6

THE CALCULATION OF LIMIT PROBABILITIES FOR MARKOV JUMP PROCESSES

By

Masami Yasuda

(Received October 5, 1977)

Reprinted from the Bulletin of Mathematical Statistics, Research Association of Statistical Sciences, Vol. 18, No. 1∼2

FUKUOKA, JAPAN
1978

THE CALCULATION OF LIMIT PROBABILITIES FOR MARKOV JUMP PROCESSES

By

Masami YASUDA*

(Received October 5, 1977)

1. Introduction

In this paper the limit probability and the total deviation are considered by introducing an artificial transition matrix in Markov jump processes. Section 2 contains a simultaneous equation which the limit probability satisfies. In a single positive recurrent class the simultaneous equation can be reduced to an ordinary one and its solution has been given by Ballow [1], Miller [11] and Feller [5]. We note that the calculation has relation to summability methods. If the state is finite, then we can get an explicit formula of the limit probability for Markov jump processes with several classes by solving the simultaneous equation. In section 3 we shall define a total deviation from the limit probability. Our results extend that of Kemeny and Snell [9] to the denumerable state case. The notion, deviation measure, in [9] is utilized for Markov decision processes (Veinott [13]).

2. Limit probabilities

Let $P(t)=\{p_{ij}(t), i, j\in S\}$, for each $t\geqq 0$, denote the transition probabilities for a Markov jump process with a denumerable state S, such that $p_{ij}(t)\to\delta_{ij}$ as $t\downarrow 0$. Then the limit

$$\pi_{ij}=\lim_{t\to\infty} p_{ij}(t) \tag{2.1}$$

exists for each i, j (Chung [3]). It is also known that the derivative

$$q_{ij}=\lim_{t\to 0}\{p_{ij}(t)-\delta_{ij}\}/t \tag{2.2}$$

exists for each i, j. We write Π, Q for the matrix whose (i, j)-th element is π_{ij}, q_{ij} respectively. By practical interests, we consider the cases; all q_{ij} is finite, $q_{ij}\neq 0$ and $\sum_j q_{ij}=0$ for all i. So the matrix Q is of the form

$$Q=C(R-I) \tag{2.3}$$

* Statistical Laboratory, College of General Education, Chiba University, Chiba 280, Japan.

where elements of the matrix are

$$[R]_{ij}=r_{ij}=\delta_{ij}-q_{ij}/p_{ii},$$

$$[C]_{ij}=c_i\delta_{ij}=-q_{ii}\delta_{ij}$$

and $[I]_{ij}=\delta_{ij}$. Since $q_{ii}\neq 0$, the diagonal matrix C is non-singular. The (i, i) element of the inverse C^{-1} is the mean sojourn time at state i. For each i, j, r_{ij} is a transition law from state i to state j and the trasition matrix R is called jump chain or embedded chain. These facts are found in Chung [6], Feller [5].

Now our problem is to calculate the limit probability Π by means of matrices R and C.

ASSUMPTION. *We assume, for elements c_i of the diagonal matrix C, that*

$$0<\inf_i c_i, \tag{2.4}$$

$$\sup_i c_i<\infty. \tag{2.5}$$

In the terminology of Chung [6], (2.4) and (2.5) imply that all the state is not absorbing and stable respectively. By (2.5), the infinitesimal transition rate Q is bounded and so we can write $P(t)=\exp(Qt)$, $t>0$. The assumption (2.5) is a key point of our following considerations. But for an example as the birth-and-death process of denumerable state (Feller [5]) this fails. On the other hand, the assumption (2.4) is not so important. Because, if $c_i=0$ for a state i, then it is absorbing and staying there for ever so the limit becomes trivial.

We introduce an artificial transition matrix following Jensen and Kendall [7]. By (2.5), we are able to define

$$R_\lambda=I+\lambda^{-1}Q=I+\lambda^{-1}C(R-I) \tag{2.6}$$

for $\lambda\geqq\sup c_i$. In the circumstances of (2.5) Q is bounded and the transition probability is given (Kendall and Reuter [10]) by

$$P(t)=\exp(Qt),\quad t\geqq 0, \tag{2.7}$$

and hence the expansion

$$P(t)=e^{-t\lambda}\sum_n\frac{(\lambda t)^n}{n!}R_\lambda^n,\quad t\geqq 0, \tag{2.8}$$

is possible (Jensen and Kendall [7]) by (2.6). The matrix R_λ is useful. One is that the classification of states of R_λ determines that of the above Markov jump process. The second is that the limit probability Π is a Borel sum of $\{R_\lambda^n, n\geqq 0\}$.

