Quandle Cocycle Invariants and Their Application to Surface Knots

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Introduction

In 1982, Joyce [25] introduced a *quandle*, which is an algebraic object consisting of a non-empty set with a self-distributive binary operation satisfying three conditions related to the Reidemeister moves I,II, and III respectively (cf. [11], [31]). At the same time, Matveev [32] independently defined the same notion, called a *distributive groupoid*. A *rack* and *rack (co)homology* were introduced in [16], [17]. By modifying rack (co)homology, Carter, Jelsovsky, Kamada, Langford, and Saito [7], [8] developed the (co)homology for quandles, called *quandle (co)homology*. Using the cocycles of quandle cohomology, they introduced state-sum invariants for classical knots in dimension 3 and surface knots in dimension 4, which is called *quandle cocycle invariants*. The quandle cocycle invariants have been applied to estimation of the triple point numbers ([23], [40]) and the detection on non-invertibility of surface knots ([6], [7]), and so on. The invariants are related to quandle colorings of classical knots and surface knots, and more precise than their coloring numbers.

This thesis has two major goals (Theorem 3.20 and Theorem 3.22). The first goal is to prove that for each non-negative integer g, there is an infinite family of non-invertible surface knots of genus g. Several studies have been made on the non-invertibility of certain twist-spun knots (cf. [15], [19], [30], [37]). Furthermore, Gordon showed more general results using more geometric methods in [21]. However, most of the known methods of proving the non-invertibility can not be applied directly to knotted surfaces of higher genus. On the other hand, the quandle cocycle invariants can detect the non-invertibility without regard to genus. Theorem 3.20 is proved by calculating the quandle cocycle invariant of certain surface knot of genus $g > 0$

The quandle cocycle invariant of a surface knots F takes value in the group ring $\mathbb{Z}[G]$, and in $\mathbb{Z} \subset \mathbb{Z}[G]$ if F admits only trivial colorings or the triple point number of F is 0, where G is the coefficient group of the quandle cohomology. For some 2-knots, the quandle cocycle invariants derived from Mochizuki's 3-cocycle were calculated concretely in [6], [7] and [24], where Mochizuki's 3-cocycle is a $\mathbb{Z}_p$-valued cocycle on the dihedral quandle R_p of order p, where p is an odd prime integer. However, it was not known whether or not there are any non-ribbon 2-knots which admit a non-trivial R_p-coloring, and all of whose cocycle invariants derived from $\mathbb{Z}_p$-valued 3-cocycles on R_p take value in $\mathbb{Z} \subset \mathbb{Z}[\mathbb{Z}_p]$ for any odd prime integer p. The second goal is to prove that there are such knotted surfaces. This is obtained by calculating the quandle cocycle invariants of twist-spun pretzel knots.

This thesis is organized as follows.

The main purpose of Chapter 1 is to give the informations, which are used in the following chapters, on quandle colorings of classical knots. We discuss dihedral quandle colorings of torus knots and pretzel knots, find all the colorings. In addition, we consider a conjecture for Alexander quandle colorings. The conjecture is related to the Kauffman-Harary conjecture which is a conjecture for dihedral quandle colorings. We prove that the new conjecture is true for twist knots.

Chapter 2 discusses quandle colorings of surface knots. In particular, we consider the colorings of twist-spun knots which are surface knots constructed from classical knots.

In Chapter 3, we calculate the quandle cocycle invariants of torus knots and pretzel knots and their twist-spins using the results in previous chapters. From these results, we obtain Theorem 3.20 (the first goal) and Theorem 3.22 (the second goal).

Chapter 1

Quandle Colorings of Classical Knots

In this chapter, we summarize the definitions of classical knots, quandles, and quandle colorings of classical knots. We discuss dihedral quandle colorings of torus knots and pretzel knots. The results are necessary for the later chapters. In addition, we consider a conjecture derived from the Kauffman-Harary conjecture.

1.1 Preliminaries

A *classical knot*, or simply *knot*, is an oriented simple closed curve embedded in the 3-space $\mathbb{R}^3$. Two knots K and K' are *equivalent* if there is an orientation-preserving homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^2$ such that $h(K) = K'$. A *diagram* of K is the image of K by a projection $\mathbb{R}^3 \to \mathbb{R}^2$ such that the singularity set of the image consists of isolated double points. Let D, D' be diagrams of K, K' respectively. Two knots K and K' are equivalent if and only if D can be transformed into D' by using a finite sequence of the three *Reidemeister moves* of types I, II, and III (see Figure 1.1) and ambient isotopies of $\mathbb{R}^3$. We denote the orientation-reversed knot and the mirror image of K by $-K$ and K^* respectively, where the mirror image is the image of K by orientation-reversing homeomorphism of $\mathbb{R}^3$. Refer to [28], [29], and [31] for more detail.

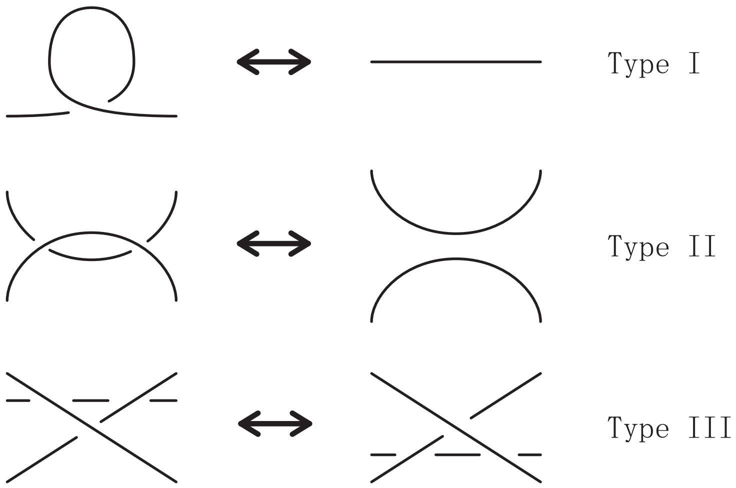


Figure 1.1: Reidemeister moves

Definition 1.1. A *quandle*, X, is a set with a binary operation $* : X \times X \to X$ satisfying the following conditions:

(Q1) For any $x \in X$, $x * x = x$.

(Q2) For any $x, y \in X$, there is a unique element $z \in X$ such that $x = z * y$.

(Q3) For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

If the condition (Q1) is removed, X is called a *rack*. The three conditions (Q1), (Q2), and (Q3) correspond to the Reidemeister moves of types I, II, and III respectively (cf. [31]). The condition (Q2) is equivalent to the following condition:

(Q2') For any $x, y \in X$, there is a binary operation $*^{-1} : X \times X \to X$ such that $(x * y) *^{-1} y = x = (x *^{-1} y) * y$.

We list some typical examples of quandles. See [11, 25, 32] for further examples of quandles.

Example 1.2. (1) Let T be a set with the operation $x * y = x$ for any $x, y \in T$. Then T is a quandle, called a *trivial quandle*.

(2) Let G be a group with the operation given by

$$x * y = y^{-1}xy$$

for $x, y \in G$. Then G is a quandle, called the *conjugate quandle* of G and denoted by $\mathrm{Conj}(G)$. We remark that $x *^{-1} y$ is equal to bab^{-1}.

(3) Let p be a positive integer. We define the binary operation $*$ on the set $\{0, 1, \ldots, p-1\}$ by

$$x * y = 2y - x \pmod{p}.$$

Then the set $\{0, 1, \ldots, p-1\}$ become a quandle, called the *dihedral quandle* of order p and denoted by R_p. The operation $*^{-1}$ is identical with the operation $*$.

(4) Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ be the Laurent polynomial ring over $\mathbb{Z}$, $J \subset \Lambda$ an ideal of Λ. Then the quotient ring Λ/J with the binary operation defined by

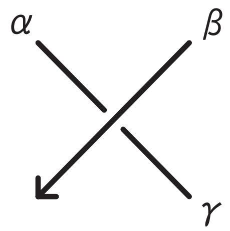
$$x * y = tx + (1-t)y \quad \text{in } \Lambda/J$$

for any $x, y \in \Lambda/J$ is a quandle called an *Alexander quandle*. The operation $*^{-1}$ is given by $x *^{-1} y = t^{-1}x + (1 - t^{-1})y$. We remark that the dihedral quandle R_p is isomorphic to $\Lambda/(m, t+1)$. Similarly, for the Laurent polynomial ring $\Lambda_p = \mathbb{Z}_p[t, t^{-1}]$ over the cyclic group of order p, we may define the quandle structure.

Definition 1.3. Let D be a diagram of an oriented classical knot K, and Σ the set of arcs of D. Given a quandle X, an *X-coloring* of D is a map $C : \Sigma \to X$ which satisfies

$$C(\gamma) = C(\alpha) * C(\beta)$$

at each crossing, where $\alpha, \gamma \in \Sigma$ are under-arcs on the right and left of the over-arc $\beta \in \Sigma$, respectively. See Figure 1.2.



$$C(\gamma) = C(\alpha) * C(\beta)$$

Figure 1.2: Coloring relation at a crossing

If an X-coloring uses only one color we say that it is *trivial*. The coloring by the dihedral quandle R_p is Fox's p-coloring, and this is independent of the orientation of a knot. It is a classical result of knot theory that for any prime integer p, a knot K has a non-trivial Fox's p-coloring, that is, a non-trivial R_p-coloring if and only if $p | \Delta_K(-1)$, where $\Delta_K(t)$ is the Alexander polynomial of K (cf. [14]).

We denote the set of all X-colorings of a classical knot diagram D by $\mathrm{Col_X(D)}$. The cardinality $\#\mathrm{Col_X(D)}$ of colorings is called the *X-coloring number* of D. If a knot K admits the only trivial X-colorings, the X-coloring number is equal to the cardinality of the quandle X.

Proposition 1.4 ([7])**.** *The X-coloring number of D is an invariant of K.*

It is easy to check that the R_3-coloring number of the trefoil and the trivial knot is 9 and 3, respectively. Hence, it is follows from Proposition 1.4 that the trefoil is not trivial.

Definition 1.5. Let D be a knot diagram of an oriented knot K and $C : \Sigma \to X$ an X-coloring of D. A *shadow X-coloring of D extending C* is a map $\widetilde{C} : \widetilde{\Sigma} \to X$, where $\widetilde{\Sigma}$ is the union of Σ and the set of the connected regions of D, satisfying the following conditions:

(i) The map $\widetilde{C}$ restricted to Σ coincides with C

(ii) If μ and ν are regions separated by an arc α, where μ is on the right of α (see Figure 1.3), then

$$\widetilde{C}(\nu) = \widetilde{C}(\mu) * \widetilde{C}(\alpha).$$

We denote the set of all shadow X-coloring of D by $\widetilde{\mathrm{Col}}_X(D)$.

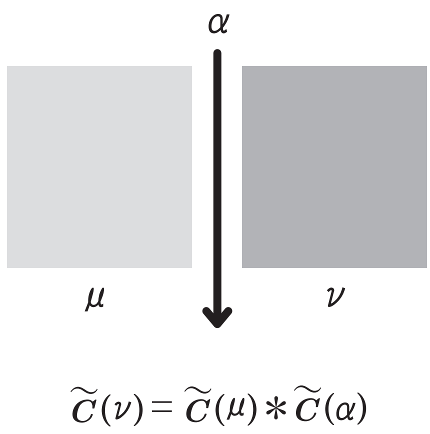


Figure 1.3: Shadow coloring relation around α

We call the ordered triple

$$(\widetilde{C}(\mu), \widetilde{C}(\alpha), \widetilde{C}(\beta)) \in X^3$$

the *quandle triple* at a crossing point of the diagram, where α is the under-arc on the right of the over-arc β, and μ is the region on the right side of both α and β. We denote the quandle triple at a crossing point x by $\widetilde{C}(x)$. See Figure 1.4.

Let T be a one-string tangle diagram on $\mathbb{R}^2_+$ of K such that $\partial T = T \cap \partial \mathbb{R}^2_+$, and $\Sigma(T)$ the set of arcs of T.

Definition 1.6. An X-*coloring* of T is a map $C : \Sigma \to X$ which satisfies

$$C(\gamma) = C(\alpha) * C(\beta)$$

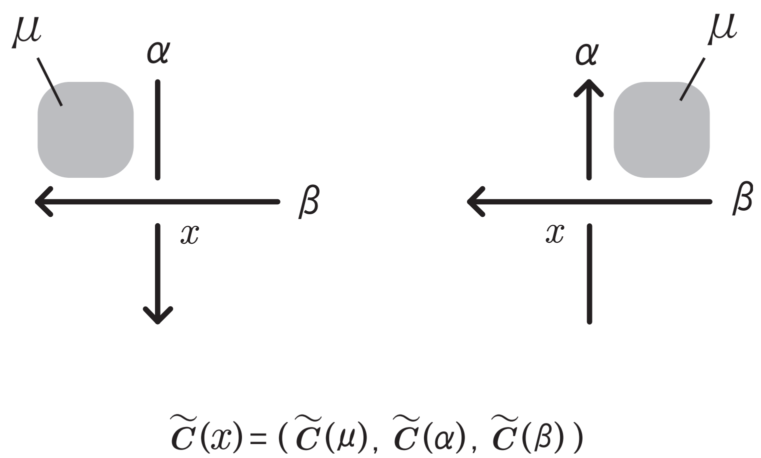


Figure 1.4: Quandle triples at a crossing

at each crossing, where $\alpha, \gamma \in \Sigma$ are under-arcs on the right an left of the over-arc $\beta \in \Sigma$, respectively.

