Multi-Objective Fuzzy Stopping in Systems with Randomness and Fuzziness

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Abstract In a stochastic and fuzzy environment, a multi-objective fuzzy stopping problem is discussed. The randomness and fuzziness are evaluated by probabilistic expectations and scalarization functions respectively. Pareto optimal fuzzy stopping times are given under the assumption of regularity for stopping rules, by using λ-optimal stopping times.

Keywords Multi-objective optimal stopping; Fuzzy stochastic systems; Fuzzy stopping; Pareto optimal.

1. Introduction

This paper deals with a multi-objective fuzzy stopping model for 'fuzzy stochastic systems' represented by sequences of fuzzy random variables. The 'fuzzy random variable', which is a fuzzy-number-valued extension of classical random variables, was studied by Puri and Ralescu [8] and has been discussed by many authors. It is one of the successful hybrid notions of randomness and fuzziness. On the other hand, stopping problems for a sequence of real-valued random variables were studied by many authors, and their applications are well-known in various fields (Chow et al. [2], Shiryayev [10]). The optimal fuzzy stopping for fuzzy random variables is discussed by Yoshida et al. [13], and also optimal stopping models for fuzzy systems without randomness are studied by Yoshida [14,15]. This paper analyzes a multi-objective stopping model for fuzzy stochastic systems, by extending the results of the classical stochastic systems (Aubin [1], Ohtsubo [7]).

In this paper, we also discuss the optimization by 'fuzzy' stopping times, which are first studied by Kacprzyk [4]. Fuzzy stopping times are introduced for dynamic fuzzy systems by Kurano et al. [6] and they are discussed by Yoshida et al. [12], and this paper applies the notion of fuzzy stopping times in a stochastic and fuzzy environment. In this paper, we evaluate the randomness and fuzziness regarding the stopped fuzzy stochastic systems respectively with probablistic expectations and scalarization functions. And we give Pareto optimal stopping times for the multi-objective model, by introducing the notion of λ-optimal stopping times.

In Section 2, the notations and definitions of fuzzy random variables are given. In Section 3, fuzzy stopping times are introduced. We formulate a multi-objective optimal stopping problem for fuzzy stochastic systems by fuzzy stopping times and we give Pareto optimal fuzzy stopping times for the problem under the assumption of regularity for stopping rules. Finally, in Section 4, a numerical example is given to illustrate our idea.

2. Fuzzy random variables

Some mathematical notations of fuzzy random variables are given in this section. Let $(\Omega, \mathcal{M}, P)$ be a probability space, where $\mathcal{M}$ is a σ-field and P is a non-atomic probability measure. Let $\mathbb{R}$ be the set of all real numbers, let $\mathcal{B}$ denote the Borel σ-field of $\mathbb{R}$ and let $\mathcal{I}$ denote the set of all bounded closed sub-intervals of $\mathbb{R}$. A fuzzy number is denoted by its membership function $\tilde{a} : \mathbb{R} \mapsto [0,1]$ which is normal, upper-semicontinuous, fuzzy convex and has a compact support. Refer to Zadeh [16] for the theory of fuzzy sets. $\mathcal{R}$ denotes the set of all fuzzy numbers. The α-cut of a fuzzy number $\tilde{a} (\in \mathcal{R})$ is given by

$$\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\} \ (\alpha \in (0,1])$$

and

$$\tilde{a}_0 := \mathrm{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\},$$

where cl denotes the closure of an interval. In this paper, we write the closed intervals by

$$[\tilde{a}]_\alpha := [[\tilde{a}]_\alpha^-, [\tilde{a}]_\alpha^+] \quad \text{for } \alpha \in [0,1].$$

A map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a fuzzy random variable if The maps $\omega \mapsto [\tilde{X}(\omega)]_\alpha^-$ and $\omega \mapsto [\tilde{X}(\omega)]_\alpha^+$ are measurable for all $\alpha \in [0,1]$, where $[\tilde{X}(\omega)]_\alpha = [[\tilde{X}(\omega)]_\alpha^-, [\tilde{X}(\omega)]_\alpha^+] := \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ is the α-cut of the fuzzy number $\tilde{X}(\omega)$ ([11]).

