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AN INFINITE FAMILY OF NON-INVERTIBLE SURFACES IN 4-SPACE

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Abstract

A proof is given that for each non-negative integer g, there is an infinite family of knotted surfaces of genus g, none of which is ambient isotopic to itself with the orientation reversed.

1. *Introduction*

Throughout this paper, surfaces embedded in the 4-space $\mathbb{R}^4$ are connected, closed, and oriented. Such a knotted surface is called *non-invertible* if it is not ambient isotopic to itself with the orientation reversed. For knotted 2-spheres, there are several studies on non-invertibility; see [**4**, **5**, **7**, **9**, **14**], for example. However, most of the known methods of proving non-invertibility cannot be applied directly to knotted surfaces of higher genus. In [**2**], Carter, Jelsovsky, Kamada, Langford and Saito introduced the quandle cocycle invariants of knotted surfaces, which can detect non-invertibility regardless of genus. Our concern in this paper is mainly with the twist-spins of classical knots, introduced by Zeeman [**16**]. For integers $g \geqslant 0$ and $m, n, r \geqslant 2$ such that m and n are relatively prime, let $F_g^r(m,n)$ denote the knotted surface of genus g obtained from the r-twist-spin of the (m,n)-torus knot by surgery along g trivial 1-handles. The following is our main theorem, proved by calculating the cocycle invariant of $F_g^r(m,n)$ associated with Mochizuki's 3-cocycle (see [**11**]).

THEOREM 1.1. *Suppose that m is odd. Then the knotted surface $F_g^r(m,n)$ is non-invertible if there is a prime factor p of m satisfying the following conditions:*
(i) $p \equiv 3 \pmod 4$,
(ii) *m is not divisible by p^2,*
(iii) *n is even, and*
(iv) *r is even, and is not divisible by p.*
Therefore, for each non-negative integer g, there is an infinite family of non-invertible knotted surfaces of genus g.

In order to calculate the cocycle invariant systematically, some difficulties must first be overcome. In Section 2, we review the definition of quandle cocycle invariants, where we reformulate Mochizuki's 3-cocycle [**11**] of a dihedral quandle (Example 2.1). In Section 3, we give a diagram representing *any* twist-spin, which is more pictorial than that given in [**15**]. Section 4 is devoted to the introduction of a modified cocycle invariant of a classical knot in $\mathbb{R}^3$, which was motivated by Rourke and Sanderson's observation [**13**]. In Section 5, we study colorings for

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twist-spins by quandles. In particular, for dihedral quandles we prove that the cocycle invariant of a twist-spin is obtained directly from that of the associated classical knot (Theorem 5.4). In Section 6, we calculate the cocycle invariants of torus knots and their twist-spins concretely, by using Mochizuki's 3-cocycle (Theorem 6.3). We conclude the paper with the proof of Theorem 1.1.

2. *Preliminaries*

A *quandle* [**8**, **10**] is a set Q with a binary operation $*$ satisfying the following conditions:

(i) $a * a = a$ for any $a \in Q$,
(ii) the map $*a : Q \longrightarrow Q$ defined by $x \mapsto x * a$ is bijective for each $a \in Q$, and
(iii) $(a * b) * c = (a * c) * (b * c)$ for any $a, b, c \in Q$.

The homology and cohomology theory for quandles is developed in [**2**]; it is similar to that for groups. For an abelian group G, let $C^n(Q; G)$ be the free abelian group generated by the maps $f : Q^n \longrightarrow G$ satisfying $f(x_1, \dots, x_n) = 0$ if $x_k = x_{k+1}$ for some $k = 1, \dots, n-1$. The coboundary map

$$\delta^n : C^n(Q; G) \longrightarrow C^{n+1}(Q; G)$$

is given by

$$(\delta^n f)(x_1, \dots, x_{n+1}) = \sum_{k=2}^{n+1} (-1)^k \Big\{ f(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}) \\ - f(x_1 * x_k, \dots, x_{k-1} * x_k, x_{k+1}, \dots, x_{n+1}) \Big\}.$$

The quandle cohomology group $H^*(Q; G)$ is defined by $\{C^*(Q; G), \delta^*\}$ in the usual manner, and the cocycle and coboundary groups are denoted by $Z^*(Q; G)$ and $B^*(Q; G)$, respectively.

