ON THE KAUFFMAN-HARARY CONJECTURE FOR ALEXANDER QUANDLE COLORINGS

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Abstract. The Kauffman-Harary conjecture is a conjecture for Fox's colorings of alternating knots with prime determinants. We consider a conjecture for Alexander quandle colorings by referring to the Kauffman-Harary conjecture. We prove that this new conjecture is true for twist knots.

1. Introduction.

We recall the following conjecture called the *Kauffman-Harary conjecture* [3].

Conjecture 1.1. *Let D be a reduced alternating knot diagram with a prime determinant p. Then every non-trivial Fox's p-coloring of D assigns different colors to different arcs of D.*

In [1], Asaeda, Przytycki and Sikora generalize the conjecture by stating it in terms of homology of the double cover of the 3-sphere S^3 branched along a link, and prove that the generalized conjecture is true for Montesinos links.

Fox's p-coloring is coincident with the coloring by the dihedral quandle R_p of order p (cf. [2]). We consider the following conjecture associated with the Alexander quandle which we can regard as a generalization of R_p.

Conjecture 1.2. *Let K be an alternating oriented knot, D be a reduced alternating diagram of K, and $\Delta_K(t)$ be the Alexander polynomial of K. If the ring $\mathbb{Z}[t, t^{-1}]/(\Delta_K(t))$ is an integral domain, then every non-trivial coloring of D by the Alexander quandle $\mathbb{Z}[t, t^{-1}]/(\Delta_K(t))$ assigns different colors to different arcs of D.*

If the Kauffman-Harary conjecture is true then Conjecture 1.2, also, is true for any alternating knots with prime determinants (see Section 2). But Conjecture 1.2 is not included in the Kauffman-Harary conjecture because there is a knot with a non-prime determinant whose Alexander polynomial is a prime element in the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$.

This paper is organized as follows. In Section 2, we review the definition of quandles and colorings by quandles of knot diagrams. In Section 3, we study colorings by Alexander quandles for the diagram of twist knots. Finally, we prove that Conjecture 1.2 is true for twist knots (Theorem 3.3).

2. Quandles and Colorings.

In this section, we review the definition of quandles and colorings by quandles.

Definition 2.1 ([4], [5])**.** A quandle, X, is a set with a binary operation $* : X \times X \to X$ satisfying the following conditions:

2000 *Mathematics Subject Classification.* 57Q25, 57M42.

(Q1) For any $x \in X$, $x * x = x$.
(Q2) For any $x, y \in X$, there is a unique element $z \in X$ such that $x = z * y$.
(Q3) For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

The condition (Q2) is equivalent to the following condition:

(Q2') For any $x, y \in X$, there is a binary operation $*^{-1} : X \times X \to X$ such that $(x * y) *^{-1} y = x = (x *^{-1} y) * y$.

We list some typical examples of quandles.

Example 2.2. (1) Let m be a positive integer. We define the binary operation $*$ on the set $\{0, 1, 2, \dots, m-1\}$ by $x*y = 2y-x \pmod{m}$ for $x, y \in \{0, 1, 2, \dots, m-1\}$. Then the set become a quandle, called the *dihedral quandle* of order m and denoted by R_m. The operation $*^{-1}$ is identical with the operation $*$.

(2) Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ be the Laurent polynomial ring over $\mathbb{Z}$, $J \subset \Lambda$ be an ideal of Λ. Then the quotient ring Λ/J with the binary operation defined by $x * y = tx + (1-t)y$ for any $x, y \in \Lambda/J$ is a quandle called an *Alexander quandle*. The operation $*^{-1}$ is given by $x *^{-1} y = t^{-1}x + (1-t^{-1})y$. We remark that the dihedral quandle R_m is isomorphic to $\Lambda/(m, t+1)$

Definition 2.3. Let D be a diagram of an oriented knot K, and Σ the set of arcs of D. Given a quandle X, an *X-coloring* for D is a map $C : \Sigma \to X$ which satisfies $C(\gamma) = C(\alpha) * C(\beta)$ at each crossing, where $\alpha, \gamma \in \Sigma$ are under-arcs on the right and left of the over-arc $\beta \in \Sigma$, respectively. If an X-coloring uses only one color we say that it is *trivial*.