Now we introduce expectations of fuzzy random variables for the description of stopping models for fuzzy stochastic systems. A fuzzy random variable $\tilde{X}$ is called integrably bounded if $\omega \mapsto [\tilde{X}(\omega)]_\alpha^\pm$ are integrable for all $\alpha \in [0,1]$. Let $\tilde{X}$ be an integrably bounded fuzzy random variable. We put closed intervals

$$[E(\tilde{X})]_\alpha := \left[\int_\Omega [\tilde{X}(\omega)]_\alpha^- \,\mathrm{d}P(\omega), \int_\Omega [\tilde{X}(\omega)]_\alpha^+ \,\mathrm{d}P(\omega)\right] \tag{1}$$

for $\alpha \in [0,1]$. Since the map $\alpha \mapsto [E(\tilde{X})]_\alpha$ is left-continuous by the monotone convergence theorem, the expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number ([5,Lemma 3]):

$$E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min\left\{\alpha, 1_{[E(\tilde{X})]_\alpha}(x)\right\}$$

for $x \in \mathbb{R}$, where 1_D is the classical indicator function of a set D.

3. A multi-objective fuzzy stopping problem

Let k be a positive integer. In this section, we formulate a multi-objective optimal 'fuzzy' stopping problem for k fuzzy stochastic systems and we give Pareto optimal solutions for the problem. Let $\{1, 2, \cdots, k\}$ denote the set of k objects which are described by fuzzy stochastic systems with the time space $\mathbb{N} := \{0, 1, 2, \cdots\}$. For an object $i = 1, 2, \cdots, k$, let $\{\tilde{X}_n^i\}_{n=0}^\infty$ be a sequence of fuzzy random variables such that $E(\max_{1 \leq i \leq k} \sup_{n \geq 0} [\tilde{X}_n^i(\omega)]_0^+]) < \infty$ and $E(\min_{1 \leq i \leq k} [\tilde{X}_n^i(\omega)]_0^-]) > -\infty$ for $n = 0, 1, 2, \cdots$, where the interval $[[\tilde{X}_n^i(\omega)]_0^-, [\tilde{X}_n^i(\omega)]_0^+]$ is the 0-cut of the fuzzy number $\tilde{X}_n^i(\omega)$. For $n = 0, 1, 2, \cdots$, $\mathcal{M}_n$ denotes the smallest σ-field on Ω generated by all random variables $[\tilde{X}_n^i(\omega)]_\alpha^-$ and $[\tilde{X}_n^i(\omega)]_\alpha^+$ ($i = 1, 2, \cdots, k; m = 0, 1, 2, \cdots, n; \alpha \in [0,1]$), and $\mathcal{M}_\infty$ denotes the smallest σ-field containing $\bigcup_{n=0}^\infty \mathcal{M}_n$. Then we call $(\{\tilde{X}_n^i\}_{n=0}^\infty, \{\mathcal{M}_n\}_{n=0}^\infty)$ the fuzzy stochastic system for an object i. A map $\tau : \Omega \mapsto \mathbb{N} \cup \{\infty\}$ is called a stopping time if it satisfies

$$\{\omega \mid \tau(\omega) = n\} \in \mathcal{M}_n$$

for $n = 0, 1, 2, \cdots$. Then we have the following lemma which is trivial from the definitions.

Lemma 1. *Let $i = 1, 2, \cdots, k$ be an object and let τ be a finite stopping time. We define*

$$\tilde{X}_\tau^i(\omega) := \tilde{X}_n^i(\omega), \quad \omega \in \{\tau = n\} \tag{2}$$

for $n = 0, 1, 2, \cdots$. Then, $\tilde{X}_\tau^i$ is a fuzzy random variable.

