$

where P, r are defined as in (2.2), (2.3) respectively.

Assumption 2. There is a probability measure $d\Phi$ on $[0,1]$ such that for any sequence $s(k;n)$, $k=1,2,\cdots,n$ with $\lim_{k,n\to\infty} s(k;n)=s(x)$ for $k/n=x$

$$\lim_{i,j,n\to\infty}\sum_{k=i+1}^n s(k;n)p_k=\int_x^y s(t)d\Phi(t)=\int_{(x,y]} s(t)d\Phi(t) \tag{2.5}$$

where $i/n=x$, $j/n=y$ for $x,y\in[0,1]$. Further we assume that $d\Phi$ satisfies the conditions

$$(1-\Phi(x))^{-1}\int_x^1 y^{-1}d\Phi(y)\to 1 \quad \text{as } x\to 1, \tag{2.5i}$$

$$x\int_x^1 y^{-1}d\Phi(y)\to 0 \quad \text{as } x\to 0. \tag{2.5ii}$$

Hereafter Assumptions 1 and 2 will always hold. But, in Section 5, (2.5i) and (2.5ii) are slightly strengthened to discuss multiple-choice problems.

Let us define

$$\begin{aligned} w(k;n)=w(k)=Pv(k)&=\sum_{j=k+1}^n k\pi_j v(j)/(j(j-1)\pi_k), \qquad k=1,\cdots,n-1,\\ w(n;n)&=w(n)=0. \end{aligned} \tag{2.6}$$

As in the previous section, this correponds to the conditional optimal value when the decision-maker rejects all objects until and including the ith relatively best. Since

$$w(k)=\{v(k+1)/(k+1)+w(k+1)k/(k+1)\}\pi_{k+1}/\pi_k$$

holds, (2.4) implies that

$$w(k+1)-w(k)=w(k+1)\pi_k^{-1}p_k-(k+1)^{-1}\{r(k+1)-w(k+1)\}^+\pi_{k+1}/\pi_k. \tag{2.7}$$

Proposition 2.1. A scaling limit of the sequence, $f(x)=\lim_{k,n\to\infty} w(k;n)$ for $k/n=x$ exists. Using the abbreviation (1.6), $f(x)$ satisfies the equation

$$\begin{aligned} df(x)&=f(x)[1-\Phi(x))^{-1}d\Phi(x)-x^{-1}(R(x)-f(x))^+dx, \qquad 0<x<1,\\ f(1)&=0, \end{aligned} \tag{2.8}$$

where

$$R(x)=x(1-\Phi(x))^{-1}\int_x^1 y^{-1}d\Phi(y), \qquad 0\leqq x\leqq 1,$$

is well defined by (2.5i) and (2.5ii).

Proof. The standard Picard iteration method implies the existence of the equation and the scaling limit. As k and n tend to ∞, provided $k/n = x$, we see that $\pi_k^{-1} p_k \to (1-\Phi(x))^{-1} d\Phi(x)$, $r(k+1) \to R(x)$, $\pi_{k+1}/\pi_k \to 1$ and $(k+1)^{-1} = n(k+1)^{-1}(1/n) \to x^{-1}dx$. Thus (2.8) is immediately obtained by taking the sum of (2.7).

Theorem 2.2. The optimal value v^* of the problem in the asymptotic form is given by $v^* = f(0+)$.

Proof. Since $w(1;n) \geqq 0$,

$$v^* = \lim_{n\to\infty} v(1;n) = \lim_{n\to\infty} \max(r(1;n), w(1;n)) = \lim_{n\to\infty} w(1;n) = f(0+).$$

Now let $h(x) = \int_x^1 y^{-1} d\Phi(y)$ and

$$\text{(2.9)} \quad H(x) = h(x) - \int_x^1 y^{-1} h(y)dy = (1+\log(x))h(x) + \int_x^1 \log(y)dh(y)$$

for $0 \leqq x \leqq 1$.

Condition (Φ). $H(x) = H(x;\Phi)$ changes its sign once from $-$ to $+$ as x varies from 0 to 1.

Define

$$\text{(2.10)} \quad \alpha^* = \begin{cases} \inf\{x; H(x) \geqq 0\} \\ 1 \text{ if empty.} \end{cases}$$

Then Condition (Φ) implies that $H(x) \geqq 0$ on $[\alpha^*, 1]$. This is important for our argument to obtain the solution exactly, and is closely related to the condition for an OLA policy in Markov decision processes. In the discrete-parameter problem, a similar condition was imposed in Presman and Sonin (1972), Derman et al. (unpublished) and Rasmussen and Robbins (1975).