In a Markov jump process or equivalently a continuous time parameter Markov chain the classification of states (Chung [3]) are follows. A state i is recurrent iff

$$\int_0^\infty p_{ii}(t)\,dt=\infty$$

and is non-recurrent iff

$$\int_0^\infty p_{ii}(t)\,dt<\infty.$$

A possitive recurrent is recurrent and $\pi_{ii}>0$.

PROPOSITION 2. 1. *A state of a Markov jump process is recurrent or non-recurrent according as the same properties in the discrete time Markov chain R_λ defined in* (2.6).

PROOF. This is straight forward from

$$\int_0^\infty P(t)\,dt=\lambda^{-1}\sum_0^\infty R_\lambda^n \tag{2.9}$$

and the definition of the classification of states.

Without (2.5), Miller [11] remarked that if the Markov jump process is recurrent so is the jump chain of discrete parameter and vice versa but it does not extend to the possitiveness of the recurrence. Under the assumption (2.5), we will prove later the positiveness is extended.

The next lemma come from Jensen and Kendall [7]. Our proof is slightly simple.

LEMMA 2. 2. *For $\lambda\geqq\sup c_i$,*

$$\Pi=R_\lambda^* \tag{2.10}$$

where the matrix R_λ^* is

$$R_\lambda^*=\lim(n+1)^{-1}\{I+R_\lambda+\cdots+R_\lambda^n\}. \tag{2.11}$$

We say that R_λ^* is a Cesaro sum of a sequence $\{R_\lambda^n, n\geqq 0\}$.

PROOF. Since (2.1) and (2.8),

$$e^{-\lambda t}\sum\frac{(\lambda t)^n}{n!}[R^n]_{ij}$$

has a limit as $t\to\infty$ for each i, j. It holds for each element so we drop indices denoting elements. Hence wo obtain, by the definition of the summability method, that the sequence of $\{R_\lambda^n\}$ is Borel summable to Π. The limit Π is the same value for any $\lambda\geqq\sup c_i$. On the other hand, since $R_\lambda^n=o(n^{1/2})$ is satisfied, it is also Cesaro summable by Hardy-Littlewood theorem (Hardy [6]). Thus a consistence of the summability proves (2.10).

In lemma 2. 2, we also proved that the limit probability Π is a Borel sum of the sequence of powers of the transition matrix R_λ.

To calculate the limit probability Π, we derive simultaneous equations for Π.

THEOREM 2. 3. *Under* (2.4), (2.5) ,*Π satisfies the following equations:*

$$C(I-R)\Pi=\Pi C(I-R)=0, \tag{2.12}$$

$$R^*C^{-1}\Pi=R^*C^{-1} \tag{2.13}$$

where the martix R^* is a Cesaro sum of $\{R^n, n\geqq 0\}$.

If, in addition, all state is possitive recurrent, then

(2.14) $$\Pi R^*C^{-1}=R^*C^{-1}.$$

PROOF. We note that all the products of above matrices are associate. It is sufficient to prove the equation for R_λ^* in stead of Π because of lemma 2.2. Let R^*, R_λ^* be Cesaro sum of $\{R^n\}$ and $\{R_\lambda^n\}$ respectively. By a well known formula (Kemeny, Snell and Knapp [9]),

(2.15) $$RR^*=R^*R=R^*R^*=R^*,$$

(2.16) $$R_\lambda R_\lambda^*=R_\lambda^*R_\lambda=R_\lambda^*R_\lambda^*=R_\lambda^*$$

hold. If (2.6) is muliplied by R_λ^* from the right hand side, (2.16) implies $[C(I-R)]R_\lambda^*=0$. Since C is diagonal, the associability holds. Hence $C(I-R)R_\lambda^*=0$. Similary we have $R_\lambda^*C(I-R)=0$. This proves (2.12). Next we show