Now, for an object i, we consider the estimation of the fuzzy stochastic system stopped at a finite stopping time τ, by the evaluation of the fuzzy random variable $\tilde{X}_\tau^i$. Let $g : \mathcal{I} \mapsto \mathbb{R}$ be a σ-additively homogeneous map, that is, g satisfies the following (3) and (4):

$$g\left(\sum_{n=0}^\infty c_n\right) = \sum_{n=0}^\infty g(c_n) \tag{3}$$

for bounded closed intervals $\{c_n\}_{n=0}^\infty \subset \mathcal{I}$ such that $\sum_{n=0}^\infty c_n \in \mathcal{I}$, and

$$g(\mu c) = \mu g(c) \tag{4}$$

for bounded closed intervals $c \in \mathcal{I}$ and real numbers $\mu \geq 0$, where the operation on closed intervals is defined ordinary as $\sum_{n=0}^\infty c_n := \mathrm{cl}\{\sum_{n=0}^\infty x_n \mid x_n \in c_n, n = 0, 1, 2, \cdots\}$ and $\mu c := \{\mu x \mid x \in c\}$. Weighting functions, which satisfy (3) and (4), are used for the evaluation of fuzzy numbers (Fortemps and Roubens [3]). From (2), for $\omega \in \Omega$, the α-cut of the fuzzy number $\tilde{X}_\tau^i(\omega)$ must be a closed interval $[\tilde{X}_\tau^i(\omega)]_\alpha$. Therefore, from the definition (1) and scalarization function g, the evaluation of the fuzzy random variable $\tilde{X}_\tau^i$ is represented by the following integral:

$$\int_0^1 g(E([\tilde{X}_\tau^i(\cdot)]_\alpha)) \,\mathrm{d}\alpha. \tag{5}$$

Lemma 2. *For an object $i = 1, 2, \cdots, k$ and a finite stopping time τ, it holds that*

$$\int_0^1 g(E([\tilde{X}_\tau^i(\cdot)]_\alpha)) \,d\alpha = E\left(\int_0^1 g([\tilde{X}_\tau^i(\cdot)]_\alpha) \,d\alpha\right).$$

In the following definition, we modify fuzzy stopping times introduced by Kurano et al. [6] in order to apply them to fuzzy random variables.

Definition 1. A map $\tilde{\tau} : \mathbb{N} \times \Omega \mapsto [0,1]$ is called a fuzzy stopping time if it satisfies the following (i) – (iii):

(i) For each $n = 0, 1, 2, \cdots$, the map $\omega \mapsto \tilde{\tau}(n, \omega)$ is $\mathcal{M}_n$-measurable.

(ii) For almost all $\omega \in \Omega$, the map $n \mapsto \tilde{\tau}(n, \omega)$ is non-increasing.

(iii) For almost all $\omega \in \Omega$, there exists an integer m such that $\tilde{\tau}(n,\omega) = 0$ for all $n \geq m$.

Regarding the grade of membership of fuzzy stopping times, '$\tilde{\tau}(n,\omega) = 0$' means 'to stop at time n' and '$\tilde{\tau}(n,\omega) = 1$' means 'to continue at time n' respectively. And the intermediate value '$0 < \tilde{\tau}(n,\omega) < 1$' is a notion of 'fuzzy stopping'. It is easy to check the following lemma regarding construction of fuzzy stopping times ([6]).

Lemma 3.

(i) *Let $\tilde{\tau}$ be a fuzzy stopping time. Define a map $\tilde{\tau}_\alpha : \Omega \mapsto \mathbb{N}$ by*

$$\tilde{\tau}_\alpha(\omega) := \inf\{n \mid \tilde{\tau}(n,\omega) < \alpha\}, \quad \omega \in \Omega \tag{6}$$

for $\alpha \in (0,1]$, where the infimum of the empty set is understood to be $+\infty$. Then, we have:

(a) $\{\tilde{\tau}_\alpha \leq n\} \in \mathcal{M}_n$ *for $n = 0, 1, 2, \cdots$;*