Proposition 2.3. If $\Phi(x)$ satisfies Condition (Φ), and if $\Phi(x)$ is continuous for $0 \leqq x < 1$, then the optimal value is given by

$$\text{(2.11)} \quad v^* = (1-\Phi(\alpha^*))f(\alpha^*) = \alpha^* h(\alpha^*).$$

The stopping island $[\alpha^*, 1]$ is determined by the unique solution of the equation

$$\text{(2.12)} \quad H(x) = 0, \qquad 0 < x < 1.$$

Proof. By (1.6), (2.8) is equivalent to

$$(1-\Phi(x))df(x) = f(x)d\Phi(x) - x^{-1}(1-\Phi(x))(R(x)-f(x))^+ dx, \qquad 0 < x < 1.$$

Since $\Phi(x) - \Phi(x-) = 0$ for every $0 < x < 1$, (1.7) implies that $(1-\Phi(x))f(x)$ is differentiable in $0 < x < 1$ and $g(x) = x^{-1}(1-\Phi(x))f(x)$ satisfies the equation

$$dg(x) = -x^{-1}\max\{h(x), g(x)\}dx, \qquad 0 < x < 1,$$
$$g(1) = 0.$$

Condition (Φ) implies that (2.12) has a unique solution and this differential equation is explicitly solved as

$$g(x) = \begin{cases} \int_x^1 y^{-1} h(y)dy & \text{on } \{H(x) \geqq 0\} = [\alpha^*, 1], \\ (\text{const})/x & \text{on } \{H(x) < 0\} = (0, \alpha^*). \end{cases}$$

Therefore, using Theorem 2.2, (2.11) is obtained immediately.

This proposition provides a solution of the problem with the random structure under Condition (Φ). From Equation (2.12), the lower bound of the stopping island, or the threshold of the acceptance region for the relatively best object is determined; the optimal value is also calculated from this threshold in (2.11).

Corollary 2.4. If the measure $d\Phi(x)$ is absolutely continuous with respect to Lebesgue measure dx and $\Phi(x)$ is its density function,

$$\text{(2.13)} \quad d\Phi(x) = \phi(x)dx,$$

then (2.8) is reduced to a differential equation:

$$\text{(2.14)} \quad \begin{aligned} df(x)/dx &= \phi(x)(1-\Phi(x))^{-1}f(x) - x^{-1}(R(x)-f(x))^+, \qquad 0 < x < 1. \\ f(1) &= 0. \end{aligned}$$

Hence α^* is a solution of the equation

$$\text{(2.15)} \quad H(x) = \int_x^1 y^{-1}(1-\log(y)+\log(x))\phi(y)dy = 0.$$

It is noted that $x \leqq R(x) \leqq 1$ for $0 \leqq x \leqq 1$. The case $R(x) = x$ for $0 \leqq x \leqq 1$, gives a model for the non-random number of objects, that is, $p_k = 1$ for $k = n$, $p_k = 0$ otherwise. Since $\Phi(x) = 0$, $0 \leqq x < 1$, (2.14) becomes the differential equation (1.4), which is known to be the simplest case. The other equality, $R(x) = 1$ for $0 < x \leqq 1$ and $R(0) = 0$, implies $f(x) = 0$ because no stopping occurs. Generally, if $R_1(x) \leqq R_2(x)$, $0 \leqq x \leqq 1$ then the corresponding optimal value is $v_1^* \geqq v_2^*$. Hence for the non-random case, $R(x) = x$, this gives the maximum value for the number of objects, when this has a distribution. The next two examples are intended to illustrate an intermediate result between the non-random and the uniformly distributed cases.

Example 2.1. Let the number of objects be uniformly distributed on a partial interval $\{n-m, n-m+1, \cdots, n\}$ of $\{1, \cdots, n\}$ for some m $(0 \leqq m < n)$.