(2.17) $$R^*C^{-1}R_\lambda=R^*C^{-1}=R_\lambda R^*C^{-1}.$$

Since C is non-singular by (2.4),

(2.18) $$R=I+\lambda C^{-1}(R_\lambda-I).$$

Multiplication of (2.18) by R^* from the left and the right hand side give $R^*C^{-1}(R_\lambda-I)=0$ and $(R_\lambda-I)R^*=0$ respectively. By (2.4) and (2.5), R^*C^{-1}, $R^*C^{-1}R_\lambda$ and $R_\lambda R^*$ are finite and so (2.17) holds clearly. Define

$$R_\lambda^{(n)}=(n+1)^{-1}\sum_{k=0}^{n}R_\lambda^k.$$

From (2.11), note that $\lim R_\lambda^{(n)}=R_\lambda^*$. Iterative applications of (2.17) imply $R^*C^{-1}R_\lambda^{(n)}=R^*C^{-1}$. We obtain that, by dominated convergence theorem,

$$\lim R^*C^{-1}R_\lambda^{(n)}=R^*C^{-1}(\lim R_\lambda^{(n)})$$

and

$$R^*C^{-1}=R^*C^{-1}R_\lambda^{(n)}.$$

So (2.11) is proved. We also have from (2.17) that

$$R_\lambda^{(n)}R^*C^{-1}=R^*C^{-1}.$$

By Fatou's theorem,

$$(\lim R_\lambda^{(n)})R^*C^{-1}\leqq\lim R_\lambda^{(n)}R^*C^{-1}$$

and thus

(2.19) $$R_\lambda^*R^*C^{-1}\leqq R^*C^{-1}.$$

If a state i is positive recurrent, that is, $\pi_{ii}>0$, then

$$\sum_j \pi_{ij}=\sum_j [R^*]_{ij}=1$$

by the result of Chung [6]. So a convergence theorem of measures implies the equality of (2.19) for i. All states are positive recurrent and thus (2.19) holds with equality. This complete the proof.

If we know that the limit probability Π dose not depend on a initial state, letting

(2.20) $$\pi_j=[\Pi]_{ij}$$

and a vector $\pi=\{\pi_i, i\in S\}$, only an equation;

(2.21) $$\pi C(I-R)=0$$

is significant among simultaneous equations. This is called a steady state equation or equilibrium equation in applications. Miller [11] and Feller [6] are studied the equation. Barlow [1] calculated the explicit solution of (2.21) as a Markov renewal process. In general case, Kendall and Reuter [10] determined a form of Π according as a classification of states. A finite approximation for Π is given by Tweedie [12] when states are possitive recurrent.

COROLLARY 2.4. *If a Cesaro sum R^* is*

(2.22) $$[R^*]_{ij}=r_j^*\geqq 0$$

for each i, j and

(2.23) $$\sum_j r_j^*=1.$$

Then Π takes the form of (2.20) and given by

(2.24) $$\Pi=\sigma^{-1}R^*C^{-1}$$

where σ is a trace of the matrix R^*C^{-1}.

PROOF. Since Π satisfies a simultaneous equation (2.12), we have $\Pi=R\Pi$ and so $\Pi=R^{(n)}\Pi$ where

$$R^{(n)}=(n+1)^{-1}\sum_{k=0}^{n}R^k.$$

Thus, from

$$\lim R^{(n)}=R^* \quad \text{and} \quad \sum_j R_{ij}^*=1$$

for each i,

(2.25) $$\Pi=\lim R^{(n)}\Pi=R^*\Pi.$$

Therfore it is clear that

$$[\Pi]_{ij}=[R^*\Pi]_{ij}=\sum_k r_k^*\pi_{kj}$$

is independent of a state i. Now it is immediate that Π is given by (2.24) from (2.13). Because a trace of the matrix R^*C^{-1} equals

$$\sigma=\sum_k r_k^* c_k^{-1}$$

and it is finite by (2.4), (2.5).

From this corollary we obtain that if the jump chain is possitive recurrent, then the Markov jump process is so. We see the converse also holds by a similar treatment of R_λ and simultaneous equations concerning with R^*. Therfore the positive recurrence between the process and the chain is extended under (2. 4) and (2. 5). Refer to Miller [11].

COROLLARY 2. 5. *The limit probability Π is given by*

(2. 26) $$\Pi=R^*$$

whenever the diagonal matrix is of from

(2. 27) $$C=cI$$

for a scaler $c>0$.

PROOF. Since C is of (2. 27), it is commutable. Hence $R^*\Pi=R^*$ by (2. 13). By (2. 25), $R^*\Pi$ equals Π. Thus (2. 26) holds clearly and independent of a scaler.