(b) $\tilde{\tau}_\alpha(\omega) \leq \tilde{\tau}_{\alpha'}(\omega)$ *a.a. $\omega \in \Omega$ if $\alpha \geq \alpha'$;*

(c) $\lim_{\alpha' \uparrow \alpha} \tilde{\tau}_{\alpha'}(\omega) = \tilde{\tau}_\alpha(\omega)$ *a.a. $\omega \in \Omega$ if $\alpha > 0$;*

(d) $\tilde{\tau}_0(\omega) := \lim_{\alpha \downarrow 0} \tilde{\tau}_\alpha(\omega) < \infty$ *a.a. $\omega \in \Omega$.*

(ii) *Let $\{\tilde{\tau}_\alpha\}_{\alpha \in [0,1]}$ be maps $\tilde{\tau}_\alpha : \Omega \mapsto \mathbb{N}$ satisfying the above (a) (b) and (d). Define a map $\tilde{\tau} : \mathbb{N} \times \Omega \mapsto [0,1]$ by*

$$\tilde{\tau}(n,\omega) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{\{\tilde{\tau}_\alpha > n\}}(\omega)\} \tag{7}$$

for $n = 0, 1, 2, \cdots$ and $\omega \in \Omega$. Then $\tilde{\tau}$ is a fuzzy stopping time.

Fuzzy stopping times are always finite from Definition 1(iii). Now, by using Lemma 3 and the weighting function g, we consider the estimation of the fuzzy stochastic system stopped at a 'fuzzy' stopping time $\tilde{\tau}$ regarding the i-th object. Let $i = 1, 2, \cdots, k$ be an object and let $\tilde{\tau}$ be a fuzzy stopping time. From Lemma 1, we have $[\tilde{X}^i_{\tilde{\tau}_\alpha}(\omega)]_\alpha := [\tilde{X}^i_n(\omega)]_\alpha$ for $\omega \in \{\tilde{\tau}_\alpha = n\}$, where $\tilde{\tau}_\alpha(\omega)$ are 'classical' stopping times given by (6). By Lemma 2, we define a random variable

$$G^i_{\tilde{\tau}}(\omega) := \int_0^1 g([\tilde{X}^i_{\tilde{\tau}_\alpha}(\omega)]_\alpha)\, \mathrm{d}\alpha, \quad \omega \in \Omega. \tag{8}$$

Note that (8) is well-defined since the function $\alpha \mapsto g([\tilde{X}^i_{\tilde{\tau}_\alpha}(\omega)]_\alpha)$ is left-continuous on $(0,1]$. Therefore the expectation $E(G^i_{\tilde{\tau}})$ is the evaluation (5) of the fuzzy random variable $\tilde{X}_{\tilde{\tau}}$. By Fubini's theorem, we have

$$E(G^i_{\tilde{\tau}}) = \int_0^1 E(g([\tilde{X}^i_{\tilde{\tau}_\alpha}(\cdot)]_\alpha))\, \mathrm{d}\alpha$$

for fuzzy stopping times $\tilde{\tau}$. Then, Pareto optimal solutions for the multi-objective stopping model are characterized as follows.

Definition 2. A fuzzy stopping time $\tilde{\tau}^*$ is called Pareto optimal if there exists no fuzzy stopping time $\tilde{\tau}$ such that

$$E(G^i_{\tilde{\tau}}) \geq E(G^i_{\tilde{\tau}^*}) \quad \text{for all objects } i = 1, 2, \cdots, k$$

and

$$E(G^i_{\tilde{\tau}}) > E(G^i_{\tilde{\tau}^*}) \quad \text{for some object } i = 1, 2, \cdots, k.$$

We introduce the following λ-optimal stopping times in order to obtain Pareto optimal stopping times. Real numbers $\{\lambda^i\}_{i=1}^k$ are called weights of objects if they satisfy

$$\sum_{i=1}^k \lambda^i = 1 \quad \text{and} \quad \lambda^i \geq 0 \ (i = 1, 2, \cdots, k).$$