Let $\Sigma^*(T)$ be the union of $\Sigma(T)$ and the set of regions of $\mathbb{R}^2_+$ separated by the underlying immersed curve of T.

Definition 1.7. A *shadow X-coloring* of T extending C is a map $C^* : \Sigma^*(T) \to X$ satisfying the following conditions:

(i) The map C^* restricted to $\Sigma(T)$ is coincident with C

(ii) If μ and ν are regions separated by an arc α, where μ is on the right of α, then

$$C^*(\nu) = C^*(\mu) * C^*(\alpha)$$

(iii) It holds that $C^*(\alpha_+) = C^*(\lambda)$, where α_+ is the initial arc of T and λ is the unbounded region.

We denote the set of all shadow X-colorings of T by $\mathrm{Col}^*_X(D)$.

In the same way, we may define the quandle triple at a crossing point of the tangle diagram. We denote the quandle triple at a crossing point x by $C^*(x)$.

Using the trick in [34], we may prove the following proposition with some generality.

Proposition 1.8. *Suppose $x * y = x * z$ always implies $y = z$ in a quandle X. Then for any X-coloring of a one-string tangle diagram T of K, the two external arcs have the same color.*

Proof. Let T_1 and T_2 be the tangle diagrams obtained from that of T as shown in Figure 1.5. Fix any coloring of T and trivially extend it to a coloring of T_1. Moving the circle under T, we get a coloring of T_2. Then the coloring of T remains unchanged. The large added circle component of T_2 is colored with two colors $x, z \in X$ such that $z = x * a_+ = x * a_-$, where $a_+, a_- \in X$ are the colors of the external arcs. Hence $a_+ = a_-$ under the assumption of the quandle X. □

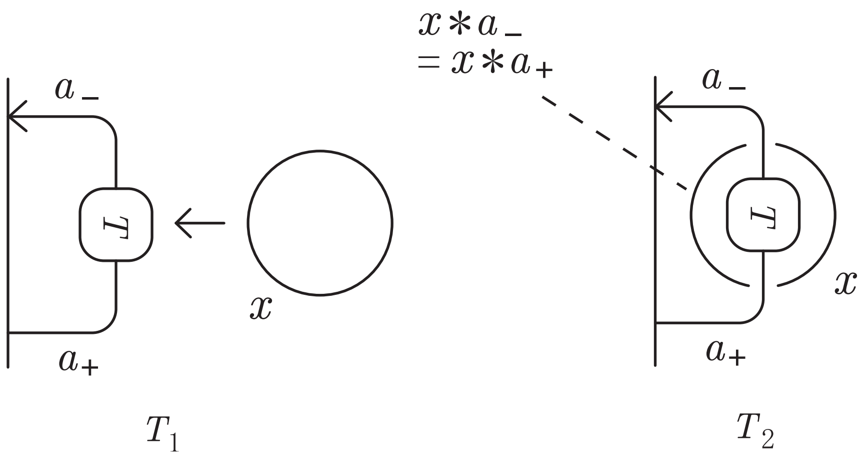


Figure 1.5: Przytycki's trick

Corollary 1.9. *Let a_+ and a_- be the colors of the initial and terminal arcs of a tangle diagram T, respectively. Then we have $x * a_+ = x * a_-$ for any $x \in X$.*

Proof. The lemma follows from Proof of Proposition 1.8. □

Corollary 1.10. *Let X be a dihedral quandle or an Alexander quandle. For any X-coloring C, the initial arc and the terminal arc of a tangle diagram T colored by C have the same color.*

Proof. The lemma follows from Proposition 1.8 and the definition of dihedral quandles and Alexander quandles. □

1.2 Examples

In this section, we consider R_p-colorings of torus knots and pretzel knots, and determine their coloring numbers.

1.2.1 Torus Knots

Let $T(m,n)$ be the (m,n)-torus knot, where m and n are relatively prime integers with $m,n \geqq 2$. A diagram $D_{T(m,n)}$ of $T(m,n)$ is obtained by closing the m fold product of an element Δ in the n-braid group B_n, where $\Delta = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$ with the standard generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ of B_n. We denote the j-th crossing point from the left of the i-th Δ by x_{ij}. See Figure 1.6.

We will explicitly find all R_p-colorings of a diagram $D_{T(m,n)}$ of the (m,n)-torus knot $T(m,n)$. Beginning with the colors $a_{01}, a_{02}, \ldots, a_{0n} \in R_p$ of the top over-arcs of the n strings, the color a_{ij} of the j-th over-arc, ($j = 1, 2, \ldots, n$, counting from left to right) after the i applications of Δ is uniquely determined. The relation between these colors are described by

$$\boldsymbol{a}_{i+1} = A\boldsymbol{a}_i$$

where

$$\boldsymbol{a}_i = \begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{pmatrix} \quad \text{and} \quad A = \left(\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \\ \hline -1 & & \text{\Large 0} & 2 \\ & \ddots & & \vdots \\ \text{\Large 0} & & -1 & 2 \end{array}\right).$$

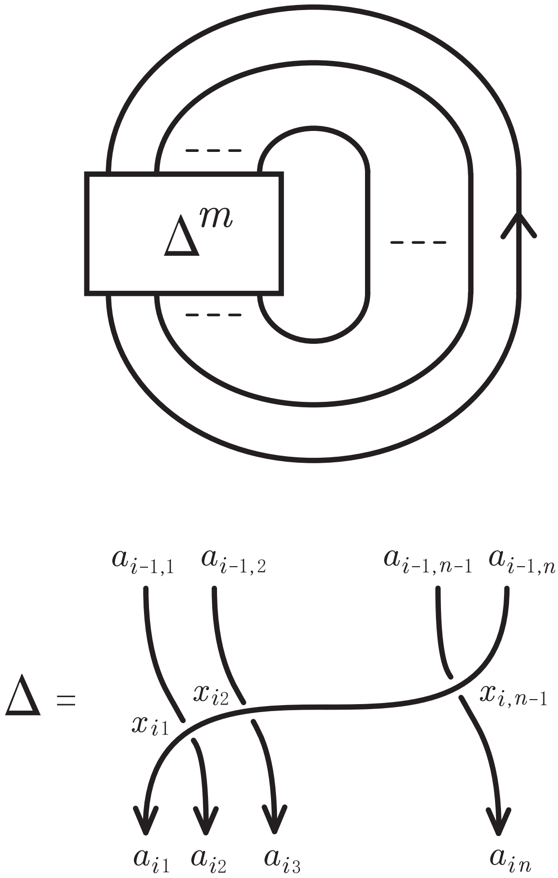


Figure 1.6: A diagram $D_{T(m,n)}$ of $T(m,n)$

Since the diagram $D_{T(m,n)}$ is the closure of Δ^m, the color a_{mj} is equal to the color a_{0j} for any j ($1 \leqq j \leqq n$). Hence we have

$$\boldsymbol{a}_0 = A^m \boldsymbol{a}_0.$$

Lemma 1.11 ([6]). *For the $n \times n$-matrix A above, we have*

$$A^i = N(I + M + \cdots + M^{i-1}) + M^i, \tag{1.1}$$

where

$$M = \left(\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \\ \hline -1 & & \text{\huge 0} & 0 \\ & \ddots & & \vdots \\ \text{\huge 0} & & -1 & 0 \end{array}\right) \quad \textit{and} \quad N = \left(\begin{array}{ccc|c} 0 & \cdots & 0 & 2 \\ \hline 0 & \cdots & 0 & 2 \\ & \vdots & & \vdots \\ 0 & \cdots & 0 & 2 \end{array}\right).$$

Proof. It is not difficult to see that

$$A = M + N, \ AN = N, \text{ and } AM^i = NM^i + M^{i+1}.$$

The lemma follows from these relations by induction on m □

For later calculations, it is convenient to consider the second index j of color a_{ij} modulo n, i.e., j as an element in $\mathbb{Z}_n$, and we assume this from now on. Hence $a_{i0} = a_{in}, a_{i,-j} = a_{i,n-j}$, etc.

Proposition 1.12 ([6])**.** *Let $D_{T(m,n)}$ be the diagram of $T(m,n)$ defined above, where m and n are relatively prime with $m, n \geqq 2$ and m is odd. Then the diagram $D_{T(m,n)}$ admits a non-trivial R_p-coloring if and only if m is divisible by p and n is even. Moreover, a non-trivial R_p-coloring of $D_{T(m,n)}$ is determined by $a_{01}, a_{02}, \ldots, a_{0n}$, which are satisfying*

$$\begin{cases} a_{01} = a_{03} = \cdots = a_{0,n-1}, \\ a_{02} = a_{04} = \cdots = a_{0n}. \end{cases}$$

Proof. In terms of the components a_{ij}, the equation (1.1) is written as

$$a_{ij} = 2\sum_{k=0}^{i-1}(-1)^k a_{0,-k} + (-1)^i a_{0,\,j-i} \quad \text{for } 1 \leqq j \leqq n.$$

In particular, the consistency condition of the coloring imposed on the colors $a_{01}, a_{02}, \ldots, a_{0n}$ are written as the equations

$$a_{0j} = f(\boldsymbol{a}_0) - a_{0,\,j-m} \quad \text{for } 1 \leqq j \leqq n,$$

where

$$f(\boldsymbol{a}_0) = 2\sum_{k=0}^{m-1}(-1)^k a_{0,-k}.$$

Note that this equation is equivalent to the equation $A^m\boldsymbol{a}_0 = \boldsymbol{a}_0$. By replacing j with $j-m$, we have

$$a_{0,j-m} = f(\boldsymbol{a}_0) - a_{0,\,j-2m} \quad \text{for } 1 \leqq j \leqq n.$$

Hence, we obtain

$$a_{0j} = a_{0,\,j-2m} \quad \text{for } 1 \leqq j \leqq n. \tag{1.2}$$

Assume that n is odd. Since m and n are relatively prime integers, it follows from the equation (1.2) that

$$a_{0j} = a_{01} \quad \text{for } 1 \leqq j \leqq n.$$

Hence, $D_{T(m,n)}$ admits only trivial R_p-colorings.

Assume that n is even. Since $(n, -2m) = 2$, we have

$$a_{0j} = \begin{cases} a_{01} & \text{if } j \text{ is odd,} \\ a_{02} & \text{if } j \text{ is even.} \end{cases}$$

Since $f(\boldsymbol{a}_0) = -(m-1)a_{01} + (m+1)a_{02}$, the equation $A^m\boldsymbol{a}_0 = \boldsymbol{a}_0$ is equivalent to

$$m(a_{01} - a_{02}) = 0.$$

Hence, $D_{T(m,n)}$ admits a non-trivial R_p-coloring if and only if m is divisible by p and n is even. □

Corollary 1.13 ([6]). *If m is divisible by p and n is even, then the R_p-coloring number of $T(m,n)$ is p^2. If m is not divisible by p, or n is odd, then the R_p-coloring number of $T(m,n)$ is p.*

Assume that $D_{T(m,n)}$ admits a non-trivial R_p-coloring, that is, m is divisible by p and n is even. Then we have

$$a_{ij} = \begin{cases} a_{i1} & \text{if } j \text{ is odd,} \\ a_{i2} & \text{if } j \text{ is even,} \end{cases}$$

for any i. Hence, the relation $\boldsymbol{a}_i = A\boldsymbol{a}_{i-1}$ reduces to

$$\begin{pmatrix} a_{i1} \\ a_{i2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} a_{i-1,1} \\ a_{i-1,2} \end{pmatrix}.$$

By induction, we obtain

$$\begin{cases} a_{i1} = a - i\delta, \\ a_{i2} = a - (i+1)\delta, \end{cases}$$

where $a = a_{01}$, and $\delta = a_{01} - a_{02}$. In particular, we have

$$a_{i-1,j} = \begin{cases} a - (i-1)\delta & \text{if } j \text{ is odd}, \\ a - i\delta & \text{if } j \text{ is even}, \end{cases}$$
$$a_{i-1,n} = a - i\delta,$$

where $a_{i-1,j}$ and $a_{i-1,n}$ are the colors of arcs around the crossing x_{ij}. See Figure 1.7.

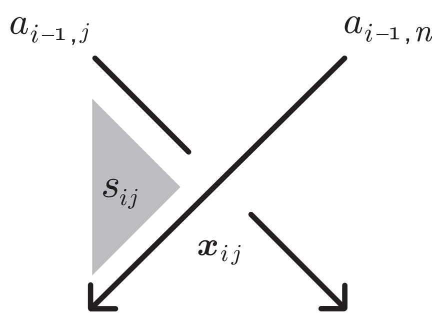


Figure 1.7: The quandle triple at x_{ij}

We consider the shadow R_p-coloring of $D_{T(m,n)}$ which admits a non-trivial R_p-coloring. We color the region on the right of x_{ij} by $s_{ij} \in R_p$. Since the regions on the right of x_{i1} are the same region for any $1 \geqq i \geqq n$, it holds that $s_{11} = s_{21} = \cdots = s_{n1}$. Put $s := s_{11} = s_{21} = \cdots = s_{n1}$. It follows from the relation $s_{i,j+1} = 2a_{i-1,j} - s_{ij}$ that

$$s_{ij} = \begin{cases} s - (j-1)\delta & \text{if } j \text{ is odd,} \\ 2a - s - (2i-j)\delta & \text{if } j \text{ is even,} \end{cases}$$

where $a = a_{01}$, $b = a_{02}$, and $\delta = a_{01} - a_{02}$. From what has been discussed above, we have the following proposition.