TABLE 1

case	stopping island	optimal value
$1-\theta \geqq e^{-2}$	$[\sqrt{1-\theta}\, e^{-1}, 1]$	$-(\sqrt{1-\theta}/\theta)\log(1-\theta)e^{-1}$
$1-\theta \leqq e^{-2}$	$[e^{-2}, 1]$	$2e^{-2}/\theta$

That is, $p_i = 1/(m+1)$ for $i = n-m, \cdots, n$, and $p_i = 0$, otherwise. Let i, m, $n \to \infty$ with $\theta = m/n$ fixed. Taking the scaling limit (2.5) of Assumption 2, we have $\phi(x) = 1/\theta$ for $1-\theta \leqq x \leqq 1$, and $\phi(x) = 0$, otherwise, and it is seen that (2.5i) and (2.5ii) are satisfied. Instead of solving the differential equation (2.14), we obtain v^* and α^* directly from (2.11) and (2.15), because each distribution $\Phi(x) = \Phi(x;\theta)$; $0<\theta<1$ satisfies Condition (Φ). The conclusions are set out in Table 1. If $\theta \to 0$, the optimal value tends to e^{-1} (non-random case). If $\theta \to 1$, it tends to $2e^{-2}$ as discussed in Presman and Sonin (1972), and Rasmussen and Robbins (1975). Stewart (1981) treated the same distribution but his model was adapted in a Bayesian sense.

Example 2.2. Now consider the limit distribution,

$$\Phi(\{1\}) = 1-\theta \text{ and } d\Phi(x) = \theta dx \text{ for } 0<x<1 \text{ with some } 0 \leqq \theta \leqq 1.$$

There is a point mass of probability at the point 1. This is another intermediate example between the non-random case and the uniformly distributed case, which is not absolutely continuous. Since it satisfies Condition (Φ) and is continuous in $0<x<1$, we can apply Proposition 2.3. We see that

$$\alpha^* = \exp((1-2\theta-\sqrt{(1-2\theta+2\theta^2)})/\theta)$$

by solving Equation (2.12). Hence the optimal value is

$$v^* = (\theta+\sqrt{(1-2\theta+2\theta^2)})\exp((1-2\theta-\sqrt{(1-2\theta+2\theta^2)})/\theta)$$

by (2.11). We observe that the optimal value is monotone decreasing as θ increases.

3. The problem with a refusal probability

One of the variations in the best-choice problem is a model which introduces a refusal probability into the decision 'acceptance'. Smith (1975) calls the secretary problem with this change 'uncertain employment'. Sakaguchi (1979) generalized this model to the multiple-choice problem, on which a random structure will also be imposed in Section 5. The optimality equation for a finite (deterministic) number of objects n with a refusal probability p is

$$(3.1) \quad v(i) = \max\left\{pi/n + (1-p)i\sum_{j=i+1}^{n} v(j)/(j(j-1)),\ i\sum_{j=i+1}^{n} v(j)/(j(j-1))\right\}$$

where p is a constant such that $0<p\leqq 1$. Following the same procedure with the scaling limit, this leads to the differential equation

$$(3.2) \quad df(x)/dx = -px^{-1}(x-f(x))^+, \quad 0<x<1, \quad f(1)=0.$$

Solving it, we obtain the optimal value $v_p^* = f(0+) = p^{1/(1-p)}$ and the stopping island $[p^{1/(1-p)}, 1]$, namely Smith's (1975) result.

Now we consider a model with a random number of objects and inducing the non-constant refusal probability $p(i) = p(i;n)$. We can describe the model by the optimality equation using the same notation as in Section 2:

$$(3.3) \quad \begin{aligned} v(i) &= \max\{p(i)r(i) + (1-p(i))Pv(i), Pv(i)\}, \quad i = 1,\cdots,n-1, \\ v(n) &= p(n). \end{aligned}$$

As in the previous section, we have the following theorem under the same assumptions.

Let

$$h(x) = \int_x^1 y^{-1}d\Phi(y)$$

and

$$(3.4) \quad H_p(x) = h(x) - q(x)\int_x^1 h(y)p(y)/(yq(y))dy$$

where

$$q(x) = \exp\left(\int_x^1 y^{-1}(1-p(y))dy\right)$$

and $p(x)$ is a scaling limit of $p(i) = p(i;n)$ with $i/n = x$. From a realistic point of view, the refusal probability should not depend on the order in which the objects are examined. In this case, (3.4) becomes

$$H_p(x) = h(x) - px^{p-1}\int_x^1 y^{-p}h(y)dy$$

where, as in Example 3.1, the refusal probability is assumed to be constant.