For a set of weights $\lambda := \{\lambda^i\}_{i=1}^k$, we define a fuzzy stochastic system $\{\tilde{X}^\lambda_n\}_{n=0}^\infty$, which is $\{\mathcal{M}_n\}_{n=0}^\infty$-adapted, by

$$\tilde{X}^\lambda_n(\omega)(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{[\tilde{X}^\lambda_n(\omega)]_\alpha}(x)\}$$

for $\omega \in \Omega$, $x \in \mathbb{R}$, where the α-cuts $[\tilde{X}^\lambda_n(\omega)]_\alpha$ are closed intervals given by

$$[\tilde{X}^\lambda_n(\omega)]_\alpha = \left[\sum_{i=1}^k \lambda^i [\tilde{X}^i_n(\omega)]^-_\alpha, \sum_{i=1}^k \lambda^i [\tilde{X}^i_n(\omega)]^+_\alpha\right]$$

for $\omega \in \Omega$. For fuzzy stopping times $\tilde{\tau}$, in the same way as (8) we define a random variable

$$G^\lambda_{\tilde{\tau}}(\omega) := \int_0^1 g([\tilde{X}^\lambda_{\tilde{\tau}_\alpha}(\omega)]_\alpha)\, \mathrm{d}\alpha \quad \text{for } \omega \in \Omega.$$

Similarly to the proof of Lemma 2, we can easily check that its expectation is reduced to the weighted sum of the expectations for objects:

$$E(G^\lambda_{\tilde{\tau}}) = \sum_{i=1}^k \lambda^i E(G^i_{\tilde{\tau}}).$$

Now we give the definition of λ-optimal stopping times as follows.

Definition 3. Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects. Then a fuzzy stopping time $\tilde{\tau}^*$ is called λ-optimal if

$$E(G^\lambda_{\tilde{\tau}^*}) \geq E(G^\lambda_{\tilde{\tau}})$$

for all fuzzy stopping times $\tilde{\tau}$.

Theorem 1. *Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects such that*

$$\sum_{i=1}^{k} \lambda^i = 1 \quad \text{and} \quad \lambda^i > 0 \ (i = 1, 2, \cdots, k). \tag{9}$$

Then a λ-optimal fuzzy stopping time $\tilde{\tau}^*$ is Pareto optimal.

Finally, in order to construct λ-optimal fuzzy stopping times, we introduce the following (λ, α)-optimal fuzzy stopping times.

Definition 4. Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects and let $\alpha \in [0,1]$. A fuzzy stopping time $\tilde{\tau}^*$ is called (λ, α)-optimal if

$$E(g([\tilde{X}^\lambda_{\tilde{\tau}^*_\alpha}(\cdot)]_\alpha)) \geq E(g([\tilde{X}^\lambda_{\tilde{\tau}_\alpha}(\cdot)]_\alpha))$$

for all fuzzy stopping times $\tilde{\tau}$.

In order to characterize (λ, α)-optimal stopping times, we let real random variables

$$\gamma^\lambda_{n,\alpha} := \operatorname*{ess\ sup}_{\tau:\ \text{stopping times}, \tau \geq n} E(g([\tilde{X}^\lambda_\tau(\cdot)]_\alpha) | \mathcal{M}_n) \tag{10}$$

for $n = 0, 1, 2, \cdots$, where the definition of the essential supremum is referred to [2,Chap.1-6]. Define a stopping time $\sigma^\lambda_\alpha : \Omega \mapsto \mathbb{N}$ by

$$\sigma^\lambda_\alpha(\omega) := \inf \left\{ n \mid g([\tilde{X}^\lambda_n(\omega)]_\alpha) = \gamma^\lambda_{n,\alpha}(\omega) \right\}$$

for $\omega \in \Omega$ and $\alpha \in [0,1]$, where the infimum of the empty set is understood to be $+\infty$. Then the following lemma can be checked easily by Chow et al. [2].