Proposition 1.14 ([6]). *Assume that m is divisible by p and n is even, then the quandle triple $\widetilde{C}(x_{ij})$ at x_{ij} is give by*

$$\widetilde{C}(x_{ij}) = \begin{cases} (s-(j-1)\delta, a-(i-1)\delta, a-i\delta) & \textit{if } j \textit{ is odd,} \\ (2a-s-(2i-j)\delta, a-i\delta, a-i\delta) & \textit{if } j \textit{ is even.} \end{cases}$$

1.2.2 Pretzel Knots

Let m be a non-negative integer, and $p_1, \ldots, p_m$ non-zero integers. We denote by $P(p_1, \ldots, p_m)$ the pretzel link of type $(p_1, \ldots, p_m)$. A diagram D_P of $P(p_1, \ldots, p_m)$ is obtained as shown in Figure 1.8, that is, m is the number of columns, p_i is the number of half-twists on the i-th column. The pretzel link $P(p_1, \ldots, p_m)$ is a knot if and only if (i) $p_1, \ldots, p_m, m$ are odd, or (ii) there is a unique p_i in $\{p_1, \ldots, p_m\}$ such that p_i is even. We say that $P(p_1, \ldots, p_m)$ is odd (or even, resp.) if it is in the case (i) (or (ii), resp.).

We will explicitly find all R_p-colorings of the diagram D_P of the pretzel knot $P(p_1, \ldots, p_m)$, where p is a prime integer. We color the arcs of i-th column by $a_{i-1,0}$, $a_{i-1,1}$,..., $a_{i-1,|p_i|} \in R_p$ from the top (See Figure 1.8). We note that $a_{i,1} = a_{i+1,0}$ if $p_i > 0$, $a_{i,|p_i|-1} = a_{i+1,|p_{i+1}|}$ if $p_i < 0$, and $a_{00} = a_{m0}$, $b_{00} = b_{m0}$. We use the notations a_i, b_i instead of $a_{i,0}, a_{i,|p_i|}$, respectively. The relations between these colors are described by

$$\begin{pmatrix} b_{i-1} \\ b_i \end{pmatrix} = A^{p_i} \begin{pmatrix} a_{i-1} \\ a_i \end{pmatrix}, \text{ where } A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

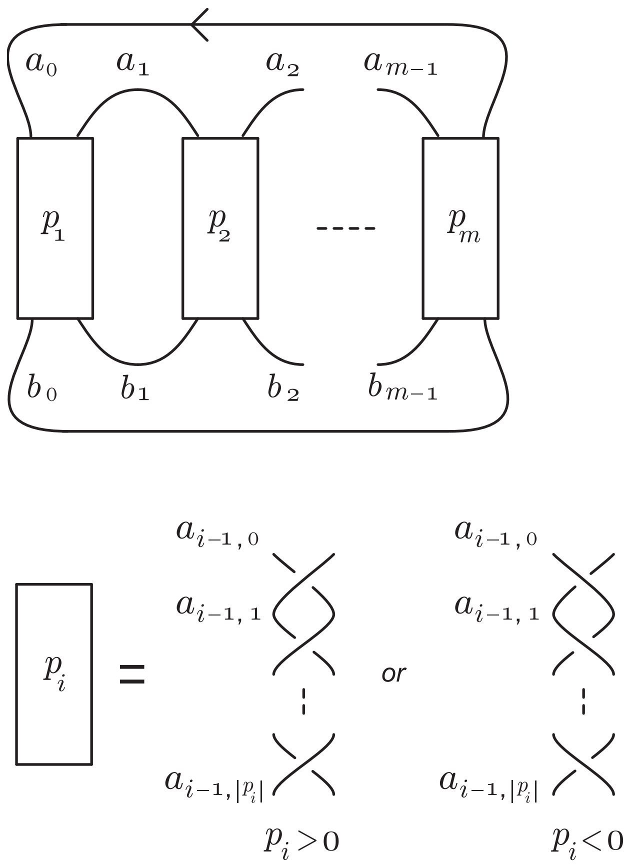


Figure 1.8: A diagram D_P of $P(p_1, \ldots, p_m)$

without regard to the sign of p_i $(1 \leqq i \leqq m)$. By induction, we have

$$\begin{pmatrix} b_{i-1} \\ b_i \end{pmatrix} = \begin{pmatrix} -p_i+1 & p_i \\ -p_i & p_i+1 \end{pmatrix} \begin{pmatrix} a_{i-1} \\ a_i \end{pmatrix}. \tag{1.3}$$

It is known that $P(p_1, \ldots, p_m)$ admits a non-trivial R_p-coloring if and only if it holds that

$$\sum_{i=1}^{m} p_1 p_2 \cdots \widehat{p_i} \cdots p_m = 0 \pmod{p}.$$

For example, if there is a unique p_i in $\{p_1, \ldots, p_m\}$ such that p_i is divisible by p, then the colorings of $P(p_1, \ldots, p_m)$ are always trivial. We consider the following two cases with respect to $p_i \pmod{p}$.

Case 1. Assume that all p_i's are not divisible by p $(1 \leqq i \leqq m)$. Then the relation (1.3) induce

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} -q_i + 1 & q_i \\ -q_i & q_i + 1 \end{pmatrix} \begin{pmatrix} a_{i-1} \\ b_{i-1} \end{pmatrix},$$

where $q_i = \frac{1}{p_i}$. By induction, we have

$$a_i = Q_i c_0 + a_0, \tag{1.4}$$

where $Q_i = \sum_{k=1}^{i} q_k$ and $c_0 = b_0 - a_0$. By the definition of the R_p-coloring, the color of each arc of the i-th column is obtained by

$$a_{i-1,j} = a_{i-1} \pm j d_i \quad (0 \leqq j \leqq |p_i|), \tag{1.5}$$

where $d_i = a_i - a_{i-1} = q_i c_0$, and the '$\pm$' is '$+$' when $p_i > 0$ and '$-$' when $p_i < 0$. Therefore, given the colors $a_0, b_0 \in R_p$, we may determine a R_p-coloring of the diagram D_P of a pretzel knot $P(p_1, \dots, p_m)$ by the relations (1.4) and (1.5).

Case 2. Assume that $p_{i_1}, \dots, p_{i_n}$ in $\{p_1, \dots, p_m\}$ are divisible by p for some $n \geqq 2$ $(i_1 < \dots < i_n)$. From the relation (1.3), we have

$$a_{i_k} = b_{i_k} = a_{i_k+1} = b_{i_k+1} = \dots = a_{i_{k+1}-1} = b_{i_{k+1}-1}.$$

For i such that $i_k < i < i_{k+1}$, since the top arcs of the i-th column have the same color a_{i_k}, all arcs of it are colored by a_{i_k} . The color of each arc of the j_k-th column is obtained from the equation (1.4) by substituting j_k for i. Thus an R_p-coloring of the above diagram D_P is determined by the colors $a_{i_1}, \dots, a_{i_n}$.

From what has been discussed above, we have the following proposition with respect to the R_p-coloring number of the pretzel knot $P(p_1, \dots, p_m)$.

Proposition 1.15. *Assume that the diagram D_P of $P(p_1, \dots, p_m)$ admit a non-trivial R_p-coloring. Then the R_p-coloring number of $P(p_1, \dots, p_m)$ is equal to p^2 if all p_i's are not divisible by p, or p^n if $p_{i_1}, \dots, p_{i_n}$ in $\{p_1, \dots, p_m\}$ are divisible by p for some $n \geqq 2$.*

Let D_P be the diagram of a pretzel knot $P(p_1,\dots,p_m)$ defined above. We consider the shadow R_p-colorings of the diagram D_P colored as shown in Figure 1.8. The shadow R_p-colorings do not depend on the orientation of a diagram (but the quandle triples depend on it). Let x_{ij} be the j-th crossing point from the top of the i-th column ($1 \leqq i \leqq m, 1 \leqq j \leqq |p_i|$). We color the region on the right, left side of x_{ij} by $s_{i-1}, s_i \in R_p$, and upper, under side by $s_{i-1,j-1}, s_{i-1,j} \in R_p$, respectively (see Figure 1.9). We note that the relations $s_{0,0} = s_{1,0} = \cdots = s_{m-1,0}$, $s_{0,|p_1|} = s_{1,|p_2|} = \cdots = s_{m-1,|p_m|}$, and $s_0 = s_m$ hold. By definition, we have the following relations.

$$\begin{cases} s_{i,0} = 2a_0 - s_0, \\ s_i = 2a_i - s_{i,0} = 2(a_i - a_0) + s_0, \\ s_{ij} = 2a_{ij} - s_i = 2(a_{ij} - a_i + a_0) - s_0. \end{cases} \tag{1.6}$$

Therefore, given the colors $s \in R_p$, we may determine a shadow R_p-coloring of the R_p-colored diagram D_P of a pretzel knot $P(p_1,\dots,p_m)$ by the equations (1.6).

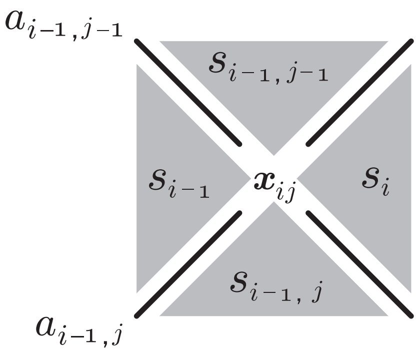


Figure 1.9: Colors around x_{ij}

1.3 The Kauffman-Harary Conjecture

The Kauffman-Harary conjecture is the following conjecture for dihedral quandle colorings.

Conjecture 1.16 ([22]). *Let D be a reduced alternating knot diagram with a prime determinant p. Then every non-trivial R_p-coloring of D assigns different colors to different arcs of D.*

In [2], Asaeda, Przytycki and Sikora generalize the conjecture by stating it in terms of homology of the double cover of the 3-sphere S^3 branched along a link, and prove that the generalized conjecture is true for Montesinos links.

We consider the following conjecture associated with the Alexander quandle which we can regard as a generalization of R_p.

Conjecture 1.17. *Let K be an alternating oriented knot, D be a reduced alternating diagram of K, and $\Delta_K(t)$ be the Alexander polynomial of K. If the ring $\mathbb{Z}[t, t^{-1}]/(\Delta_K(t))$ is an integral domain, then every non-trivial coloring of D by the Alexander quandle $\mathbb{Z}[t, t^{-1}]/(\Delta_K(t))$ assigns different colors to different arcs of D.*

If the Kauffman-Harary conjecture is true then Conjecture 1.17, also, is true for any alternating knots with prime determinants. But Conjecture 1.17 is not included in the Kauffman-Harary conjecture because there is a knot with a non-prime determinant whose Alexander polynomial is a prime element in the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$.

We consider Alexander quandle colorings of twist knots. A diagram D_n of an n-twist knot K_n is pictured in the left of Figure 1.10, where $|n|$ is the number of crossings in the "twist" part. The twists are right-handed if $n > 0$ and left-handed if $n < 0$. The left of Figure 1.10 shows the case $n > 0$. We orient K_n by the orientation indicated in the left of Figure 1.10.

Let Λ/J be an Alexander quandle. We color the arcs of the "twist" part in the diagram D_n by $a_0, b_0, b_1, \ldots, b_{|n|} \in \Lambda/J$ as shown in Figure 1.10. By

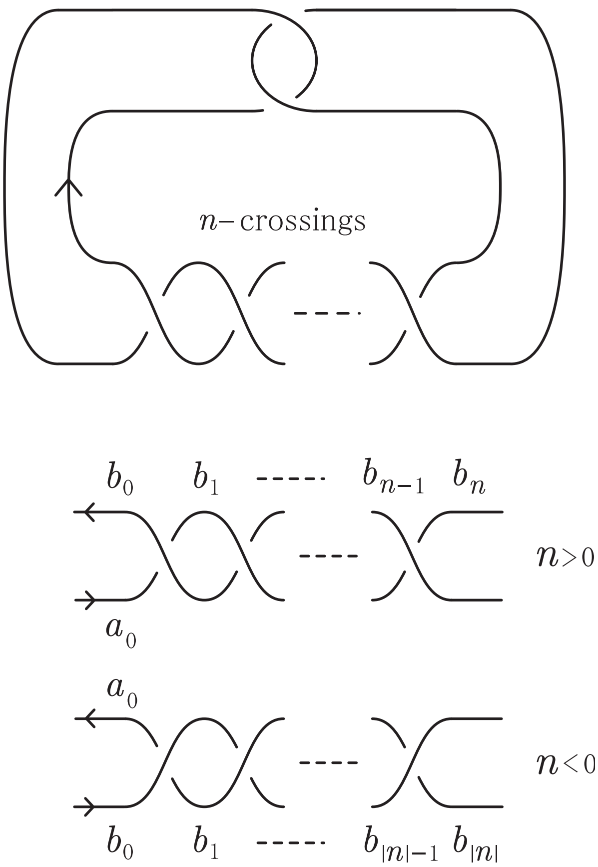


Figure 1.10: A diagram D_n of K_n

the definition of Alexander quandle colorings, the relations between these colors are described by

$$\begin{pmatrix} b_{i-1} \\ b_i \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 1 \\ t & 1-t \end{pmatrix} \begin{pmatrix} b_{i-2} \\ b_{i-1} \end{pmatrix} & \text{if } i \text{ is even,} \\ \begin{pmatrix} 0 & 1 \\ t^{-1} & 1-t^{-1} \end{pmatrix} \begin{pmatrix} b_{i-2} \\ b_{i-1} \end{pmatrix} & \text{if } i \text{ is odd,} \end{cases}$$

without regard to the sign of n, where $i = 0, 1, \ldots, n$ and $b_{-1} = a_0$. By induction, we have

$$\begin{pmatrix} b_{i-1} \\ b_i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i(t^{-1}-1)+2 & i(1-t^{-1}) \\ i(t^{-1}-1) & i(1-t^{-1})+2 \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \tag{1.7}$$

if i is even, and

$$\begin{pmatrix} b_{i-1} \\ b_i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (i-1)(t^{-1}-1) & (i-1)(1-t^{-1})+2 \\ (i+1)(t^{-1}-1)+2 & (i+1)(1-t^{-1}) \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \tag{1.8}$$

if i is odd. Furthermore, the colorings $a_0, b_0, b_{|n|-1}, b_{|n|}$ of the four arcs in the "clasp" part of D_n have the following relations:

$$b_0 = \begin{cases} t^{-1}a_0 + (1-t^{-1})b_n & \text{if } n \text{ is positive, even,} \\ ta_0 + (1-t)b_n & \text{if } n \text{ is positive, odd,} \\ ta_0 + (1-t)b_{|n|-1} & \text{if } n \text{ is negative, even,} \\ t^{-1}a_0 + (1-t^{-1})b_{|n|-1} & \text{if } n \text{ is negative, odd,} \end{cases} \tag{1.9}$$

and

$$b_{|n|-1} = \begin{cases} t^{-1}b_n + (1-t^{-1})a_0 & \text{if } n \text{ is positive,} \\ tb_{|n|} + (1-t)b_0 & \text{if } n \text{ is negative.} \end{cases} \tag{1.10}$$