Condition (Φ_p). $H_p(x)$ changes its sign once from $-$ to $+$ as x increases. Define, similarly,

$$(3.5) \quad \alpha_p^* = \begin{cases} \inf\{x; H_p(x) \geqq 0\} \\ 1 \text{ if empty.} \end{cases}$$

Theorem 3.1. The integral equation of the problem is

$$df(x)=f(x)(1-\Phi(x))^{-1}d\Phi(x)-x^{-1}p(x)(R(x)-f(x))^{+}dx,\qquad 0<x<1,$$
$$f(1)=0. \tag{3.6}$$

If $\Phi(x)$ is continuous for $0<x<1$ with Condition (Φ_p), then the optimal value v_p^* with a refusal probability $p(x)$, $0\leqq x\leqq 1$ is given by

$$v_p^*=f(0+)=(1-\Phi(\alpha_p^*))f(\alpha_p^*)=\alpha_p^*q(\alpha_p^*)\int_{\alpha_p^*}^{1}h(y)p(y)/(yq(y))dy. \tag{3.7}$$

The stopping island $[\alpha_p^*,1]$ is determined by the solution α_p^* of $H_p(x)=0$.

Example 3.1. We consider the case of $\Phi(x)=x$, $0\leqq x\leqq 1$, where the number of objects is uniformly distributed on $\{1,2,\cdots,n\}$ and $p(x)=p$ for $0\leqq x\leqq 1$. Since $d\Phi(x)=dx$, (3.6) leads to a differential equation:

$$df(x)/dx=(1-x)^{-1}f(x)-px^{-1}(R(x)-f(x))^{+},\qquad 0<x<1,\quad f(1)=0$$

where $R(x)=-x(1-x)^{-1}\log(x)$. Since $h(x)=-\log(x)$ and $q(x)=x^{p-1}$, the equation $H_p(x)=0$ becomes

$$p(x^{p-1}-1)+(1-p)\log(x)=0.$$

Hence α_p^* is the unique solution of this transcendental equation in $0<x<1$. We see immediately that $\{x;H_p(x)\geqq 0\}=[\alpha_p^*,1]$ holds, and hence $v_p^*=-\alpha_p^*\log(\alpha_p^*)$ by (3.7). Some numerical results are given in Table 2. We note that $p=1.0$ corresponds to the non-refusal case with a uniformly distributed number of objects discussed in Section 2 (see Rasmussen and Robbins (1975)).

4. A multiple-choice problem (I)

Another variation in the best-choice problem is the case where the decision is allowed to make r-object choices (i.e., r stops) and one wants to choose the best among these (see Gilbert and Mosteller (1966)). Sakaguchi (1978) has solved this by using the OLA policy and Tamaki (1979a) has discussed the case where the

TABLE 2

refusal probability p	stopping island $[\alpha_p^*,1]$	optimal value $-\alpha_p^*\log(\alpha_p^*)$
0.5	[0.0810, 1]	0.2036
0.7	[0.1052, 1]	0.2369
0.9	[0.1260, 1]	0.2610
0.99	[0.1344, 1]	0.2698
1.0	$[e^{-2}=0.1353,1]$	$2e^{-2}=0.2707$

number of objects is a uniformly distributed random variable, and has obtained an explicit value in the asymptotic form for the case of $r=2$.

As in the previous sections, we derive an integral equation in the case of r-object choices with a random number of objects for the optimality equation. Following Presman and Sonin (1972) and Tamaki (1979a), the optimality equation becomes

$$v_r(i)=\max\{r(i)+Pv_{r-1}(i),Pv_r(i)\},\qquad r=1,2,\cdots$$
$$v_0(i)=0. \tag{4.1}$$

As in (2.4), let $w_r(k)=Pv_r(k)$, $k=1,2,\cdots,n-1$ and $w_r(n)=0$ for each r. This denotes the conditional optimal value, as before. The same Assumptions 1 and 2 hold as in Section 2.

Theorem 4.1. A scaling limit $f_r(x)$ of $w_r(k;n)$ provided $k/n=x$ in the multiple-choice problem satisfies the equation

$$df_r(x)=(1-\Phi(x))^{-1}f_r(x)d\Phi(x)-x^{-1}(R(x)+f_{r-1}(x)-f_r(x))^{+}dx,$$
$$f_r(1)=0,\qquad r=1,2,\cdots, \tag{4.2}$$
$$f_0(x)=0\quad \text{for } 0<x<1.$$

The optimal value v_r^* equals $f_r(0+)$.