For example, the coloring by the dihedral quandle R_m is coincident with Fox's m-coloring. An Alexander quandle coloring C satisfies $C(\gamma) = tC(\alpha) + (1-t)C(\beta)$ and $C(\alpha) = t^{-1}C(\gamma) + (1-t^{-1})C(\beta)$ at each crossing, where α, β and γ are above-mentioned.

Let D be a diagram of an oriented knot K with a prime determinant p, $\Delta_K(t)$ be the Alexander polynomial of K, and $\Sigma = \{\alpha_i\}_i$ be the set of arcs of D. Since the determinant p is equal to $|\Delta_K(-1)|$, an R_p-coloring of D corresponds to a coloring by $\Lambda/(\Delta_K(t), t+1) = \mathbb{Z}/(\Delta_K(-1))$. For any color $f_i(t)$ of α_i colored by $\Lambda/(\Delta_K(t))$, there is a color a_i of α_i colored by R_p, such that $f_i(-1) = a_i \pmod{p}$. If a_i is not equal to a_j in $\mathbb{Z}_p$ for $i \neq j$, then $f_i(t)$ is not equal to $f_j(t)$ in $\Lambda/(\Delta_K(t))$. Therefore, if the Kauffman-Harary conjecture is true then Conjecture 1.2, also, is true for any alternating knots with prime determinants. We remark that there is a knot with non-prime determinants whose Alexander polynomial is a prime element in the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$. See Section 3.

3. Alexander quandle colorings of twist knots.

In this section, we consider Alexander quandle colorings of twist knots.

The diagram D_n of an n-twist knot K_n is pictured in the left of Figure 1, where $|n|$ is the number of crossings in the "twist" part. The twists are right-handed if $n > 0$ and left-handed if $n < 0$. The left of Figure 1 shows the case $n > 0$. We orient K_n by the orientation indicated in the left of Figure 1.

Let Λ/J be an Alexander quandle. We color the arcs of the "twist" part in the diagram D_n by $a_0, b_0, b_1, \dots, b_{|n|} \in \Lambda/J$ as shown in the right of Figure 1. By the definition of Alexander quandle colorings, the relations between these colors are

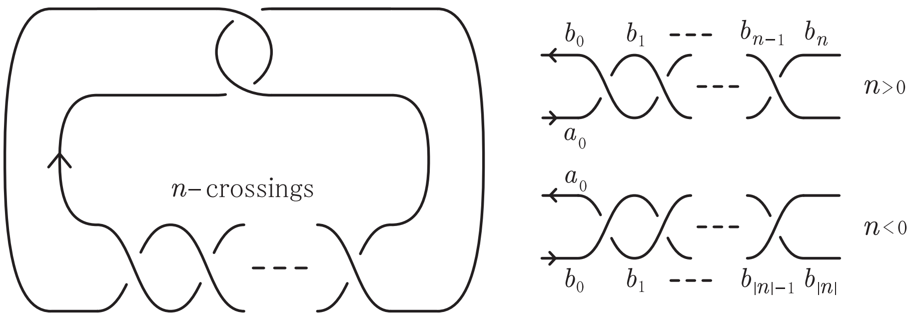

FIGURE 1. The diagram D_n of K_n

described by

$$\begin{pmatrix} b_{i-1} \\ b_i \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 1 \\ t & 1-t \end{pmatrix} \begin{pmatrix} b_{i-2} \\ b_{i-1} \end{pmatrix} & \text{if } i \text{ is even,} \\ \begin{pmatrix} 0 & 1 \\ t^{-1} & 1-t^{-1} \end{pmatrix} \begin{pmatrix} b_{i-2} \\ b_{i-1} \end{pmatrix} & \text{if } i \text{ is odd,} \end{cases}$$

without regard to the sign of n, where $i = 0, 1, \ldots, n$ and $b_{-1} = a_0$. By induction, we have

$$\begin{pmatrix} b_{i-1} \\ b_i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i(t^{-1}-1)+2 & i(1-t^{-1}) \\ i(t^{-1}-1) & i(1-t^{-1})+2 \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \tag{1}$$

if i is even, and

$$\begin{pmatrix} b_{i-1} \\ b_i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (i-1)(t^{-1}-1) & (i-1)(1-t^{-1})+2 \\ (i+1)(t^{-1}-1)+2 & (i+1)(1-t^{-1}) \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \tag{2}$$