Lemma 4. *Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects. Suppose*

$$P(\sigma^\lambda_\alpha < \infty) = 1 \quad \text{for all } \alpha \in [0,1]. \tag{11}$$

Then, for $\alpha \in [0,1]$, the following (i) and (ii) hold:

(i) $\gamma^\lambda_{n,\alpha}(\omega) = \max\{g([\tilde{X}^\lambda_n(\omega)]_\alpha), \gamma^\lambda_{n+1,\alpha}(\omega)\}$ a.a. $\omega \in \Omega$ *for* $n = 0, 1, 2, \cdots$;

(ii) σ^λ_α *is a (λ, α)-optimal stopping time and* $E(\gamma^\lambda_{0,\alpha}) = E(g([\tilde{X}^\lambda_{\sigma^\lambda_\alpha}(\cdot)]_\alpha))$.

In order to construct an optimal fuzzy stopping time from the (λ, α)-optimal stopping times $\{\sigma^\lambda_\alpha\}_{\alpha \in [0,1]}$, we need the following regularity condition.

Assumption A (Regularity). The map $\alpha \mapsto \sigma^\lambda_\alpha(\omega)$ is non-increasing for almost all $\omega \in \Omega$.

Under Assumption A, we can define a map $\tilde{\sigma}^\lambda : \mathbb{N} \times \Omega \mapsto [0,1]$ by

$$\tilde{\sigma}^\lambda(n, \omega) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{\{\sigma^\lambda_\alpha > n\}}(\omega)\}$$

for $n = 0, 1, 2, \cdots$ and $\omega \in \Omega$. Put the α-cut (6) of $\tilde{\sigma}^\lambda(n, \omega)$ by $\tilde{\sigma}^\lambda_\alpha(\omega)$. Then $\tilde{\sigma}^\lambda_\alpha(\omega)$ and $\sigma^\lambda_\alpha(\omega)$ may not equal only at most countable many $\alpha \in (0,1]$, so we obtain the following result.

Theorem 2. *Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for objects satisfying (9). Suppose (11) and Assumption A hold. Then $\tilde{\sigma}^\lambda$ is a λ-optimal fuzzy stopping time and it is also Pareto optimal.*

4. A numerical example

An example is given to illustrate our idea of the multi-objective optimal fuzzy stopping problem in Section 3. In this example, k objects mean k assets in a financial market $\{B_n\}_{n=0}^\infty$ which is a sequence of real random variables. We assume that $\{B_n\}_{n=0}^\infty$ are decsribed by sums of real random variables:

$$B_n := \sum_{m=0}^{n} W_m \quad n = 0, 1, 2, \cdots,$$

where $\{W_n\}_{n=0}^\infty$ is a sequence of independent random variables on $[-1/2^n, 1/2^n]$. The price of each asset $i (= 1, 2, \cdots, k)$ is described by a fuzzy stochastic system $\{\tilde{X}^i_n\}_{n=0}^\infty$ constructed as follows: We put real random variables

$$Y^i_n := p^i + r^i n + v^i B_n, \quad i = 1, 2, \cdots, k.$$

where p^i, r^i and v^i are constants such that p^i is the initial price of an asset i, r^i is the appreciate rate of the asset i, and v^i is the volatility of the asset i. Let c^i and d^i be constants satisfying $0 < d^i < 3(c^i - r^i)$. Then we put

$$M^i_n := Y^i_n - c^i(n+1) \quad \text{for } n = 0, 1, 2, \cdots.$$

where $c^i(n+1)$ means the maintenance cost for asset i. Hence we take a sequence of fuzzy random variables $\{\tilde{X}^i_n\}_{n=0}^\infty$ by

$$\tilde{X}^i_n(\omega)(x) := \begin{cases} L((M^i_n(\omega) - x)/a^i_n) & \text{if } x \leq M^i_n(\omega) \\ L((x - M^i_n(\omega))/a^i_n) & \text{if } x \geq M^i_n(\omega) \end{cases}$$