Example 2.1. The set $\{0, 1, \dots, p-1\}$ becomes a quandle under the binary operation $a * b \equiv 2b - a \pmod{p}$, which is called the *dihedral quandle* of order p, and is denoted by R_p. Mochizuki [**11**] proved that $H^3(R_p; \mathbb{Z}_p) \cong \mathbb{Z}_p$ for any odd prime p, and gave an explicit presentation of its generator. By an easy calculation, his 3-cocycle turns out to be the sum of two polynomials θ_p and ν_p, given by

$$\begin{cases} \theta_p(x, y, z) = (x - y) \dfrac{(2z - y)^p + y^p - 2z^p}{p}; \\ \nu_p(x, y, z) = 4(x - y)(y - z)z^{p-1} + (x - y)^2 \big\{ (2z - y)^{p-1} - y^{p-1} \big\}. \end{cases}$$

Note that the coefficients of the polynomial $(2z - y)^p + y^p - 2z^p$ are divisible by p. Since it can be checked by hand that $\delta^3 \theta_p = 0$ and $\nu_p = \delta^2 f$ for $f(x, y) = (x - y)^2 y^{p-1} \in C^2(R_p; \mathbb{Z}_p)$, we have $\theta_p \in Z^3(R_p; \mathbb{Z}_p)$ and $\nu_p \in B^3(R_p; \mathbb{Z}_p)$. Hence $[\theta_p] \in H^3(R_p; \mathbb{Z}_p)$ is also a generator. We adopt this presentation θ_p as the Mochizuki 3-cocycle. Note that θ_p satisfies

$$\theta_p(x * z, y * z, z) = -\theta_p(x, y, z), \qquad \text{for any } x, y, z \in R_p.$$

To describe a knotted surface, we use a fixed projection of $\mathbb{R}^4$ onto $\mathbb{R}^3$, as well as a description of a classical knot into the plane. A (*surface*) *diagram* of a knotted surface $F \subset \mathbb{R}^4$ is the image of F by the projection that has double-point curves,

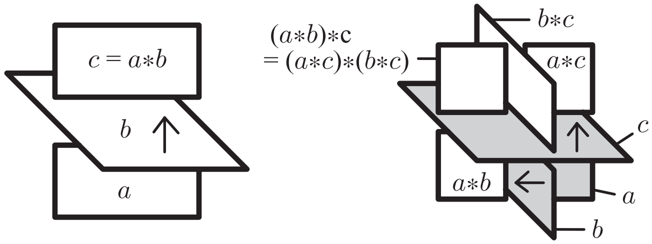

FIGURE 1.

isolated triple points, and isolated branch points as singularities, where we indicate crossing information in the usual manner. (Refer to [**1**], for example.) In particular, two sheets intersect along a double curve locally; one is underneath the other, relative to the projection direction. The under-sheet is shown broken in a diagram, so that the diagram consists of connected regions separated by over-sheets.

Let D be a diagram of F, and $\Sigma(D)$ the set of such connected regions of D. A map $C : \Sigma(D) \longrightarrow Q$ into a quandle Q is a *Q-coloring of D* if it satisfies the following condition along every double-point curve: if $a = C(\alpha_1)$ and $c = C(\alpha_2)$ are the colors of the under-sheets α_1 and α_2 separated by the over-sheet β colored by $b = C(\beta)$, where the orientation normal of β points from α_1 to α_2, then $a * b = c$ holds (see the left-hand diagram in Figure 1). We denote the set of such Q-colorings of D by $\mathrm{Col}_Q(D)$.

Each triple point t of D is assigned the sign $\varepsilon(t) = \pm 1$ induced from the orientation; specifically, $\varepsilon(t) = +1$ if and only if the ordered triple of the orientation normals of, respectively, the top, middle, and bottom sheets agrees with the orientation of $\mathbb{R}^3$. Given a Q-coloring $C \in \mathrm{Col}_Q(D)$, the colors of the sheets near t are characterized by three colors $a = C(\alpha)$, $b = C(\beta)$ and $c = C(\gamma)$, where γ is the top sheet, β is the middle sheet from which the orientation normal of γ points, and α is the bottom sheet from which the orientation normals of β and γ point. The ordered triple (a, b, c) is denoted by $C(t) \in Q^3$ (see the right-hand diagram in Figure 1, where the sheets α, β, and γ are shaded.

Assume that Q is a finite quandle. Given a 3-cocycle $f \in Z^3(Q; G)$, we define the *Boltzmann weight* at t by $W_f(t; C) = f(a, b, c)^{\varepsilon(t)} \in G$, where $C(t) = (a, b, c)$ and G is written multiplicatively. Then the *cocycle invariant* of F associated with f is the state-sum

$$\Phi_f(F) = \sum_{C \in \mathrm{Col}_Q(D)} \Big[\prod_{t \in \mathrm{X}_3(D)} W_f(t; C) \Big] \in \mathbb{Z}[G],$$

valued in the group-ring $\mathbb{Z}[G]$, where $\mathrm{X}_3(D)$ denotes the set of triple points of D. This is proved in [**2**] to be an invariant of F that does not depend on the choice of a diagram D of F.