The result (2. 26) is interesting. It is intuitively probable that the limit probability Π will consist with that of R^n tending $n\to\infty$. Because each state has same mean sojourn time and the motion law obeys R. A limit of R^n does not always exists but the Markov process always has the limit Π which equals the Cesaro sum of R^n. With the aid of a terminology of summability methods calculating Π from the jump chain R is seen as follows.

Let be $\lambda>c$, then

$$R_\lambda=pR+qI, \quad 0<p<1,$$

where $p=\lambda^{-1}c$ and $q=1-p$. By the binomial expansion,

$$R_\lambda^n=\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}R^k.$$

We can prove that each transition matrix with denumerable states is Euler summable for any parameter except a trivial case. Hence R^n as $n\to\infty$ has a limit and this is a limit probability Π by (2. 8). Since an Euler sum equals a Cesero sum (Hardy [6]), (2. 26) are obtained. If $\lambda=c$, then

$$R_\lambda=R.$$

So we consider a Borel sum of a sequence R^n. A Borel sum also equals a Cesaro one by Proposition 2. 1. In conclusion, if (2. 27) holds, that is, if each state has same mean sojourn time, the calculation of Π means to seek an Euler sum or a Borel one of the jump chain, which are consisted with the Cesaro one R^*.

The remainder of this section we restrict to consider a finite state Markov jump process. Therefore the assumptions (2. 4) and (2. 5) become only

$$\min_i c_i>0$$

and thus we do not require anything in a finite state except a trivial case.

Next lemma is proved by Blackwell [2] in case of $C=I$ using a discount factor.

LEMMA 2. 6. *The matrix*

(2. 28) $$C(I-R)+R^*C^{-1}$$

is non-singular.

PROOF. First we show that the following matrix is non-singular (Blackwell [2]).

(2. 29) $$I-R+R^*$$

For any finite matrix A,

$$(I-A)\{I+A+A^2+\cdots+A^n\}=I-A^{n+1}, \quad n>0.$$

Therefore if a sequence $\{A^n\}$ converges to a zero matrix in a summability method, then $I-A$ is non-singular and equals a sum of the same summability of the sequence $\{I+A+A^2+\cdots+A^n\}$. Let $A=R-R^*$. So $A^n=R^n-R^*$ converges to a zero matrix in a Cesaro summability. Thus $I-A=I-R+R^*$ is non-singular.

Let matrices denote a linear mapping on a N-dimensional vector space E^N where N is a number of states of Markov chain. Define subspaces

$$W_1=\{C(I-R)x\,;\,x\in E^N\} \quad\text{and}\quad W_2=\{R^*C^{-1}x\,;\,x\in E^N\}.$$

It is sufficient to prove

(2. 30) $$W_1\cap W_2=\{0\}.$$

Because this implies $\dim(W_1+W_2)=\dim(W_1)+\dim(W_2)$ for a dimension of a subspace and so

$$\operatorname{rank}\{C(I-R)+R^*C^{-1}\}=\operatorname{rank}\{C(I-R)\}+\operatorname{rank}\{R^*C^{-1}\}=\operatorname{rank}\{I-R\}+\operatorname{rank}\{R^*\}.$$

This equals N by (2. 29). Thus (2. 28) is non-singular. To prove (2. 29), we show that

(2. 31) $$C(I-R)z=0,$$

(2. 32) $$R^*C^{-1}z=0$$

imply $z=0$. Suppose, without loss of generality, the transition matrix R is classified as

$$R=\begin{bmatrix} M & 0 \\ U & T \end{bmatrix}$$

where M and T are corresponding a recurrent and a transient class repectively. M^* is a Cesaro sum of a sequence of powers of M and its element is strictly positive. Let C_M, z_M and else denote the component of corresponding classes of matrices or vectors. By (2. 32),

$$M^*C_M^{-1}z_M=0.$$

Since coefficients are strictively positive, we have $z_M=0$. By (2. 31),

$$-C_TUz_M+C_T(I_T-T)z_T=C_T(I_T-T)z_T=0.$$

Since $C_T(I_T-T)$ is non-singuler, it must $z_T=0$. Thus we obtain $z=0$ and this completes the proof.

THEOREM 2. 7. *The limit probability Π in a finite state is given by*

(2.33) $$\Pi=R^*C^{-1}[C(I-R)+R^*C^{-1}]^{-1}=[C(I-R)+R^*C^{-1}]^{-1}R^*C^{-1}.$$

PROOF. The commutative property R^*C^{-1} and $C(I-R)+R^*C^{-1}$ implies the right hand side equality in (2.33). By Lemma 2.6, a solution of (2.12) and (2.13) in Theorem 2.3 is uniquely solved and so (2.33) is immediate.