Lemma 1.18. *Assume that the ring Λ/J is an integral domain. The diagram D_n admits a non-trivial Λ/J-coloring if and only if it holds that*

$$\begin{cases} n(t^{-1}-2+t) = 2 & \textit{if } n \textit{ is positive, even,} \\ t^{-1} + \frac{1}{2}(n+1)(t^{-2}-2t^{-1}+1) = 0 & \textit{if } n \textit{ is positive, odd,} \\ |n|(t^{-1}-2+t) = -2 & \textit{if } n \textit{ is negative, even,} \\ t^{-1} - \frac{1}{2}(|n|-1)(t^{-2}-2t^{-1}+1) = 0 & \textit{if } n \textit{ is negative, odd.} \end{cases}$$

Proof. We assume that n is positive, even. From the relations (1.7), (1.9) and (1.10), we obtain $(a_0 - b_0)(n(t^{-1}-2+t)-2) = 0$. If the color a_0 is equal to the color b_0 then D_n has nothing but trivial Λ/J-colorings. Since

Λ/J is an integral domain, D_n admits a non-trivial Λ/J-coloring if and only if it holds that $n(t^{-1} - 2 + t) = 2$.

In the same way, we can prove this lemma for other cases. □

The Alexander polynomial $\Delta_{K_n}(t)$ of the twist knot K_n is equal to

$$\begin{cases} \frac{n}{2}(1 - 2t + t^2) - t & \text{if } n \text{ is positive, even,} \\ t + \frac{1}{2}(n+1)(1 - 2t + t^2) & \text{if } n \text{ is positive, odd,} \\ \frac{|n|}{2}(1 - 2t + t^2) + t & \text{if } n \text{ is negative, even,} \\ t - \frac{1}{2}(|n| - 1)(1 - 2t + t^2) & \text{if } n \text{ is negative, odd,} \end{cases} \tag{1.11}$$

up to multiplication by a unit $\pm t^{\pm k}$. There is an integer n such that, although the determinant $|\Delta_{K_n}(-1)|$ is not prime, the Alexander polynomial $\Delta_{K_n}(t)$ is prime in the Laurent polynomial ring $\Lambda = \mathbb{Z}[t, t^{-1}]$, that is, the ring $\Lambda/(\Delta_{K_n}(t))$ is an integral domain. For example, if n is odd then $\Delta_{K_n}(t)$ is always prime. We have the following proposition from Lemma 1.18 and the equations (1.11).

Proposition 1.19. *For any integer n the diagram D_n of the twist knot K_n admits a non-trivial $\Lambda/(\Delta_{K_n}(t))$-coloring.*

We consider the case that n is positive, that is, we suppose that the diagram D_n is alternating. Assume that D_n is colored by a non-trivial $\Lambda/(\Delta_{K_n}(t))$-coloring. If the color b_l is equal to the color b_m for integers l, m $(-1 \leqq l, m \leqq n)$ then from the relations (1.7), (1.8) we obtain $l = m$. In other words, different arcs of D_n are colored by different colors. Accordingly, we have the following theorem.

Theorem 1.20. *Conjecture 1.17 is true for twist knots.*

In the same way, we can prove that for a negative integer n, that is, for a non-alternating diagram D_n, every non-trivial $\Lambda/(\Delta_{K_n}(t))$-coloring of D_n assigns different colors to different arcs of D_n. Possibly we may remove the condition that "a diagram is alternating" from Conjecture 1.17. At the present the author does not know a counterexample.

Chapter 2

Quandle Colorings of Surface Knots

In this chapter, we consider quandle colorings of surface knots, and construct diagrams of twist-spun knots, For torus knots and pretzel knots, we obtain the informations used to calculate the cocycle invariants of their twist-spins in the next chapter.

2.1 Preliminaries

A *surface knot* is a connected, oriented, and closed surface embedded in the 4-space $\mathbb{R}^4$. In particular, we call an embedded 2-sphere a 2-*knot*. Two surface knots F and F' are *equivalent* if there is an orientation-preserving homeomorphism $h : \mathbb{R}^4 \to \mathbb{R}^4$ such that $h(F) = F'$. Two surface knots F and F' are *ambient isotopic* if there is an ambient isotopy $\{h_t\}$ of $\mathbb{R}^4$ such that $h_1(F) = F'$. Two surface knots are equivalent if and only if they are ambient isotopic. A surface knot F is *trivial* (*unknotted*) if there an embedded handlebody H in $\mathbb{R}^4$ with $\partial H = F$, where a handlebody is a 3-manifold obtained from a 3-ball by attaching some 1-handles. For a surface knot F we denote the same surface with the orientation reversed by $-F$, and the mirror image of F, which is the image of F by orientation-reversing homeomorphism of $\mathbb{R}^4$, by F^*. We say that F is *non-invertible* if $F \ncong -F$. Refer to [11], [12], and [26] for more detail.

A projection $p : F \to \mathbb{R}^3$ is *generic* if the singularity set of the projection consists of double points, triple points and branch points (see the top of Figure 2.1). Crossing information is indicated in the image $p(F)$ of F by a generic projection as follows: Along every double point curve, two sheets intersect locally, one of which is under the other relative to the projection direction of p. Then the under-sheet is broken by the over-sheet. A *diagram* of F is the image $p(F)$ with such crossing information (see the bottom of Figure 2.1). Hence, a diagram is regarded as a union of disjoint compact, connected sheets. Two surface diagrams represent equivalent surface knots if and only if one can be transformed into the other by using finite sequence of the *Roseman moves* and ambient isotopies of $\mathbb{R}^4$ (cf. [7], [11], [12], and [26]).

The *triple point number* of a surface-knot F, denoted by $t(F)$, is the minimum number of triple points among all possible generic projections of F into the 3-space $\mathbb{R}^3$. By definition, the triple point number $t(F)$ is an invariant of F.

In the same way as the coloring of a classical knot diagram, we may define a coloring of a surface knot diagram.

Definition 2.1. Let D be a diagram of a surface knot F, Σ the set of connected sheets of D, and X a quandle. A coloring of D is a map $C : \Sigma \to X$ satisfying

$$C(\gamma) = C(\alpha) * C(\beta)$$

at each double curve, where $\alpha, \beta, \gamma \in \Sigma$ are the three sheets meeting at the double curve such that β is the over-sheet, α, γ are the under-sheets, which the normal direction of β points α to γ (see Figure 2.2).

We denote the set of all X-colorings of D by $\mathrm{Col}_X(D)$. The cardinality $\#\mathrm{Col}_X(D)$ of colorings is called the *X-coloring number* of D.

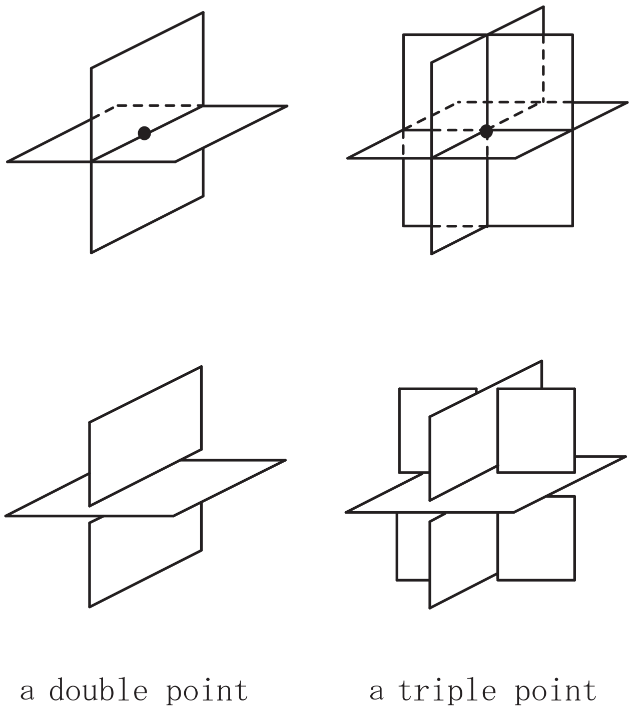


Figure 2.1: Singular points

Proposition 2.2 ([35]). *The X-coloring number $\#\mathrm{Col}_X(D)$ of D is a surface knot invariant.*

Each triple point t of D is assigned the sign $\varepsilon(t) = \pm 1$ induced from the orientation in such a way that $\varepsilon(t) = +1$ if and only if the ordered triple of the orientation normals of the top, middle, and bottom sheets, respectively, agree with the orientation of $\mathbb{R}^3$. The colors of the sheets near t are determined by the three colors $C(\alpha)$,$C(\beta)$ and $C(\gamma)$, where γ is the top sheet, β is the middle sheet in the back-side region of γ, and α is the bottom sheet in the back-side regions of both β and γ (see Figure 2.3). The ordered triple

$$(C(\alpha), C(\beta), C(\gamma)) \in X^3$$

is called the *color* of t and denoted by $C(t)$.

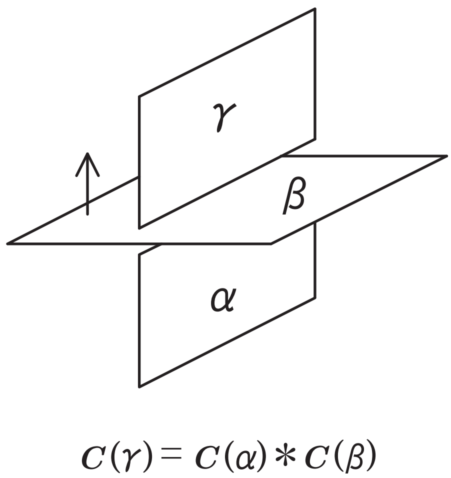


Figure 2.2: Coloring relation at a double point curve

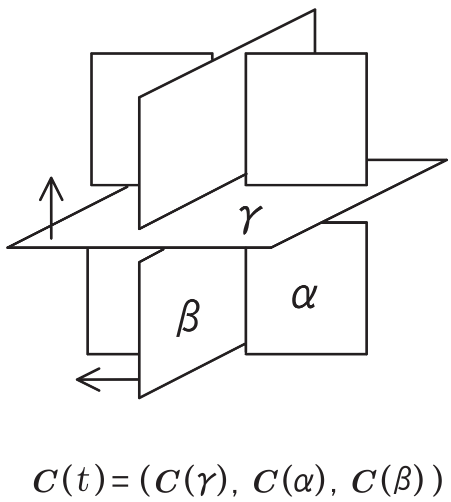


Figure 2.3: The color of a triple point

2.2 Twist-Spun Knots

In this section, we consider twist-spun knots and their diagrams.

A *spun knot* is a 2-knot introduced by Artin [1], and constructed from an oriented classical knot K as follows: Let K_0 be an embedded arc in $\mathbb{R}^3_+ = \{(x, y, z, w) \mid z \geqq 0,\ w = 0\}$ such that its closure is K and $\partial K_0 = K_0 \cap \partial \mathbb{R}^3_+$.

The subset

$$\{(x, y, z\cos\theta, z\sin\theta) \mid (x, y, z) \in K_0,\ \theta \in [0, 2\pi)\}$$

of $\mathbb{R}^4$ is a 2-sphere embedded in $\mathbb{R}^4$. This 2-knot is a spun knot of K. The spun knot is obtained by routing K_0 around $\mathbb{R}^2 = \{(x, y, z, w) \mid z = 0,\ w = 0\}$. See Figure 2.4.