3. *Diagrams of twist-spun knots*

For each non-negative integer r, Zeeman [**16**] constructed a knotted 2-sphere from a classical knot K. This is called the *r-twist-spin of K*, and is denoted by $\tau^r K$. Let T be a tangle diagram whose closure presents K. In this section, we construct a diagram of $\tau^r K$ from T; this is more pictorial than that given in [**15**].

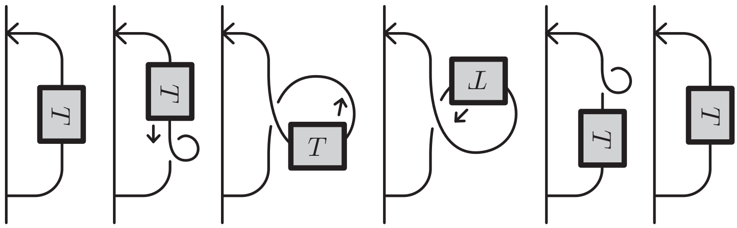

FIGURE 2.

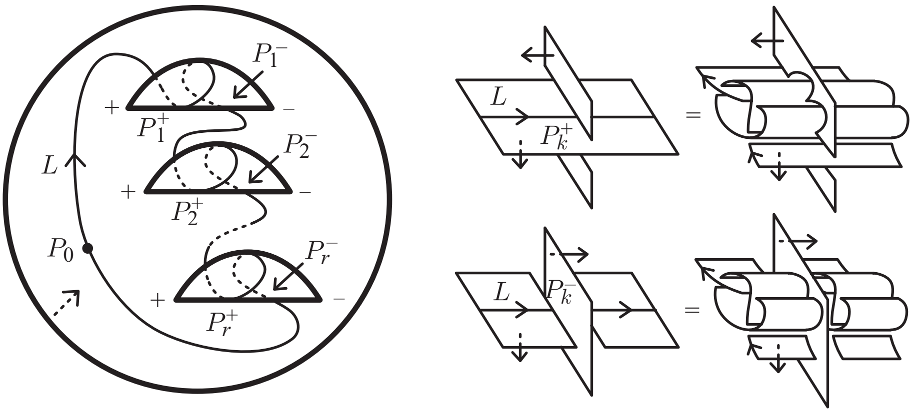

FIGURE 3.

Consider a motion picture of tangle diagrams in the upper half-plane $\mathbb{R}^2_+$, as shown in Figure 2 (reading from left to right), with the boundary-points of T fixed. We give $\mathbb{R}^3$ an open book structure $\mathbb{R}^2_+ \times \mathbb{S}^1$, where its binding is the boundary of $\mathbb{R}^2_+$, and we regard each page $\mathbb{R}^2_+$ as a frame of the motion. Since the motion presents one full twist of T, the trace of the motion, repeated r times, describes a surface diagram of $\tau^n K$ in $\mathbb{R}^2_+ \times \mathbb{S}^1$. We denote the diagram of $\tau^r K$ thus obtained by $\mathrm{D}^r T \subset \mathbb{R}^3$.

The diagram $\mathrm{D}^r T$ is directly constructed as follows. First, we consider a 2-sphere in $\mathbb{R}^3$ with r kinks, as shown on the left-hand side of Figure 3, and we give crossing information along the double-point curves as in the middle; for the signs of the branch points, refer to [**1**], for example. We fix an orientation of the sphere such that the orientation normal points toward the inside, except for the kinks. Let L be an oriented simple closed curve with a base point P_0, which goes around the kinks. Then the diagram $\mathrm{D}^r T$ is obtained by replacing an annulus-neighborhood of L with $T \times L$. Let P_k^+ and P_k^- be the intersections of L and the double-point curve at the kth kink ($1 \leqslant k \leqslant r$). Near the point P_k^+, the sheet containing L is over the transverse sheet. Hence, crossing information at $T \times \{P_k^+\}$ is given such that $T \times L$ is over the transverse sheet. On the other hand, crossing information at $T \times \{P_k^-\}$ is given such that $T \times L$ is under the transverse sheet. See the right-hand side of Figure 3 as an example of a trefoil, where coherent orientations of T, L, and $T \times L$ are also depicted.