if i is odd. Furthermore, the colorings $a_0, b_0, b_{|n|-1}, b_{|n|}$ of the four arcs in the "clasp" part of D_n have the following relations:

$$b_0 = \begin{cases} t^{-1}a_0 + (1-t^{-1})b_n & \text{if } n \text{ is positive, even,} \\ ta_0 + (1-t)b_n & \text{if } n \text{ is positive, odd,} \\ ta_0 + (1-t)b_{|n|-1} & \text{if } n \text{ is negative, even,} \\ t^{-1}a_0 + (1-t^{-1})b_{|n|-1} & \text{if } n \text{ is negative, odd,} \end{cases} \tag{3}$$

and

$$b_{|n|-1} = \begin{cases} t^{-1}b_n + (1-t^{-1})a_0 & \text{if } n \text{ is positive,} \\ tb_{|n|} + (1-t)b_0 & \text{if } n \text{ is negative.} \end{cases} \tag{4}$$

Lemma 3.1. *Assume that the ring Λ/J is an integral domain. The diagram D_n admits a non-trivial Λ/J-coloring if and only if it holds that*

$$\begin{cases} n(t^{-1}-2+t) = 2 & \textit{if } n \textit{ is positive, even,} \\ t^{-1} + \frac{1}{2}(n+1)(t^{-2}-2t^{-1}+1) = 0 & \textit{if } n \textit{ is positive, odd,} \\ |n|(t^{-1}-2+t) = -2 & \textit{if } n \textit{ is negative, even,} \\ t^{-1} - \frac{1}{2}(|n|-1)(t^{-2}-2t^{-1}+1) = 0 & \textit{if } n \textit{ is negative, odd.} \end{cases}$$

Proof. We assume that n is positive, even. From the relations (1),(3) and (4), we obtain $(a_0 - b_0)(n(t^{-1} - 2 + t) - 2) = 0$. If the color a_0 is equal to the color b_0 then D_n has nothing but trivial Λ/J-colorings. Since Λ/J is an integral domain, D_n admits a non-trivial Λ/J-coloring if and only if it holds that $n(t^{-1} - 2 + t) = 2$.

In the same way, we can prove this lemma for other cases. $\square$

The Alexander polynomial $\Delta_{K_n}(t)$ of the twist knot K_n is equal to

$$\begin{cases} \frac{n}{2}(1-2t+t^2) - t & \text{if } n \text{ is positive, even,} \\ t + \frac{1}{2}(n+1)(1-2t+t^2) & \text{if } n \text{ is positive, odd,} \\ \frac{|n|}{2}(1-2t+t^2) + t & \text{if } n \text{ is negative, even,} \\ t - \frac{1}{2}(|n|-1)(1-2t+t^2) & \text{if } n \text{ is negative, odd,} \end{cases} \tag{5}$$

up to multiplication by a unit $\pm t^{\pm k}$. There is an integer n such that, although the determinant $|\Delta_{K_n}(-1)|$ is not prime, the Alexander polynomial $\Delta_{K_n}(t)$ is prime in the Laurent polynomial ring $\Lambda = \mathbb{Z}[t, t^{-1}]$, that is, the ring $\Lambda/(\Delta_{K_n}(t))$ is an integral domain. For example, if n is odd then $\Delta_{K_n}(t)$ is always prime. We have the following proposition from Lemma 3.1 and (5).

Proposition 3.2. *For any integer n the diagram D_n of the twist knot K_n admits a non-trivial $\Lambda/(\Delta_{K_n}(t))$-coloring.*

We consider the case that n is positive, that is, we suppose that the diagram D_n is alternating. Assume that D_n is colored by a non-trivial $\Lambda/(\Delta_{K_n}(t))$-coloring. If the color b_l is equal to the color b_m for integers l, m $(-1 \leqq l, m \leqq n)$ then from the relations (1), (2) we obtain $l = m$. In other words, different arcs of D_n are colored by different colors. Accordingly, we have the following theorem.

Theorem 3.3. *Conjecture 1.2 is true for twist knots.*

In the same way, we can prove that for a negative integer n, that is, for a non-alternating diagram D_n, every non-trivial $\Lambda/(\Delta_{K_n}(t))$-coloring of D_n assigns different colors to different arcs of D_n. Possibly we may remove the condition that "a diagram is alternating" from Conjecture 1.2. At the present the author does not know a counterexample.

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