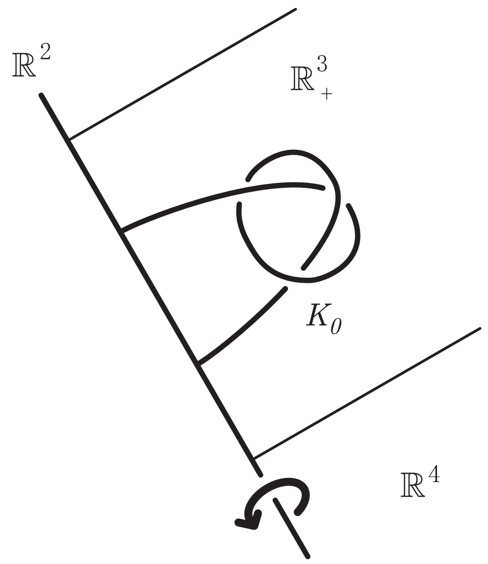


Figure 2.4: Construction of a spun knot

There is another definition of a spun knot. Let B be a 3-ball, K_0 an embedded arc in B such that its closure is K and $\partial K_0 = K_0 \cap \partial B$. We define an equivalence relation $\sim$ on $B \times S^1$ by

$$(x, \theta) \sim (x, \theta') \quad \text{for } x \in \partial B \text{ and } \theta, \theta' \in S^1.$$

Then $B \times S^1/\sim$ is a 4-sphere, and $K_0 \times S^1/\sim$ is a 2-sphere in the 4-sphere. Therefore, the 2-sphere $K_0 \times S^1/\sim$ is a 2-knot in $\mathbb{R}^4 = (B \times S^1/\sim) \setminus \{point\}$. We call this 2-knot a spun knot of K. The two definitions give the same 2-knot up to equivalence.

A spun knot was generalized by Zeeman [42] as follows: Let K_0 be an embedded arc in $\mathbb{R}^3_+$ such that its closure is an oriented classical knot K and $\partial K_0 = K_0 \cap \partial \mathbb{R}^3_+$ and its knotted part is contained in a 3-ball B^3 (see Figure 2.5). By twisting the 3-ball B^3 r-times while the 3-space $\mathbb{R}^3_+$ goes around $\mathbb{R}^2$, we obtain a 2-knot in $\mathbb{R}^4$. This 2-knot is called an *r-twist-spun knot* of K. A 0-twist-spun knot is a spun knot. See Figure 2.5.

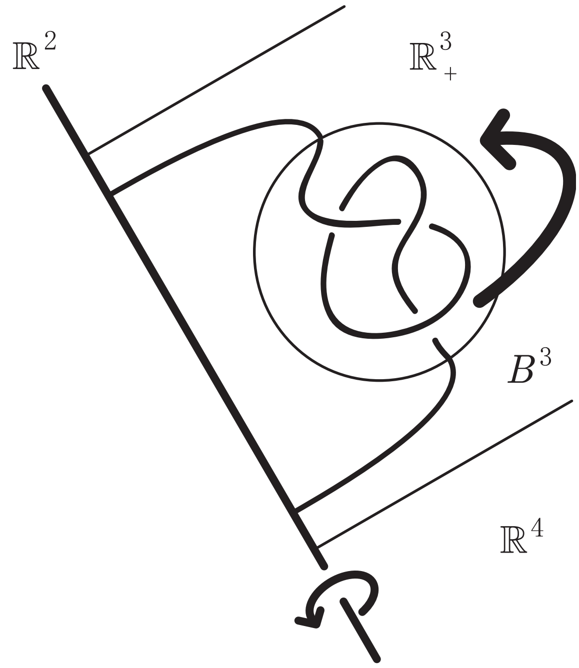


Figure 2.5: Construction of a twist-spun knot

Similarly to a spun knot, there is another definition of a twist-spun knot. Let B be a 3-ball, K_0 an embedded arc in B such that its closure is K and $\partial K_0 = K_0 \cap \partial B$. Let R_θ be rotation by the angle θ in the meridian direction. Then $B \times S^1/\sim$ is a 4-sphere, and $\bigcup_{\theta \in S^1} \left(R_{r\theta}(K_0) \times \{e^{i\theta}\}\right)/\sim$ is a 2-sphere in the 4-sphere, where $\sim$ is the equivalence relation defined above. Therefore, the 2-sphere $\bigcup_{\theta \in S^1} \left(R_{r\theta}(K_0) \times \{e^{i\theta}\}\right)/\sim$ is a 2-knot in $\mathbb{R}^4 = (B \times S^1/\sim) \setminus \{point\}$. We call this 2-knot a r-twist-spun knot of K, denote by $\tau^r(K)$. The two definitions give the same 2-knot up to equivalence.

Proposition 2.3 ([32]). *For any oriented classical knot K and positive integer r, we have*

(1) $\tau^r(-K) \cong \tau^r(K)^*$,

(2) $\tau^r(K^*) \cong -\tau^r(K)$,

(3) $\tau^r(-K^*) \cong -\tau^r(K)^*$.

In [41], Yajima introduced the concept of a *ribbon* 2-*knot*, and prove that a 2-knot F is a ribbon 2-knot if and only if $t(F) = 0$, where $t(F)$ denote the triple point number of F.

Theorem 2.4 ([13]). *For any non-trivial classical knot K, and $r \geqq 2$, the r-twist-spun knot $\tau^r(K)$ of K is a non-ribbon knot.*

The r-twist-spin $\tau^r K$ of K has a natural diagram $\tau^r D$ constructed in [6] which has $2r$-triple points for each crossing point of T. The diagram has the following properties.

(1) The $2r$-triple points $t_1^+(x), t_1^-(x)$, $t_2^+(x), t_2^-(x)$, $\ldots$, $t_r^+(x), t_r^-(x)$ have the sign $\varepsilon(t_k^+(x)) = \varepsilon(x)$ and $\varepsilon(t_k^-(x)) = -\varepsilon(x)$ $(1 \leqq k \leqq r)$, where $\varepsilon(x)$ is the sign of a crossing point x of T.

(2) When r is odd. The diagram of $\tau^r K$ admits only trivial R_p-colorings.

(3) When r is even. There is a one-to-one correspondence between shadow colorings of T and colorings of the diagram. Let $\widetilde{C}(x) = (s, a, b) \in R_p^3$ be the quandle triple at x, and $c \in R_p$ be the color of the terminal arc of T. The colors $C(t_k^+(x))$ and $C(t_k^-(x))$ of $t_k^+(x)$ and $t_k^-(x)$ $(1 \leqq k \leqq r)$ are given by

$$C(t_k^+(x)) = (s * c^k, a * c^k, b * c^k) \in R_p^3,$$
$$C(t_k^-(x)) = (a * c^k, b * c^k, c) \in R_p^3$$

respectively, where $*$ is the binary operation of R_p and $*c^k$ is k-times composite of $*c : R_p \to R_p$.

The diagram $\tau^r D$, which is called the *Satoh's diagram*, is more pictorial than that in [39]. Refer to [6] for more detail.

Chapter 3

Quandle Cocycle Invariants

In this chapter, we define quandle cocycle invariants, and calculate the invariants of torus knots, pretzel knots, and their twist-spins. Using this results, we consider the non-invertibility of surface knots and the integrality of quandle cocycle invariants.

3.1 Quandle Homology and Cohomology

We recall homology and cohomology of quandles. Let $C_n^{\mathrm{R}}(X)$ the free abelian group generated by n-tuples $(x_1, x_2, \ldots, x_n)$ of elements of a rack/quandle X. A boundary homomorphism $\partial_n(X) : C_n^{\mathrm{R}}(X) \to C_n^{\mathrm{R}}(X)$ is defined by

$$\begin{aligned}
&\partial_n(x_1, x_2, \ldots, x_n) \\
&= \sum_{i=1}^{n} (-1)^i \; [\; (x_1, x_2, \ldots, \widehat{x_i}, \ldots, x_n) \\
&\quad - (x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n) \;]
\end{aligned}$$

for $n \geqq 2$, and $\partial_n = 0$ for $n \leqq 1$. It can be checked by hand that $\partial_{n-1}\partial_n = 0$. Hence $C_*^{\mathrm{R}}(X) = \{C_n^{\mathrm{R}}(X), \partial_n\}$ is a chain complex.

Let $C_n^{\mathrm{D}}(X)$ be the subgroup of $C_n^{\mathrm{R}}(X)$ generated by n-tuples of elements of X with

$$x_i = x_{i+1}$$

for some $i \in \{1, 2, \ldots, n-1\}$ if $n \geqq 2$, and let $C_n^{\mathrm{D}} = 0$ if $n \leqq 1$. If X is a quandle it is follows from the properties of a quandle that $\partial_n(C_n^{\mathrm{D}}(X)) \subset C_{n-1}^{\mathrm{D}}(X)$. Hence $C_*^{\mathrm{D}}(X) = \{C_n^{\mathrm{D}}(X), \partial_n\}$ is a sub-complex of $C_*^{\mathrm{R}}(X)$. We define the quotient group $C_n^{\mathrm{Q}}(X)$ and the quotient complex $C_*^{\mathrm{Q}}(X)$ by

$$C_n^{\mathrm{Q}}(X) = C_n^{\mathrm{R}}(X)/C_n^{\mathrm{D}}, \quad C_*^{\mathrm{Q}}(X) = \{C_n^{\mathrm{Q}}(X), \partial'_n\},$$

where ∂'_n is the induced homomorphism. In the following, we denote all boundary maps by ∂_n.

Let G an abelian group. We define the chain complex

$$C_*^{\mathrm{W}}(X; G) = C_*^{\mathrm{W}}(X) \otimes G, \quad \partial = \partial \otimes \mathrm{id}$$

and cochain complex

$$C^*_{\mathrm{W}}(X; G) = \mathrm{Hom}(C_*^{\mathrm{W}}(X), G), \quad \delta = \mathrm{Hom}(\partial, \mathrm{id})$$

in the usual way, where $\mathrm{W} = \mathrm{D}, \mathrm{R}, \mathrm{Q}$. Then we have the homology group

$$H_n^{\mathrm{W}}(X; G) = H_n^{\mathrm{W}}(C_*^{\mathrm{W}} \otimes G)$$

and the cohomology group

$$H^n_{\mathrm{W}}(X; G) = H^n(\mathrm{Hom}(C_*^{\mathrm{W}}(X), G)).$$

We note that only $H_n^{\mathrm{R}}(X; G)$ and $H^n_{\mathrm{R}}(X; G)$ are well defined even if X is a rack. We call $H_n^{\mathrm{R}}(X; G)$ (resp. $H^n_{\mathrm{R}}(X; G)$) the n-th *rack homology group* (resp. *rack cohomology group*) of a rack/quandle X with coefficient group G, $H_n^{\mathrm{D}}(X; G)$ (resp. $H^n_{\mathrm{D}}(X; G)$) the n-th *degenerate homology group* (resp. *degenerate cohomology group*) of a quandle X with coefficient group G, and $H_n^{\mathrm{Q}}(X; G)$ (resp. $H^n_{\mathrm{Q}}(X; G)$) the n-th *quandle homology group* (resp. *quandle cohomology group*) of a quandle X with coefficient group G.

The cycle and boundary groups (resp. cocycle and coboundary groups) are denoted by $Z_n^{\mathrm{W}}(X;G)$ and $B_n^{\mathrm{W}}(X;G)$ (resp. $Z^n_{\mathrm{W}}(X;G)$ and $B^n_{\mathrm{W}}(X;G)$), so that

$$H_n^{\mathrm{W}}(X;G) = Z_n^{\mathrm{W}}(X;G)/B_n^{\mathrm{W}}(X;G),$$
$$H^n_{\mathrm{W}}(X;G) = Z^n_{W}(X;G)/B^n_{\mathrm{W}}(X;G),$$

where $\mathrm{W} = \mathrm{D}, \mathrm{R}, \mathrm{Q}$.

Let X be a quandle. By definition, a 2-cocycle ϕ in $Z^2_{\mathrm{Q}}(X;G)$ satisfies the following conditions.

(1) $\phi(x,x) = 0 \quad$ for any $x \in X$,

(2) $\phi(x,z) - \phi(x,y) = \phi(x*y,z) - \phi(x*z,y*z) \quad$ for any $x,y,z \in X$.

Similarly, a 3-cocycle θ in $Z^3_{\mathrm{Q}}(X;G)$ satisfies the following conditions.

(i) $\theta(x,x,y) = \theta(x,y,y) = 0 \quad$ for any $x,y \in X$,

(ii) $\theta(x,z,w) - \theta(x,y,w) + \theta(x,y,z)$
$= \theta(x*y,z,w) - \theta(x*z,y*z,w) + \theta(x*w,y*w,z*w)$
for any $x,y,z,w \in X$.

In [33], Mochizuki has proved that the third quandle cohomology group $H^3_{\mathrm{Q}}(R_p;\mathbb{Z}_p)$ of R_p with coefficient group $\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_p$ for any odd prime integer p. Additionally he has given an explicit presentation of its generator.

Theorem 3.1 ([33]). *We define the map $f : \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$ by*

$$\begin{aligned} f(x,y,z) = (y-x)\Bigg(&\frac{2}{p}z^p + \frac{4}{p-1}z^{p-1}(y-z) \\ &- \frac{1}{p-1}(y^{p-1}x + (-x+2z)(-y+2z)^{p-1}) \\ &+ \frac{1}{p(p-1)}(y^p + (-y+2z)^p)\Bigg). \end{aligned}$$

Then the map f is a generator of $H^3_{\mathrm{Q}}(R_p;\mathbb{Z}_p)$.