3. A total deviation from the limit probability

Let $P(t)$ and Π are a transition probability and its limit probability as the section 2. A total deviation from the limit probability is defined by

(3.1) $$H=\int_0^\infty [P(t)-\Pi]\,dt.$$

This quantity exists for a Markov jump process with a finite state space. But it fails in denumerable cases. Kemeny and Snell [8] utilized the quality, in a finite case, for expressing a second moment of the recurrent time. Concerning recurrent events of denumerable states in a discrete time parameter, Feller [4] studied a similar deviation measure and it is expressed by the second moment of a recurrent time. Therefore the existence of H will be connected with the second moment of a recurrence time. Also it is a refinement of (2.1). We consider the existence of H and an expression in Theorem 3.4. Markov decision processe with a discount factor (Blackwell [6] and Veinott [13]) refers to the deviation.

Let define H_α for $\alpha>0$ by

(3.2) $$H_\alpha=\int_0^\infty e^{-\alpha t}[P(t)-\Pi]\,dt.$$

Clearly (3.2) is always finite and is given as follows.

LEMMA 3. 1. *Uuder* (2.4) *and* (2.5),

(3.3) $$H_\alpha=(\alpha+\lambda)^{-1}[Z_{\lambda,\alpha}-\Pi]$$

where

(3.4) $$Z_{\lambda,\alpha}=\lim_n \sum_{k=0}^n \left(\frac{\lambda}{\lambda+\alpha}\right)^k [R_\lambda-\Pi]^k$$

and R_λ is defined by (2.6). *The right hand side of* (3.3) *does not depend on λ.*

PROOF. It is clear from (2.8) and Lemma 2. 2 that

$$H_\alpha=\int_0^\infty e^{-(\alpha+\lambda)t}\sum_n \frac{(\lambda t)^n}{n!}[R_\lambda^n-\Pi]\,dt.$$

The dominated convergence theorem implies (3.3). Since it does not depend on a choice of parameters in (2.8) and Lemma 2.2, H_α is also independent of λ.

THEOREM 3. 2. *$\{H_\alpha,\ \alpha>0\}$ satisfies that*

(3.5) $$[\alpha I+C(I-R)]H_\alpha=H_\alpha[\alpha I+C(I-R)]=I-\Pi,$$

(3.6) $$\Pi H_\alpha=H_\alpha\Pi=0.$$

PROOF. For (3.6) we have

$$Z_{\lambda,\alpha}R_\lambda^*=R_\lambda^*Z_{\lambda,\alpha}=R_\lambda^*$$

from (2.10) and (2.11). Hence

$$[Z_{\lambda,\alpha}-\Pi]R_\lambda^*=R_\lambda^*[Z_{\lambda,\alpha}-\Pi]=0.$$

Thus (3.6) is proved by (3.3). For (3.5) the definition of $Z_{\lambda,\alpha}$ implies

$$Z_{\lambda,\alpha}\left[I-\frac{\lambda}{\lambda+\alpha}(R_\lambda-\Pi)\right]=\left[I-\frac{\lambda}{\lambda+\alpha}(R_\lambda-\Pi)\right]Z_{\lambda,\alpha}=I.$$

If we substitute $Z_{\lambda,\alpha}$ for $(\lambda+\alpha)H_\alpha+\Pi$ which is derived from (3.3), then

$$(\lambda+\alpha)H_\alpha-\lambda H_\alpha R_\lambda=(\lambda+\alpha)H_\alpha-\lambda R_\lambda H_\alpha=I-\Pi$$

are obtained by (3.6). Therefore the definition of R_λ in (2.6) yields (3.6).