for $n = 0, 1, 2, \cdots$, $\omega \in \Omega$ and $x \in \mathbb{R}$, where $\{a^i_n\}_{n=0}^\infty$ is a sequence given by $a^i_n := d^i(n+1)$ $(n = 0, 1, 2, \cdots)$ and the shape function is given by $L(x) := \max\{1 - |x|, 0\}$ $(x \in \mathbb{R})$. The corresponding σ-field $\mathcal{M}_n$ is the smallest σ-field generated by the random variables $W_0, W_1, W_2, \cdots, W_n$. Then their α-cuts are

$$[\tilde{X}^i_n(\omega)]_\alpha = [M^i_n(\omega) - (1-\alpha)a^i_n, M^i_n(\omega) + (1-\alpha)a^i_n],$$

$\omega \in \Omega$ for $n = 0, 1, 2, \cdots$ and $\alpha \in [0,1]$.

Now we take a weighting function by $g([x,y]) := (x+2y)/3$ for $x, y \in \mathbb{R}$ satisfying $x \le y$. Then g satisfies the properties (3) and (4), and we can easily check

$$G_n^i(\omega) = \int_0^1 g([\tilde{X}_n^i(\omega)]_\alpha)\,\mathrm{d}\alpha = M_n^i(\omega) + \frac{1}{6}a_n^i,$$

$\omega \in \Omega$. Let $\lambda := \{\lambda^i\}_{i=1}^k$ be a set of weights for assets satisfying (9). It means a kind of portfolio for the assets. Then we have

$$G_n^\lambda(\omega) = \sum_{i=1}^k \lambda^i M_n^i(\omega) + \frac{1}{6}\sum_{i=1}^k \lambda^i a_n^i, \quad \omega \in \Omega.$$

Hence Assumption A and the finiteness (11) are fulfilled. Putting $p^\lambda := \sum_{i=1}^k \lambda^i \left(p^i - r^i\right)$, $v^\lambda := \sum_{i=1}^k \lambda^i v^i$ and $c_\alpha^\lambda := \sum_{i=1}^k \lambda^i \left(c^i - r^i - \frac{1-\alpha}{3}d^i\right)$, the finite (λ, α)-optimal stopping times $\sigma_\alpha^*(\omega)$ for the problem are

$$\inf\{n \mid \operatorname*{ess\,sup}_{\tau \ge n+1} E(\sum_{m=n+1}^{\tau}(v^\lambda W_m - c_\alpha^\lambda) \mid \mathcal{M}_n)(\omega) \le 0\}$$

for $\omega \in \Omega$, where $\gamma_{n,\alpha}^\lambda$ is given by (10). By Theorem 1, λ-optimal fuzzy stopping time, which is also one of Pareto optimal stopping times, is given by

$$\tilde{\sigma}^*(n,\omega) = \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{\{\sigma_\alpha^* > n\}}(\omega)\},$$

for $n = 0, 1, 2, \cdots$ and $\omega \in \Omega$. We can easily check that the corresponding optimal expected value for the fuzzy stopping problem is

$$E(G_{\tilde{\sigma}^*}^\lambda) = p^\lambda + \int_0^1 E(\sum_{m=0}^{\sigma_\alpha^*}(v^\lambda W_m - c_\alpha^\lambda))\,\mathrm{d}\alpha$$

for a portfolio $\lambda := \{\lambda^i\}_{i=1}^k$ for the assets.

Finally, when letting d^i to zero especially in this example, we note that the fuzzy random variables $\tilde{X}_n^\lambda$ are reduced to the 'classical' random variables.

5. Concluding Remarks

In this paper, we introduced fuzzy rewards in a stochastic and fuzzy environment for fuzzy mathematical multi-object economic models, and we estimated them by probabilistic expectations and weighting functions. A Pareto optimal stopping time is given as a λ-optimal stopping time under a regularity assumption. This approach is only one of multi-object fuzzy stopping models with fuzzy rewards, and in the nest step, the study of the other types of multi-object modeling will be expected by use of fuzzy random variables.

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