Proposition 4.2. Let $g_r(x)=x^{-1}(1-\Phi(x))f_r(x)$ for $r=1,2,\cdots$. If $\Phi(x)$ is continuous for $0<x<1$, then they are differentiable and satisfy

$$dg_r(x)=-x^{-1}\max\{h(x)+g_{r-1}(x),g_r(x)\}dx,\qquad g_r(1)=0 \tag{4.3}$$

where $h(x)$ is defined in (2.9).

Let $h_r(x)=h(x)+g_{r-1}(x)$ and

$$H_r(x)=h_r(x)-\int_x^1 y^{-1}h_r(y)dy\quad \text{for } r=1,2,\cdots. \tag{4.4}$$

Condition (Φ_r). $H_r(x)$ changes its sign once from $-$ to $+$ as x increases. Let $\alpha_r^*=\inf x;\{H_r(x)\geqq 0\}$.

Theorem 4.3. The optimal value v_r^* of permitting r-object choices is $(1-\Phi(\alpha_r^*))f_r(\alpha_r^*)$, and the stopping islands are determined by the sequence $(\alpha_k^*;k=1,2,\cdots,r)$.

In the rest of this section it is restricted to the uniform distribution: $p_k=1/n$, $k=1,2,\cdots,n$. Then (4.2) implies

$$df_r(x)/dx=(1-x)^{-1}f_r(x)-x^{-1}(R(x)+f_{r-1}(x)-f_r(x))^{+},\qquad 0<x<1,$$
$$f_r(1)=0 \tag{4.5}$$

where $R(x) = -x(1-x)^{-1}\log(x)$. We now use Proposition 4.2. From (4.3), we have that

$$(4.6)\qquad g_r(x) = \int_x^1 y^{-1}h_r(y)dy = \int_x^1 y^{-1}g_{r-1}(y)dy + \int_x^1 y^{-1}\left(\int_y^1 z^{-1}d\Phi(z)\right)dy$$

on $\{x; -\log(x) + g_{r-1}(x) \geqq g_r(x)\}$ and, in the neighborhood of $x = 0$,

$$(4.7)\qquad g_r(x) = (\text{const})/x.$$

From (4.6) and (4.7), $f_r(x)$ is solved. To denote this solution explicitly, we set inductively

$$(4.8)\qquad \begin{aligned} K_{i+1} &= L_i^3/(3!) + (c_{i-1} - c_i)\exp(L_i) + K_iL_i, \\ c_i &= c_{i-1} + L_i\exp(-L_i), \qquad i = 1, 2, \cdots, r \end{aligned}$$

where $L_i = 1 + \sqrt{(1-2K_i)}$ and $K_1 = 0$ and $c_0 = 0$. It is seen that, from the continuity of the solution, that

$$(4.9)\qquad f_r(x) \begin{cases} = c_r/(1-x), & 0 < x \leqq x_r, \\ = x/(1-x) * \{\log^2(x)/(2!) + c_{r-1}/x + K_r\}, & x_r \leqq x \leqq x_{r-1}, \\ = x/(1-x) * \{-\log^3(x)/(3!) + \log^2(x)/(2!) & \\ \qquad - K_{r-1}\log(x) + c_{r-2}/x\}, & x_{r-1} \leqq x \leqq x_{r-2}, \\ = \cdots & \end{cases}$$

where $x_i = \exp(-L_i)$, $i = 1, 2, \cdots$ and $0 < x_r < x_{r-1} \cdots < x_1 = e^{-1} < 1$. The optimal value v_r^* of r-object choices is $v_r^* = f_r(0+) = c_r$. Therefore we can determine the optimal value for every r by the iteration (4.8). For example, $c_1 = 2e^{-2} = 0.2707$ and $c_2 = c_1 + (1 + \sqrt{21}/3)\exp(-(1 + \sqrt{21}/3)) = 0.4725$. The first two terms are consistent with Presman and Sonin (1972), and Tamaki (1979a) respectively. Numerical calculation for different values of r gives the results shown in Table 3.

It seems here as if the optimal value converges, but in the original model of the situation it must tend to unity as r increases. The cause of this may be that we have taken the limit n to ∞ for a prefixed number r.