We define two maps θ_p and $\nu_p : \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$ by

$$\begin{cases} \theta_p(x, y, z) = (x - y)\dfrac{(2z - y)^p + y^p - 2z^p}{p}, \\ \nu_p(x, y, z) = 4(x - y)(y - z)z^{p-1} + (x - y)^2\big((2z - y)^{p-1} - y^{p-1}\big). \end{cases}$$

We note that all the coefficients of the polynomial $(2z - y)^p + y^p - 2z^p$ are divided by p. By using the equation $\dfrac{1}{p(p-1)} = \dfrac{1}{p-1} - \dfrac{1}{p}$, we have

$$f = \theta_p + \nu_p.$$

Both θ_p and ν_p satisfy the above-mentioned conditions (i) and (ii), that is, these also are 3-cocycles in $Z^3_{\mathrm{Q}}(R_p; \mathbb{Z}_p)$. given in Section 3.2.

Proposition 3.2 ([6])**.** *The cohomology classes defined by the* 3*-cocycles* f*,* θ_p*, and* $\nu_p \in Z^3_{\mathrm{Q}}(R_p; \mathbb{Z}_p)$ *satisfy*

$$[f] = [\theta_p], \quad [\nu_p] = 0$$

in $H^3_{\mathrm{Q}}(R_p; \mathbb{Z}_p)$*.*

Proof. The proof is given in the next section. □

The presentation of θ_p is more convenient than that of the original Mochizuki's 3-cocycle f.

3.2 Quandle Cocycle Invariants of Classical Knots

In this section, we define the quandle cocycle invariants of a oriented classical knot, and calculate concretely the cocycle invariants of torus knots and pretzel knots.

Let K be a oriented classical knot, D a diagram of K. Let X be a finite quandle, G an abelian group, and θ a quandle 3-cocycle in $Z^3_{\mathrm{Q}}(X; G)$. For each shadow X-coloring $\widetilde{C} \in \widetilde{\mathrm{Col}}_X(D)$, and a crossing x of D, we define the weight $\widetilde{W}_\theta(x; \widetilde{C}) \in G$ by

$$\widetilde{W}_\theta(x; \widetilde{C}) = \varepsilon(x)\theta(\widetilde{C}(x)),$$

where $\varepsilon(x) = \pm 1$ is the sign of x, $\widetilde{C}(x)$ is the quandle triple at x (see Section 2.1). Next, we define the element $\widetilde{W}_\theta(\widetilde{C}) \in G$ by

$$\widetilde{W}_\theta(\widetilde{C}) = \sum_x \widetilde{W}_\theta(x; \widetilde{C}),$$

where x runs all crossings of D. We consider the state-sum

$$\Psi_\theta(D) = \sum_{\widetilde{C} \in \widetilde{\mathrm{Col}}_X(D)} \widetilde{W}_\theta(\widetilde{C})$$

which takes value in $\mathbb{Z}[G]$.

Theorem 3.3 ([7]). *The state-sum $\Psi_\theta(D)$ is an invariant of K.*

We denote the invariant by $\Psi_\theta(K)$. The invariant $\Psi_\theta(K)$ has following properties (cf. [7], [10], and [36]).

Proposition 3.4 ([7]). *Let $\theta, \theta' \in Z^3_{\mathrm{Q}}(X; G)$ be 3-cocycles. If θ and θ' cohomologous, then $\Psi_\theta(K) = \Psi_{\theta'}(K)$ for any knot K.*

Proposition 3.5 ([7]). *If $\theta \in Z^3_{\mathrm{Q}}(X; G)$ is a trivial, then $\Psi_\theta(K)$ is equal to the number of shadow X-colorings, that is, $\Psi_\theta(K) = \#\widetilde{\mathrm{Col}}_X(D)$.*

There exists another invariant derived from 3-cocycle in $Z^3_{\mathrm{Q}}(X; G)$ of K. This invariant is defined as follows: Let T be a one-string tangle diagram of K. For each shadow X-coloring $C^* \in \mathrm{Col}^*(T)$, and a crossing x of T, we define the weight $W^*_\theta(x; C^*) \in G$ by

$$W^*_\theta(x; C^*) = \varepsilon(x)\theta(C^*(x)),$$

where $\varepsilon(x) = \pm 1$ is the sign of x, $C^*(x)$ is the quandle triple at x (see Section 2.1). Next, we define the element $W_\theta(C^*) \in G$ by

$$W^*_\theta(C^*) = \sum_{x \in P(T)} W^*_\theta(x; C^*),$$

where $P(T)$ denotes the set of crossing pints of T. We consider the state-sum

$$\Psi_\theta^*(T) = \sum_{C^* \in \mathrm{Col}^*(T)} W_\theta^*(C^*)$$

which takes value in the group ring $\mathbb{Z}[G]$.

Proposition 3.6 ([6])**.** *The state-sum $\Psi_\theta^*(T)$ dose not depend on the choice of a tangle diagram T of K.*

Proof. It is easy to check that $\Psi_\theta^*(T)$ is invariant under the Reidemeister moves with boundary points of T fixed.

Let T_1, T_2, and T_3 be tangle diagrams of K as shown in Figure 3.1. The tangle diagrams T_2 and T_3 are obtained from T_1 by reversing the boundary points of T_1. We prove that

$$\sum_{x \in P(T_1)} W_\theta^*(x; C^*) = \sum_{x \in P(T_2)} W_\theta^*(x; C^*) = \sum_{x \in P(T_3)} W_\theta^*(x; C^*),$$

where $P(T_1)$, $P(T_2)$, and $P(T_3)$ denote the set of crossing points of T_1, T_2, and T_3, respectively. Let a_+ and a_- be the colors of initial and terminal arcs of T colored by $C^* \in \mathrm{Col}^*(T_1)$. By lemma 1.9, we have $a_+ * a_- = a_+$ and $a_- * a_+ = a_-$. Hence, T_2 and T_3 are colored as shown in Figure 3.1, where colors enclosed by a circle denote colors of regions. We note that colors of a part of enclosed by a square in T_1, T_2, and T_3 are the same. Since $\theta(a_+, a_+, a_-) = 0$, we have

$$\begin{aligned}\sum_{x \in P(T_2)} W_\theta^*(x; C^*) &= \theta(a_+, a_+, a_-) + \sum_{x \in P(T_1)} W_\theta^*(x; C^*) \\ &= \sum_{x \in P(T_1)} W_\theta^*(x; C^*).\end{aligned}$$

Similarly, we may prove that

$$\sum_{x \in P(T_1)} W_\theta^*(x; C^*) = \sum_{x \in P(T_3)} W_\theta^*(x; C^*).$$

□

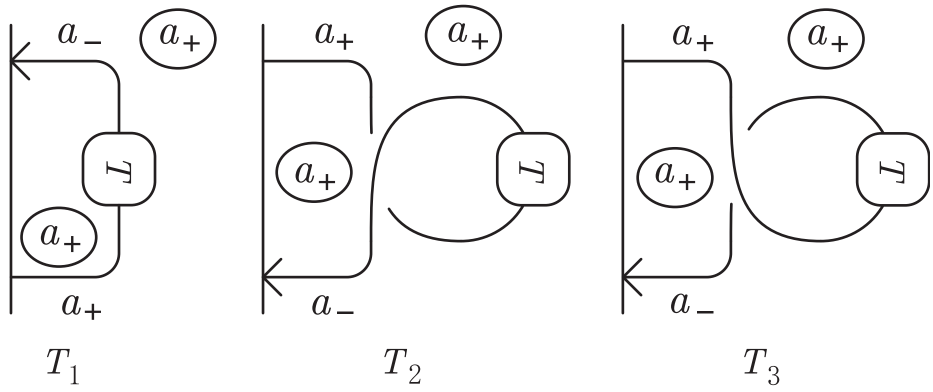


Figure 3.1: Tangle diagrams

Hence the state-sum $\Psi_\theta^*(T)$ is an invariant of K. We denote the invariant by $\Psi_\theta^*(K)$.

We consider the case $X = R_p$ and $G = \mathbb{Z}_p$. To clarify the meaning of the group ring $\mathbb{Z}[\mathbb{Z}_p]$, we identify $\mathbb{Z}[\mathbb{Z}_p]$ with the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]/(t^p - 1)$ by taking the coefficient group $\mathbb{Z}_p = \langle t | t^p = 1 \rangle$. Then we have

$$\Psi_\theta(K) = \sum_{\widetilde{C} \in \widetilde{\mathrm{Col}}(D)} t^{\widetilde{W}_\theta(\widetilde{C})} \in \mathbb{Z}[t^{\pm 1}]/(t^p - 1),$$

$$\Psi_\theta^*(K) = \sum_{C^* \in \mathrm{Col}^*(T)} t^{W_\theta^*(C^*)} \in \mathbb{Z}[t^{\pm 1}]/(t^p - 1).$$

If $\widetilde{W}_\theta(\widetilde{C})$ (resp. $W_\theta^*(C^*)$) is equal to $0 \in \mathbb{Z}_p$ for any $\widetilde{C} \in \widetilde{\mathrm{Col}}_X(D)$ (resp. $C^* \in \mathrm{Col}_X^*(T)$), then $\Psi_\theta(K)$ (resp. $\Psi_\theta^*(K)$) take value in $\mathbb{Z}$.

3.2.1 Torus Knots

We calculate the quandle cocycle invariants $\Psi_{\theta_p}(T(m,n))$ and $\Psi_{\theta_p}^*(T(m,n))$ of the torus knot $T(m,n)$, where θ_p is the 3-cocycle defined in Section 4.1. Since m and n are relatively prime and $T(m,n)$ and $T(n,m)$ are the same knot, we may assume that m is odd without loss of generality.

Let $D_{T(m,n)}$ be the diagram of $T(m,n)$ defined in Section 2.2. Fix a point e on the top arc α of $D_{T(m,n)}$. We obtain a tangle diagram of $T(m,n)$ by cutting the arc at e. From Corollary 1.10, it follows that the two external arcs of the tangle diagram have the same color. When we restrict the coloring of $D_{T(m,n)}$ such that $a = s$, the color of each arc or region of the tangle diagram coincide with the color of corresponding arc or region of $D_{T(m,n)}$, where $a = a_{01}$, $s = s_{11}$ $(= s_{21} = \cdots = s_{n1})$. Hence, we obtain

$$\Psi^*_{\theta_p}(T(m,n)) = \sum_{a=s} t^{\widetilde{W}_\theta(\widetilde{C})},$$

where $\widetilde{C}$ runs all shadow R_p-colorings of $D_{T(m,n)}$ such that $a = s$. Calculating $\widetilde{W}_{\theta_p}(\widetilde{C})$, we have $\Psi_{\theta_p}(T(m,n))$ and $\Psi^*_{\theta_p}(T(m,n))$.

First, we consider the case that $m = p$ and n is even.

Lemma 3.7 ([6]). *It holds that*

$$\sum_{j=1}^{n-1}\sum_{i=1}^{p} \widetilde{W}_{\theta_p}(x_{ij};\widetilde{C}) = -\frac{n}{2}(a-b)^2,$$

where $a = a_{01}$, $b = a_{02}$.

Proof. If $\widetilde{C}$ is trivial, that is, $a_{01} - a_{02}$, the equation holds. Hence, we may assume that $\widetilde{C}$ is not trivial. Further, if j is even, then we obtain

$$\begin{aligned}\widetilde{W}_{\theta_p}(x_{ij};\widetilde{C}) &= \varepsilon(x_{ij})\theta_p(\widetilde{C}(x_{ij}))\\ &= \theta_p(2a - s - (2i-j)\delta, a - i\delta, a - i\delta)\\ &= 0\end{aligned}$$

from Proposition 1.14, where $\delta = a - b$, $s = s_{11}$ $(= s_{21} = \cdots = s_{n1})$. Note that $\theta_p(x, y, y) = 0$ for any $x, y \in R_p$ by definition. Hence, we may assume that the sum is taken for odd $j = 1, 3, \ldots, n-1$. Then it holds that

$$\begin{aligned}&\sum_{i=1}^{p} \widetilde{W}_{\theta_p}(x_{ij};\widetilde{C}) = \varepsilon(x_{ij})\sum_{i=1}^{p}\theta_p(\widetilde{C}(x_{ij}))\\ &= \sum_{i=1}^{p}(s - a + (i-j)\delta)\frac{(a-(i+1)\delta)^p + (a-(i-1)\delta)^p - 2(a-i\delta)^p}{p}\\ &= \delta\sum_{i=1}^{p} i\frac{X_i}{p} + (s - a - j\delta)\sum_{i=1}^{p}\frac{X_i}{p},\end{aligned}$$

where

$$X_i = (a-(i+1)\delta)^p + (a-(i-1)\delta)^p - 2(a-i\delta)^p \pmod{p^2}.$$

Since $x^p = y^p \pmod{p^2}$ if $x = y \pmod{p}$, we have

$$\begin{aligned}
\sum_{i=1}^{p} i\frac{X_i}{p} &= \sum_{i=2}^{p+1}(i-1)(a-i\delta)^p + \sum_{i=0}^{p-1}(i+1)(a-i\delta)^p - 2\sum_{i=1}^{p} i(a-i\delta)^p \\
&= \{(p-1)(a-p\delta)^p + p(a-(p+a)\delta)^p\} \\
&\quad + \{a^p + 2(a-\delta)^p\} - 2\{(a-\delta)^p + p(a-p\delta)^p\} \\
&= p(b^p - a^p).
\end{aligned}$$

Furthermore, it holds that

$$\begin{aligned}
\sum_{i=1}^{p} \frac{X_i}{p} &= \sum_{i=1}^{p}(a-(i+1)\delta)^p + \sum_{i=1}^{p}(a-(i-1)\delta)^p - 2\sum_{i=1}^{p}(a-i\delta)^p \\
&= \{(a-p\delta)^p + (a-(p+1)\delta)^p\} \\
&\quad + \{a^p + (a-\delta)^p\} - 2\{(a-\delta)^p + (a-p\delta)^p\} \\
&= 0.
\end{aligned}$$

Therefore, we have

$$\sum_{i=1}^{p} \widetilde{W}_{\theta_p}(x_{ij};\widetilde{C}) = \delta\cdot(b^p - a^p) - j\delta\cdot 0 = (b-a)\delta = -\delta^2.$$

Taking the sum for odd $j = 1, 3, \ldots, n-1$, we obtain

$$\sum_{j=1}^{n-1}\sum_{i=1}^{p} \widetilde{W}_{\theta_p}(x_{ij};\widetilde{C}) = -\frac{n}{2}(a-b)^2.$$

□

Proposition 3.8 ([6])**.** *If n is even, the quandle cocycle invariants $\Psi_{\theta_p}(T(p,n))$ and $\Psi^*_{\theta_p}(T(p,n))$ are given by*

$$\Psi_{\theta_p}(T(p,n)) = p^2\left(\sum_{i=0}^{p-1} t^{-ni^2}\right) \in \mathbb{Z}[t^{\pm 1}]/(t^p-1),$$

$$\Psi^*_{\theta}(T(p,n)) = p\left(\sum_{i=0}^{p-1} t^{-ni^2}\right) \in \mathbb{Z}[t^{\pm 1}]/(t^p-1),$$

where p is an odd prime integer.