COROLLARY 3. 3. *$\{H_\alpha,\ \alpha>0\}$ satisfies a resolvent equation*

(3.7) $$H_\beta-H_\alpha+(\beta-\alpha)H_\beta H_\alpha=0,\quad \alpha,\ \beta>0.$$

PROOF.

$$\begin{aligned}H_\beta&=H_\beta\{[\alpha I+C(I-R)]H_\alpha+\Pi\}=H_\beta\{(\alpha-\beta)I+\beta I+C(I-R)\}H_\alpha\\&=(\alpha-\beta)H_\beta H_\alpha+H_\beta[\beta I+C(I-R)]H_\alpha=(\alpha-\beta)H_\beta H_\alpha+[I-\Pi]H_\alpha\\&=(\alpha-\beta)H_\beta H_\alpha+H_\alpha.\end{aligned}$$

Now we consider the total deviation H. It is seen from (3.3) in Lemma 3. 1 that if $Z_{\lambda,0}$ with $\alpha=0$ is finite, then $H=H_0$ is finite-valued. Let Z_λ be a Cesaro sum of a sequence

(3.8) $$\left\{\sum_{k=0}^n [R_\lambda-\Pi]^k;\ n\geqq 0\right\}.$$

For $\lambda>\sup c_i$, R_λ is aperiodic and so it need no Cesaro sum but a usual sum, that is,

$$Z_\lambda=Z_{\lambda,0}.$$

THEOREM 3. 4. *If R_λ defined by* (2.6) *contains only strong ergodic classes and a transient class, then Z_λ is finite-valued and the total deviation H is given by*

(3.9) $$H=\lambda^{-1}[Z_\lambda-\Pi].$$

PROOF. Without loss of a generality we assume the matrix of the jump chain R is classified as

$$R=\begin{bmatrix}M & 0\\ U & T\end{bmatrix}\quad\text{and}\quad R^*=\begin{bmatrix}M^* & 0\\ U^* & 0\end{bmatrix}$$

is a Cesaro sum of a sequence $\{R^n\}$. Let, corresponding classes in R_λ,

$$M_\lambda=I+\lambda^{-1}C_M(I-M),\quad T_\lambda=I+\lambda^{-1}C_T(I-T)$$

and

$$U_\lambda=-\lambda^{-1}C_TU$$

where a unit matrix I has suitable states and C_M, C_T are components of C corresponding M and T. Thus, by the condition of the theorem, M_λ is a recurrent class which is strong ergodic and T_λ is a transient class. Further, let $Z_{M(\lambda)}$, $Z_{T(\lambda)}$ and $Z_{U(\lambda)}$ are components of Z_λ corresponding M, T and U respectively. For $Z_{M(\lambda)}$, if $\lambda>\sup c_i$, then M_λ is aperiodic and strong ergodic. Hence $Z_{M(\lambda)}$ is finite by Kemeny, Snell and Knapp [9]. If $\lambda=\sup c_i$, it may be periodic but a Cesaro sum exists and is finite. For $Z_{T(\lambda)}$ it holds that

$$Z_{T(\lambda)}=\lim_n \sum_{k=0}^{n} T_\lambda^k$$

is finite because T_λ is transient. From $Z_{U(\lambda)}=Z_{T(\lambda)}(U-U^*)Z_{M(\lambda)}$, it is also finite. Thus we have Z_λ is finite for every class and so the total deviation H is finite-valued. Letting $\alpha=0$ in (3.3), (3.9) is given and this completes the proof.

COROLLARY 3.5. *For a finite state the total deviation H is always finite-valued and*

$$H=[C(I-R)+R^*C^{-1}]^{-1}(I-\Pi) \tag{3.10}$$

$$=[C(I-R)+R^*C^{-1}]^{-2}C(I-R). \tag{3.11}$$

The matrices $[C(I-R)+R^*C^{-1}]$, $I-\Pi$ and $C(I-R)$ are commutable and so H has other forms of products of these matrices.

PROOF. Since all finite recurrent chain are strong ergodic (Kemeny, Snell and Knapp [9]), the condition of Theorem 3.3 is satisfied. Hence H is finite-valued. Now by letting $\alpha=0$ in (3.5) and (3.6) we have

$$C(I-R)H=HC(I-R)=I-\Pi, \tag{3.12}$$

$$\Pi H=H\Pi=0. \tag{3.13}$$

Further, from (2.13) and (3.13),

$$R^*C^{-1}H=R^*C^{-1}\Pi H=0. \tag{3.14}$$

Hence (3.10) is obtained by the simultaneous equation (3.12) and (3.14) using Lemma 2.2. Since the limit probability is calculated by (2.33),

$$I-\Pi=[C(I-R)+R^*C^{-1}]^{-1}C(I-R).$$

Thus (3.11) is derived.