5. A multiple-choice problem (II)

A multiple-choice problem which is to select the best and the second best

TABLE 3

Times of choice r	1	2	3	4	5	6	7
Optimal value v_r^*	0.2707	0.4725	0.6208	0.7149	0.7552	0.7609	0.7610

objects, permitting a two-object choice, is considered by Nikolaev (1977) and Sakaguchi (1979). Sakaguchi treats the uncertain employment problem i.e. with a refusal probability, in our terminology, which we have discussed in Section 3. While this model is not considered here, we shall discuss the case of a random number of objects, and calculate the uniformly distributed special case as previously.

The optimality equation obtained by Sakaguchi (1979) and Tamaki (1979b) is as follows:

$$u_1(j) = j(j-1)/(n(n-1)),$$

$$u_2(j) = \max\left\{u_1(j), \sum_{k=j+1}^n j(j-1)/(k(k-1)(k-2)) * \sum_{s=1}^2 u_s(k)\right\}, \quad j = 2, \cdots, n-1,$$

$$u_1(n) = u_2(n) = 1,$$

$$v(1) = \max\left\{(u_1(2) + u_2(2))/2, \sum_{k=2}^n 1/(k(k-1)) * v(k)\right\},$$

$$v(i) = \max\left\{\sum_{k=i+1}^n i(i-1)/(k(k-1)(k-2)) * \sum_{s=1}^2 u_s(k), \sum_{k=i+1}^n i/(k(k-1)) * v(k)\right\},$$

$$i = 2, \cdots, n-2, \quad v(n) = 0.$$

The solution of $v(1) = v(1; n)$ is the optimal value, that is, the maximum probability of stopping twice which includes the best and the second-best objects.

Similarly as in the previous sections, we are concerned with the bounded random number of objects and the same notations are used. Because two stops are required, it is enough to assume that $N \geqq 3$, that is, $p_1 = p_2 = 0$. Hence $\pi_1 = \pi_2$ holds. We then have the following optimality equation:

$$u_1(j) = \sum_{k=j}^n j(j-1)p_k/(k(k-1)\pi_j), \qquad j = 2, \cdots, n-1,$$

$$u_2(j) = \max\left\{u_1(j), \sum_{k=j+1}^n j(j-1)\pi_k/(k(k-1)(k-2)\pi_j) * \sum_{s=1}^2 u_s(k)\right\},$$

$$u_1(n) = u_2(n) = 1,$$

$$(5.1)\qquad v(1) = \max\left\{(u_1(2) + u_2(2))/2, \sum_{k=2}^n \pi_k/(k(k-1)\pi_1) * v(k)\right\},$$

$$v(i) = \max\left\{\sum_{k=i+1}^n i(i-1)\pi_k/(k(k-1)(k-2)\pi_i) * \sum_{s=1}^2 u_s(k), \right.$$

$$\left.\sum_{k=i+1}^n i\pi_k/(k(k-1)\pi_i) * v(k)\right\}, \qquad i = 2, \cdots, n-1,$$

$$v(n) = 0.$$

Define the conditional optimal value $w(k) = w(k;n)$, $k = 1, 2, \cdots, n$ by

$$w(1) = \sum_{s=1}^{2} u_s(2)/2,$$

$$w(k) = \sum_{s=k+1}^{n} k(k-1)\pi_s/(s(s-1)(s-2)\pi_k) * \sum_{t=1}^{2} u_t(s), \qquad k = 2, \cdots, n-1,$$

$$w(n) = 0.$$

Then

$$w(k+1) - w(k) = p_k w(k+1)/\pi_k - \pi_{k+1}/((k+1)\pi_k)$$
$$(5.2) \qquad * \left[\sum_{s=k+1}^{n} k(k+1)p_s/(s(s-1)\pi_{k+1}) - w(k+1) + \left\{ \sum_{s=k+1}^{n} k(k+1)p_s/(s(s-1)\pi_{k+1}) - w(k+1) \right\}^+ \right].$$

Also define

$$\tilde{w}(k) = \sum_{s=k+1}^{n} k\pi_s/(s(s-1)\pi_k) * v(s), \qquad k = 1, \cdots, n-1,$$

$$\tilde{w}(n) = 0.$$

Then this satisfies

$$(5.3) \qquad \tilde{w}(k+1) - \tilde{w}(k) = p_k \tilde{w}(k+1)/\pi_k - \pi_{k+1}/((k+1)\pi_k) * (w(k+1) - \tilde{w}(k+1))^+.$$

Hence, if Assumptions 1 and 2 of Section 2, and if

$$(1-\Phi(x))^{-1} \int_x^1 y^{-2} d\Phi(y) \to 1 \quad \text{as } x \to 1$$

and

$$x^2 \int_x^1 y^{-2} d\Phi(y) \to 0 \quad \text{as } x \to 0$$

hold, we have the next two integral equations by taking the scaling limit.