Proof. As discussed in Chapter 1, the shadow R_p-coloring $\widetilde{C}$ of the diagram $D_{T(m,n)}$ is determined by $(a,b,s) \in \mathbb{Z}_p^3$. Hence, from Lemma 3.7, it follows that

$$\begin{aligned}\Psi_{\theta_p}(T(p,n)) &= \sum_{(a,b,s)\in\mathbb{Z}_p^3} t^{\widetilde{W}_{\theta_p}(\widetilde{C})} = \sum_{s\in\mathbb{Z}_p}\sum_{(a,b)\in\mathbb{Z}_p} t^{-\frac{n}{2}(a-b)^2} \\ &= \sum_{s\in\mathbb{Z}_p}\left(p\left(\sum_{i=0}^{p-1} t^{-\frac{n}{2}i^2}\right)\right) = p^2\left(\sum_{i=0}^{p-1} t^{-\frac{n}{2}i^2}\right)\end{aligned}$$

and

$$\begin{aligned}\Psi^*_{\theta_p}(T(p,n)) &= \sum_{(a,b)\in\mathbb{Z}_p^2} t^{\widetilde{W}_{\theta_p}(\widetilde{C})} = \sum_{(a,b)\in\mathbb{Z}_p} t^{-\frac{n}{2}(a-b)^2} \\ &= p\left(\sum_{i=0}^{p-1} t^{-\frac{n}{2}i^2}\right).\end{aligned}$$

□

From Proposition 3.8, we obtain the quandle cocycle invariants $\Psi_{\theta_p}(T(m,n))$ and $\Psi^*_{\theta_p}(T(m,n))$ of the torus knot $T(m,n)$.

Theorem 3.9 ([6])**.** *If $T(m,n)$ admits a non-trivial R_p-coloring, then we have*

$$\begin{aligned}\Psi_{\theta_p}(T(m,n)) &= p^2\left(\sum_{i=0}^{p-1} t^{-\frac{mn}{2p}i^2}\right), \\ \Psi^*_{\theta_p}(T(m,n)) &= p\left(\sum_{i=0}^{p-1} t^{-\frac{mn}{2p}i^2}\right)\end{aligned}$$

and otherwise $\Psi_{\theta_p}(T(m,n)) = p^2$ and $\Psi^*_{\theta_p}(T(m,n)) = p$.

Proof. Since $T(m,n)$ has non-trivial R_p-coloring if and only if m is divisible by p and n is even (under the assumption that m is odd), the invariants $\Psi_{\theta_p}(T(m,n))$ and $\Psi^*_{\theta_p}(T(m,n))$ are obtainted from $\Psi_{\theta_p}(T(p,n))$ and $\Psi^*_{\theta_p}(T(p,n))$ by replacing t with $t^{\frac{m}{p}}$. Hence, the theorem follows from Proposition 3.8. □

3.2.2 Pretzel Knots

In the following, we assume that the pretzel knot $P = P(p_1, \ldots, p_m)$ is alternating, odd, and oriented by the orientation indicated in Figure 1.8. We calculate the quandle cocycle invariants $\Psi_{\theta_p}(P)$ and $\Psi^*_{\theta_p}(P)$ of the pretzel knots. Using this results, we prove that the invariants take value in $\mathbb{Z}$ for any 3-cocycle in $Z^3_{\mathrm{Q}}(R_p; \mathbb{Z}_p)$.

Let D_P be the diagram of $P(p_1, \ldots, p_m)$ defined in Section 2.2. Fix a point e on the top arc α of D_P. We obtaint a tangle diagram of $P(p_1, \ldots, p_m)$ by cutting the arc at e. From Corollary 1.10, it follows that the two external arcs of the tangle diagram have the same color. When we restrict the coloring of D_P such that $a_0 = s_0$, the color of each arc or region of the tangle diagram concide with the color of corresponding arc or region of D_P. Hence, we obtain

$$\Psi^*_{\theta_p}(P) = \sum_{a_0 = s_0} t^{\widetilde{W}_\theta(\widetilde{C})},$$

where $\widetilde{C}$ runs all shadow R_p-colorings of D_P such that $a_0 = s_0$. Calculating $\widetilde{W}_\theta(\widetilde{C})$, we have $\Psi_\theta(P)$ and $\Psi^*_\theta(P)$.

Lemma 3.10. *For any shadow R_p-coloring $\widetilde{C}$ of the diagram D_P of the alternating odd pretzel knot $P(p_1, \ldots, p_m)$, we have $\widetilde{W}_{\theta_p}(\widetilde{C}) = 0$, that is,*

$$\sum_{i=1}^{m} \sum_{j=1}^{|p_i|} W_{\theta_p}(x_{ij}, \widetilde{C}) = 0,$$

where x_{ij} is the j-th crossing from the top of the i-th column of D_P.

Proof. Assume that all p_i's are positive ($1 \leqq i \leqq m$). The quandle triple

$\widetilde{C}(x_{ij})$ of a crossing point x_{ij} of D_P is given by

$$\widetilde{C}(x_{ij}) = \begin{cases} (s_{i-1,j-1}, a_{i-1,j-1}, a_{i-1,j}) & \text{if } j \text{ is even,} \\ (s_{i-1,j}, a_{i-1,j+1}, a_{i-1,j}) & \text{if } j \text{ is odd,} \end{cases}$$
$$= \begin{cases} (2(a_i + (j-2)d_i) - s_{i-1}, a_i + (j-2)d_i, a_i + (j-1)d_i) \\ \hfill \text{if } j \text{ is even,} \\ (2(a_i + (j-1)d_i) - s_{i-1}, a_i + jd_i, a_i + (j-1)d_i) \\ \hfill \text{if } j \text{ is odd,} \end{cases}$$

from the equations (1.5), (1.6) and Figure 1.9. Since $\varepsilon(x_{ij}) = -1$,

$$\widetilde{W}_{\theta_p}(x_{ij}, \widetilde{C}) = -\theta_p(\widetilde{C}(x_{ij}))$$
$$= (a_i - jd_i + s_0 - 2a_0)\frac{(a_i + (j-2)d_i)^p + (a_i + jd_i)^p - 2(a_i + (j-1)d_i)^p}{p}$$
$$= \frac{1}{p}\left(a_i X_{ij} - jd_i X_{ij} + (s_0 - 2a_0)X_{ij}\right)$$

with no regard to the parity of j, where

$$X_{ij} = (a_i + (j-2)d_i)^p + (a_i + jd_i)^p - 2(a_i + (j-1)d_i)^p.$$

Then it holds that

$$\begin{aligned} \sum_{j=1}^{p_i} X_{ij} &= (a_i - d_i)^p - (a_i)^p - (a_i + (p_i - 1)d_i)^p + (a_i + p_i d_i)^p, \\ \sum_{j=1}^{p_i} a_i X_{ij} &= a_i((a_{i-1})^p - (a_i)^p) - a_i((a_{i-1} + p_i d_i)^p - (a_i + p_i d_i)^p), \\ \sum_{j=1}^{p_i} jd_i X_{ij} &= d_i(a_{i-1})^p - d_i(a_{i-1} + p_i d_i)^p \\ &\qquad + p_i d_i((a_i + p_i d_i)^p - (a_{i-1} + p_i d_i)^p). \end{aligned} \tag{3.1}$$

If all p_i's are not divisible by p ($1 \leqq i \leqq m$), then it holds that $d_i = a_i - a_{i-1} = q_i c_0 = \frac{c_0}{p_i}$. From equations (3.1), we obtain

$$\sum_{i=1}^{m}\sum_{j=1}^{p_i} X_{ij} = \sum_{i=1}^{m}\{(a_{i-1})^p - (a_i)^p\} - \sum_{i=1}^{m}\{(a_{i-1} + c_0)^p - (a_i + c_0)^p\}$$
$$= 0 - 0 = 0.$$

Furthermore, it holds that

$$\sum_{i=1}^{m}\sum_{j=1}^{p_i} a_i X_{ij} = \sum_{i=1}^{m} d_i(a_{i-1})^p - \sum_{i=1}^{m} d_i(a_{i-1}+c_0)^p,$$
$$\sum_{i=1}^{m}\sum_{j=1}^{p_i} j d_i X_{ij} = \sum_{i=1}^{m} d_i(a_{i-1})^p - \sum_{i=1}^{m} d_i(a_{i-1}+c_0)^p.$$

Therefore, we have

$$\widetilde{W}_{\theta_p}(\widetilde{C}) = \sum_{i=1}^{m}\sum_{j=1}^{p_i} \widetilde{W}_{\theta_p}(x_{ij},\widetilde{C}) = 0.$$

If $p_{i_1},\dots,p_{i_n}$ in $\{p_1,\dots,p_m\}$ are divisible by p for some $n \geqq 1$ $(i_1 < \cdots < i_n)$, it holds that $W_{\theta_p}(x_{ij},\widetilde{C}) = 0$ for $i \neq i_1,\cdots,i_n$, because all arcs of i-th column are colored by a same color. Hence we may assume that p_i is divisible by p for any i $(1 \leqq i \leqq m = n)$. Since $p_i d_i = 0$, it follows from the equations (3.1) that $\widetilde{W}_{\theta_p}(\widetilde{C}) = 0$.

Assume that all p_i's are negative. Then in the same way we may find the quandle triple and the sign of a crossing x_{ij}, calculate $W_{\theta_p}(\widetilde{C})$, and get the same result, that is, $\widetilde{W}_{\theta_p}(\widetilde{C}) = 0$. □

Proposition 3.11. *Let $\widetilde{C}$ be a shadow R_p-coloring of the diagram D_P of the alternating odd pretzel knot $P(p_1,\dots,p_m)$. For any $\mathbb{Z}_p$-valued 3-cocycle θ on R_p, we have $\widetilde{W}_\theta(\widetilde{C}) = 0$.*

Proof. The cohomology class $[\theta_p]$ is a generator of $H^3(R_p,\mathbb{Z}_p)$. Hence, for any $\mathbb{Z}_p$-valued 3-cocycle θ on R_p, there is $k \in \mathbb{Z}_p$ such that $[\theta] = k[\theta_p] \in H^3(R_p,\mathbb{Z}_p)$. Then we have $\widetilde{W}_\theta(\widetilde{C}) = k\widetilde{W}_{\theta_p}(\widetilde{C}) = 0$.