We note here that if $C=cI$ same as (2.27), then

$$H=c^{-1}\{[I-R+R^*]^{-1}-R^*\}.$$

Veinott [13] considered processes which jump chain is $R=I+Q$. In this case it is trivially $c=1$, so

$$H=[I-R+R^*]^{-1}-R^*.$$

This is a quite same form as the discrete time parameter Markov chain (Blackwell [2]).

The following measure is fundamental to Markov decision processes with discount factors.

$$V_\alpha=\int_0^\infty e^{-\alpha t}P(t)\,dt \tag{3.15}$$

It is finite for $\alpha>0$ and we will consider relations with H.

LEMMA 3.6. *For $\alpha, \beta>0$ and an integer $N<\infty$,*

$$[\alpha I+C(I-R)]V_\alpha=V_\alpha[\alpha I+C(I-R)]=I, \tag{3.16}$$

$$V_\alpha=\alpha^{-1}\Pi+H_\alpha, \tag{3.17}$$

$$V_\alpha=\alpha^{-1}\Pi+\sum_{n=0}^{N}(\beta-\alpha)^nH_\beta^{n+1}+(\beta-\alpha)^{N+1}H_\beta^{N+1}H\alpha \tag{3.18}$$

where H_β^n, $n\geqq 1$ are powers of H_β.

PROOF. The second statement (3.17) is immediate from the definition of (3.15) and (3.2). For (3.16), it is proved directly or by (3.17), (3.5) and (2.12). The third follows from the resolvent equation (3.7).

Next theorem gives a relation between V_α and H. It is a approximate Laurant expansion of V_α about $\alpha=0$.

THEOREM 3.7. *If H^{N+1} exists for some N, then*

$$V_\alpha=\alpha^{-1}\Pi+\sum_{n=0}^{N}(-\alpha)^nH^{n+1}+(-\alpha)^{N+1}H^{N+1}H_\alpha \tag{3.19}$$

for any $\alpha>0$.

PROOF. By the condition of the theorem, H_β is defined for $\beta=0$ and so (3.18) implies (3.19).

If $(-\alpha H)$ satisfies that $(-\alpha H)^N\to 0$ as $N\to\infty$, then it is possible to obtain a Laurant expansion of V_α about $\alpha=0$. Veinott [9] gave this result in a finite state case. To utilize V_α for a discounted Markov decision process, the approximate form (3.19) is sufficient. From (2.8) and (3.10), we have

$$H^n=\int_0^\infty \frac{t^n}{n!}[P(t)-\Pi]\,dt=[C(I-R)+R^*C^{-1}]^{-n}[I-\Pi]$$

for an integer n.

References

[1] BARLOW, R., *Application of semi-Markov processes to counter problems*, Studies in Applied Probability and Management Science, Chapter 3, Stanford Univ. Press (1962).

[2] BLACKWELL, D., *Discrete dynamic programming*, Ann, Math. Statist., **33** (1962), 719-726.

[3] CHUNG, K.L., Markov Chains with Stationary Transition Probabilities, Springer, New York, (1960).

[4] FELLER, W., *Fluctuation theory of recurrent events*, Trans. Amer. Math. Soc., **67** (1949), 98–119.

[5] FELLER, W., An Introduction to Probability Theory and its Applications, Vol. 2, Wiley, New York, (1966).

[6] HARDY, G.H., Divergent Series, Clarendon Press, Oxford, (1949).

[7] JENSEN, A. and KENDALL, D., *Denumerable Markov processes with bounded generators: A routine for calculating $p(\infty)$*, J. Appl. Prob., **8** (1971), 423–427.

[8] KEMENY, J.G. and SNELL, J.L., *Finite continuous time Markov chains*, Theor. Probability Appl., **6** (1961), 101–105.

[9] KEMENY, J.G., SNELL, J.L. and KNOPP, A.W., Denumerable Markov Chains, Springer, New York, (1966).

[10] KENDALL, D. and REUTER, G.E.H., *The calculation of the ergodic projection for Markov chains and processes with a countable infinity of states*, Acta. Math., **97** (1957), 103–144.

[11] MILLER, R.G., *Stationary equations in continuous time Markov chains*, Trans. Amer. Math. Soc., **109** (1962), 35–44.

[12] TWEEDIE, R.L., *The calculation of limit probabilities for denumerable Markov processes from infinitesimal properties*, J. Appl. Prob., **10** (1973), 84–99.

[13] VEINOTT, A.F., *Discrete dynamic programming with sensitive discounted optimality criteria*. Ann. Math. Statist., **40** (1969), 1635–1660.