Proposition 5.1. Let $f(x) = \lim_{k,n\to\infty} w(k;n)$ and $\tilde{f}(x) = \lim_{k,n\to\infty} \tilde{w}(k;n)$ provided $k/n = x$. Then these satisfy

$$(5.4) \qquad df(x) = f(x)(1-\Phi(x))^{-1} d\Phi(x) - x^{-1}\{R_2(x) - f(x) + (R_2(x) - f(x))^+\}dx, \qquad f(1) = 0$$

where

$$R_2(x) = x^2(1-\Phi(x))^{-1} \int_x^1 y^{-2} d\Phi(y), \qquad 0 \leqq x \leqq 1,$$

and

$$(5.5) \qquad d\tilde{f}(x) = \tilde{f}(x)(1-\Phi(x))^{-1} d\Phi(x) - x^{-1}(f(x) - \tilde{f}(x))^+ dx, \qquad \tilde{f}(1) = 0.$$

Theorem 5.2. The optimal value $\tilde{v}^* = \lim_{n\to\infty} v(1;n)$ in the asymptotic form is given by the solution $\tilde{f}(0+)$ of (5.5).

Proof. From $v(1;n) = \max\{w(1;n), \tilde{w}(1;n)\}$, we have $\tilde{v}^* = \max\{f(0+), \tilde{f}(0+)\}$. By (5.4), $f(0+) = 0$ implies the result, $\tilde{v}^* = \tilde{f}(0+)$.

Example 5.1. We calculate the optimal value and the stopping island for the case of $p_i = 1/n$ for $i = 1, \cdots, n$, that is, the uniform distribution $d\Phi(x) = dx$. By the same method as in previous sections,

$$(5.6) \qquad f(x) \begin{cases} = 2x/(1-x) * (1 + x\log(x) - x), & \alpha_1^* \leqq x < 1, \\ = x/(1-x) * \{1 - \log(x) + x - 2\alpha_1^* + \log(\alpha_1^*)\}, & 0 < x \leqq \alpha_1^*, \end{cases}$$

where α_1^* $(= 0.28467)$ is a unique solution x of $1 - x = 2(1 + x\log(x) - x)$ in $0 < x < 1$. Also,

$$(5.7) \qquad \tilde{f}(x) \begin{cases} = -2x/(1-x) * ((1+x)\log(x) - 2x + 2), & \alpha_1^* \leqq x < 1, \\ = x/(1-x) * \{\log^2(x)/2 - (\log(\alpha_1^*) - 2_1^* + 1)\log(x) - x \\ \quad + \log^2(\alpha_1^*)/2 - 4\alpha_1^* \log(\alpha_1^*) - \log(\alpha_1^*) + 5\alpha_1^* - 4\}, & \alpha_2^* \leqq x \leqq \alpha_1^*, \\ = \alpha_2^*/(1-x) * \{1 - \log(\alpha_2^*) + \alpha_2^* - 2\alpha_1^* + \log(\alpha_1^*)\}, & 0 < x \leqq \alpha_2^*, \end{cases}$$

where α_2^* $(= 0.09610)$ is a unique solution x of

$$x - \log(x) + \log(\alpha_1^*) - 2\alpha_1^* + 1 = \log^2(x)/2 - (\log(\alpha_1^*) - 2\alpha_1^* + 1)\log(x) - x + \log^2(\alpha_1^*)/2 - 4\alpha_1^* \log(\alpha_1^*) - \log(\alpha_1^*) + 5\alpha_1^* - 4.$$

Hence the optimal value $\tilde{v}^* = \tilde{f}(0+)$ equals

$$(5.8) \qquad \tilde{f}(0+) = \alpha_2^*\{1 - \log(\alpha_2^*) + \alpha_2^* - 2\alpha_1^* + \log(\alpha_1^*)\} \qquad (= 0.15498).$$

The optimal strategy in the asymptotic form is that

(i) on $[0, \alpha_2^*]$, we pass,
(ii) on $[\alpha_2^*, \alpha_1^*]$, we make the first stop if the relative best object appears,
(iii) on $[\alpha_1^*, 1]$, we make the second stop if the relative best or second-best object appears.

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