□

Theorem 3.12. *If the alternating odd pretzel knot $P(p_1,\dots,p_m)$ admits a non-trivial R_p-coloring, the cocycle invariants $\Psi_\theta(P)$, $\Psi^*_\theta(P)$ of $P(p_1,\dots,p_m)$*

are given by

$$\Psi_\theta(P) = \begin{cases} p^3 & \textit{Case 1,} \\ p^{n+1} & \textit{Case 2,} \end{cases}$$

$$\Psi^*_\theta(P) = \begin{cases} p^2 & \textit{Case 1,} \\ p^n & \textit{Case 2,} \end{cases}$$

where Case 1, 2 means that $P(p_1, \ldots, p_m)$ belong to **Case 1***,* **2** *in Section* 1.2.2*, respectively. If $P(p_1, \ldots, p_m)$ admits only trivial R_p-colorings, then $\Psi_\theta(P) = p^2$, $\Psi^*_\theta(P) = p$.*

Proof. Assume that $P(p_1, \ldots, p_m)$ admits a non-trivial R_p-coloring. As discussed in Chapter 1, the shadow R_p-coloring $\widetilde{C}$ of the diagram D_P is determined by $(a_0, b_0, s_0) \in \mathbb{Z}_p^3$, $(a_{i_1}, \ldots, a_{i_n}, s_0) \in \mathbb{Z}_p^{n+1}$ in **Case 1**, **2** respectively. Since $\widetilde{W_\theta}(\widetilde{C}) = 0$ for any $\widetilde{C}$, we have

$$\Psi_\theta(P) = \begin{cases} \displaystyle\sum_{(a_0,b_0,s_0)\in\mathbb{Z}_p^3} t^{\widetilde{W_\theta}(\widetilde{C})} = p^3 & \text{Case 1,} \\ \displaystyle\sum_{(a_{i_1},\ldots,a_{i_n},s_0)\in\mathbb{Z}_p^{n+1}} t^{\widetilde{W_\theta}(\widetilde{C})} = p^{n+1} & \text{Case 2.} \end{cases}$$

We may assume that $a_{i_1} = a_0$ without loss of generality. Fix a base point of $P(p_1, \ldots, p_m)$ on the top arc colored by a_0. The color s_0 is the color of an adjacent region to the arc colored by a_0. By the definition of Ψ^*_θ, we obtain

$$\Psi^*_\theta(P) = \begin{cases} \displaystyle\sum_{(a_0,b_0)\in\mathbb{Z}_p^2} t^{\widetilde{W_\theta}(\widetilde{C})} = p^2 & \text{Case 1,} \\ \displaystyle\sum_{(a_{i_1},\ldots,a_{i_n})\in\mathbb{Z}_p^{n}} t^{\widetilde{W_\theta}(\widetilde{C})} = p^{n} & \text{Case 2.} \end{cases}$$

Assume that $P(p_1, \ldots, p_m)$ admits only trivial R_p-colorings. The shadow R_p-coloring $\widetilde{C}$ of the diagram D_P is determined by $(a_0, s_0) \in \mathbb{Z}_p^2$. In the same way, we obtain $\Psi_\theta(P) = p^2$, $\Psi^*_\theta(P) = p$. This completes the proof. $\square$

In this section, we have proved that $\Psi_{\theta_p} = p\Psi^*_{\theta_p}$ holds for torus knots and pretzel knots. It has been known that the equality holds for also 2-bridge knots (cf. [24]), 3-braid knots (cf. [38]). However, it is unknown whether the equality holds for any classical knots or not.

3.3 Quandle Cocycle Invariants of Surface Knots

In this section, we define the quandle cocycle invariant of surface knots, and calculate concretely the quandle cocycle invariants of twist-spun knots of torus knots and pretzel knots. Using the result, we consider non-invertibility of surface knots.

Let F be a surface knot, D a diagram of F. Let X be a finite quandle, G an abelian group, and θ a quandle 3-cocycle in $Z^3_{\mathrm{Q}}(X;G)$. For each X-coloring $C \in \mathrm{Col}_X(D)$, and a triple point t, we define $W_\theta(t,C)$ and $W_\theta(C)$ by

$$W_\theta(t,C) = \varepsilon(t)\theta(C(t)) \in G,$$
$$W_\theta(C) = \sum_t W_\theta(t,C) \in G,$$

where t runs all triple points of D. We consider the state-sum $\Phi_\theta(D)$ defined by

$$\Phi_\theta(D) = \sum_C W_\theta(C),$$

which takes value in the group ring $\mathbb{Z}[G]$.

Theorem 3.13 ([7]). *The state-sum $\Phi_\theta(D)$ is independent of the choice of a diagram D of a surface knot F.*

Hence, the state-sum $\Phi_\theta(D)$ is an invariant of a surface knot. We denote the invariant by $\Phi_\theta(F)$. By definition, the invariant $\Phi_\theta(F)$ is equal to the coloring number $|\mathrm{Col}_X(D)| \in \mathbb{Z} \subset \mathbb{Z}[G]$ if F is a ribbon 2-knot or admits only trivial colorings.

Proposition 3.14 ([7]). *If θ and θ' are cohomologous, then $\Phi_\theta(F) = \Phi_{\theta'}(F)$ for any surface knot F.*

For an element $\sum a_i g_i \in \mathbb{Z}[G]$, we denote the element $\sum a_i g_i^{-1} \in \mathbb{Z}[G]$ by $\overline{\sum a_i g_i}$.

Proposition 3.15 ([9]). *For any surface knot F and any 3-cocycle $\theta \in Z^3_Q(X;G)$, we have*

$$\Phi_\theta(-F^*) = \overline{\Phi_\theta(F)}.$$

Since $T(m,n) \cong -T(m,n)$, it follows form Proposition 2.3 and Proposition 3.15 that

$$\Phi_\theta(-\tau^r(T(m,n))) \cong \overline{\Phi_\theta(\tau^r(T(m,n)))}.$$

Hence, we obtain $\Phi_\theta(-\tau^r(T(m,n)))$ from $\Phi_\theta(\tau^r(T(m,n)))$ without calculating the invariant.

We consider the case that $X = R_p$, $G = \mathbb{Z}_p$, and $F = \tau^r(K)$, where K is a classical knot, and r is a non-negative integer. Let θ_p be the 3-cocycle defined in Chapter 2, and $\rho^r : \mathbb{Z}[t^{\pm 1}]/(t^p - 1) \to \mathbb{Z}[t^{\pm 1}]/(t^p - 1)$ be the map induced by $t \to t^r$.

In [6], Satoh have proved the following theorem.

Theorem 3.16 ([6]). (i) *If r is odd, then we have $\Phi_{\theta_p}(\tau^r(K)) = p$.*

(ii) *If r is even, then we have $\Phi_{\theta_p}(\tau^r(K)) = \rho^r \Psi^*_{\theta_p}(K)$.*

Using Theorem 3.16 and Theorem 3.9, we obtain the quandle cocycle invariants $\Phi_{\theta_p}(\tau^r(T(m,n)))$ of the torus knot $T(m,n)$.

Theorem 3.17 ([6])**.** *If $\tau^r(T(m,n))$ admits a non-trivial R_p-coloring, then*

$$\Phi_{\theta_p}(\tau^r(T(m,n))) = p\left(\sum_{i=1}^{p-1} t^{-\frac{mnr}{2p}i^2}\right)$$

and otherwise $\Phi_{\theta_p}(\tau^r(T(m,n))) = p$.

Proof of Proposition 3.2. By Theorem 3.17, we have

$$\Phi_{\theta_p}(\tau^2(T(m,2))) = p\left(\sum_{i=1}^{p-1} t^{-2i^2}\right).$$

If the cohomology class $[\theta_p]$ is not a generator of $H^3_{\mathrm{Q}}(R_p;\mathbb{Z}_p) \cong \mathbb{Z}_p$, the cocycle invariant must be p^2 by definition. Hence, $[\theta_p]$ is a generator. Therefore, it holds that $[\nu_p] = k[\theta_p] \in H^3_{\mathrm{Q}}(R_p;\mathbb{Z}_p)$ for some $k \in \mathbb{Z}_p$. By definition, we have

$$\Phi_{\nu_p}(\tau^2(T(m,2))) = p\left(\sum_{i=1}^{p-1} t^{-2ki^2}\right).$$

On other hand, by calculating $\Phi_{\nu_p}(\tau^2(T(m,2)))$, we obtain

$$\Phi_{\nu_p}(\tau^2(T(m,2))) = p^2.$$

Hence, we have $k = 0$, that is, $[\nu_p] = 0$ and $[f] = [\theta_p] + [\nu_p] = [\theta_p] \in H^3_{\mathrm{Q}}(R_p;\mathbb{Z}_p)$. □

Let $F^r_g(m,n)$ denote the surface knot of genus g obtained from $\tau^r(T(m,n))$ by surgery along $g \geqq 0$ trivial 1-handles (cf. [26]). It is easy to prove the following lemma.

Lemma 3.18 ([6])**.** $\Phi_{\theta_p}\big(F^r_g(m,n)\big) = \Phi_{\theta_p}\big(\tau^r T(m,n)\big)$

Theorem 3.19 ([6])**.** *Suppose that m is odd. Then the surface knot $F^r_g(m,n)$ is non-invertible if there is a prime factor p of m satisfying the following;*

(i) $p \equiv 3 \pmod 4$,

(ii) *m is not divisible by p^2,*

(iii) *n is even, and*

(iv) *r is even and not divisible by p.*

Proof. By Lemma 3.18, it is sufficient to prove that

$$\Phi_p\big(\tau^r T(m,n)\big) \neq \rho^{-1}\Phi_p\big(\tau^r T(m,n)\big).$$

if m, n, r and p satisfy the conditions (i)–(iv). This can be seen easily from Theorem 3.17 and the fact that

$$p\big(\sum_{i=0}^{p-1} t^{-Ni^2}\big) \neq p\big(\sum_{i=0}^{p-1} t^{Ni^2}\big)$$

in $\mathbb{Z}[t^{\pm 1}]/(t^p - 1)$ if and only if $p \equiv 3 \pmod 4$ and N is not divisible by p. $\square$

The following is a main theorem in this thesis.

Theorem 3.20 ([6])**.** *For each non-negative integer g, there is an infinite family of non-invertible surface knots of genus g.*

Proof. Among the non-invertible surface knot $F_g^r(m,n)$ satisfying the conditions (i)–(iv), we consider the ones given by $m = p$ and $n = r = 2$, for example. Then we see that the family $\big\{F_g^2(p,2)\big| p = 3, 7, 11, 19, \dots\big\}$ is infinite, for if $p \neq p'$, then

$$\Phi_p\big(F_g^2(p,2)\big) = p\big(\sum_{i=0}^{p-1} t^{-2i^2}\big),$$
$$\Phi_p\big(F_g^2(p',2)\big) = p$$

by Theorem 3.17, and hence $F_g^2(p,2)$ is not ambient isotopic to $F_g^2(p',2)$. $\square$

In the same way, we may calculate the cocycle invariant $\Phi_\theta(\tau^r P)$ of the alternating odd pretzel knot $P = P(p_1, \dots, p_m)$

Theorem 3.21. *If the alternating odd pretzel knot $P(p_1, \ldots, p_m)$ admits a non-trivial R_p-coloring and r is even, then the cocycle invariant $\Phi_\theta(\tau^r P)$ of the r-twist-spin $\tau^r P$ of $P(p_1, \ldots, p_m)$ is given by*

$$\Phi_\theta(\tau^r P) = \begin{cases} p^2 & \textit{Case 1}, \\ p^n & \textit{Case 2}. \end{cases} \tag{3.2}$$

for any 3-cocycle $\theta \in Z^3_{\mathrm{Q}}(R_p; \mathbb{Z}_p)$. If $P(p_1, \ldots, p_m)$ admits only trivial R_p-colorings or r is odd, then we have $\Phi_\theta(\tau^r P) = p$ for any 3-cocycle $\theta \in Z^3_{\mathrm{Q}}(R_p; \mathbb{Z}_p)$.

Proof. Assume that $P(p_1, \ldots, p_m)$ admits a non-trivial R_p-coloring and r is even. If $\theta = \theta_p$, then the equation (3.2) follows from Theorem 3.16 and Theorem 3.9. Let $\widetilde{C}$ be a restricted shadow R_p-coloring of the diagram D_P of $P(p_1, \ldots, p_m)$ such that $a_0 = s_0$. Let C be the R_p-coloring of the diagram $\tau^r D_P$ determined by $\widetilde{C}$. Then we obtain $W_{\theta_p}(C) = r\widetilde{W}_{\theta_p}(\widetilde{C})$. Theorem 3.16 (ii) follows from the equation (cf. [6]). By Proposition 3.11, we have $W_{\theta_p}(C) = 0$. Since the cohomology class $[\theta_p]$ is a generator of $H^3(R_p, \mathbb{Z}_p)$, it holds that $W_\theta(C) = W_{\theta_p}(C) = 0$ for any $\mathbb{Z}_p$-valued 3-cocycle θ on R_p (cf. Proof of Proposition 3.11). Hence $\Phi_\theta(\tau^r P) = \Phi_{\theta_p}(\tau^r P)$. Therefore, the equation (3.2) holds for any 3-cocycle $\theta \in Z^3_{\mathrm{Q}}(R_p; \mathbb{Z}_p)$.

Assume that $P(p_1, \ldots, p_m)$ admits only trivial R_p-colorings and r is odd. Then $\tau^r P$ admits only trivial R_p-colorings. Therefore, we have $\Phi_\theta(\tau^r P) = p$ for any 3-cocycle $\theta \in Z^3_{\mathrm{Q}}(R_p; \mathbb{Z}_p)$.

□

The following is another main theorem in this thesis.

Theorem 3.22. *There exists a non-ribbon 2-knot F which admit a non-trivial R_p-coloring, and whose quandle cocycle invariant $\Phi_\theta(F)$ takes value in $\mathbb{Z} \subset \mathbb{Z}[\mathbb{Z}_p]$ for any 3-cocycle $\theta \in Z^3_{\mathrm{Q}}(R_p; \mathbb{Z}_p)$.*

Proof. Let $P(p_1, \ldots, p_m)$ be an alternating odd pretzel knot. If $m \geqq 3$ then $P(p_1, \ldots, p_m)$ is not trivial (cf. [29]). From Theorem 2.4, it follows that the r-twist-spun knot $\tau^r P$ of $P(p_1, \ldots, p_m)$ is a non-ribbon 2-knot for any $m \geqq 3$, $r \geqq 2$. On the other hand, its quandle cocycle invariant $\Phi_\theta(\tau^r P)$ takes value in $\mathbb{Z} \subset \mathbb{Z}[\mathbb{Z}_p]$ for any 3-cocycle $\theta \in Z^3_{\mathrm{Q}}(R_p; \mathbb{Z}_p)$ by Theorem 3.21. This completes the proof. $\square$

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