For a crossing x of T, we denote the sign of x by $\varepsilon(x) = \pm 1$. Corresponding to x, the diagram $\mathrm{D}^r T$ has $2r$ triple points at $T \times \{P_k^+\}$ and $T \times \{P_k^-\}$, denoted by

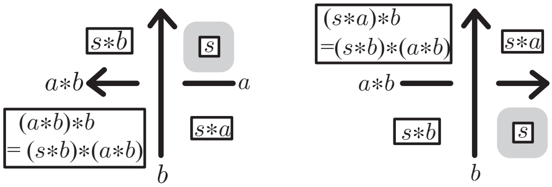

FIGURE 4.

$t_k^+(x)$ and $t_k^-(x)$ ($1 \leqslant k \leqslant r$), respectively. By the coherent orientations of T and D^rT, we immediately have $\varepsilon(t_k^+(x)) = \varepsilon(x)$ and $\varepsilon(t_k^-(x)) = -\varepsilon(x)$.

4. *Shadow colorings of tangles*

Let T be a tangle diagram of a classical knot K, and let $\Sigma(T)$ be the set of arcs of T separated by over-arcs. A map $C : \Sigma(T) \longrightarrow Q$ into a quandle Q is a *Q-coloring of T* if it satisfies the following condition at each crossing x: if α_1 and α_2 are under-arcs separated by the over-arc β, where α_1 is on the right-hand side of β, then $C(\alpha_1) * C(\beta) = C(\alpha_2)$. We denote the pair $(C(\alpha_1), C(\beta))$ by $C(x) \in Q^2$, and the set of Q-colorings of T by $\mathrm{Col}_Q(T)$. A *shadow Q-coloring of T* extending C is a map $C^* : \Sigma^*(T) \longrightarrow Q$, where $\Sigma^*(T)$ is the union of $\Sigma(T)$, and the set of regions of $\mathbb{R}^2_+$ separated by the underlying immersed curve of T, satisfying the following conditions:

(i) C^* restricted to $\Sigma(T)$ is coincident with C,
(ii) if R_1 and R_2 are regions separated by an arc α, where R_1 is on the right-side of α, then $C^*(R_1) * C^*(\alpha) = C^*(R_2)$, and
(iii) $C^*(\alpha_+) = C^*(R_0)$, where α_+ is the initial arc of T and R_0 is the unbounded region.

Note that C^* always exists uniquely for a given C (see [**3**]), and we denote C^* by C also. Let $f \in Z^3(Q; G)$ be a 3-cocycle of a finite quandle Q. Define the Boltzmann weight at a crossing x by $W_f^*(x; C) = f(s, a, b)^{\varepsilon(x)} \in G$, where $C(x) = (a, b)$ and s is the color of the region being distinguished, as shown in Figure 4. Consider the state-sum

$$\Phi_f^*(T) = \sum_{C \in \mathrm{Col}_Q(T)} \left[\prod_{x \in \mathrm{X}_2(T)} W_f^*(x; C) \right] \in \mathbb{Z}[G],$$

where $\mathrm{X}_2(T)$ denotes the set of crossings of T.

PROPOSITION 4.1. *The state-sum $\Phi_f^*(T)$ does not depend on the choice of a tangle diagram T of K.*

Proof. It is easy to check that $\Phi_f^*(T)$ is invariant under Reidemeister moves with boundary points of T fixed. Consider the moves among T_1, T_2, and T_3 reversing the boundary points; see the left of Figure 5. Let $C \in \mathrm{Col}_Q(T_1)$ be a (shadow) Q-coloring of T_1, and let a_+ and $a_- \in Q$ be the colors of the initial and terminal arcs of T_1, respectively. Since $a_+ * a_- = a_+$ and $a_- * a_+ = a_-$, the (shadow) colors of T_2 and T_3 induced from C are as shown in the figure. (In general, we have $a * a_+ = a * a_-$ for any $a \in Q$. This can be seen by taking a circle under T with the outside arc colored by a; see the right of Figure 5. Then the color of the inside arc

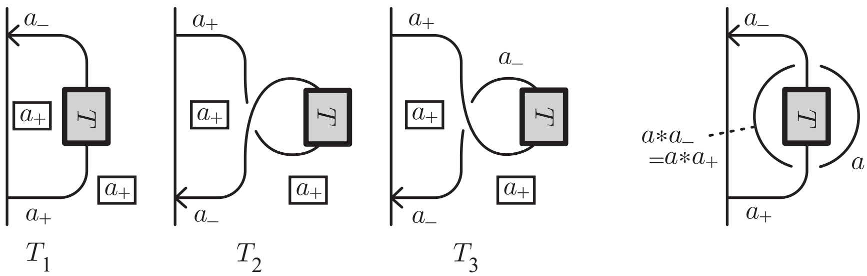

FIGURE 5.

is $a * a_+ = a * a_-$ (see [**12**]).) Then we have

$$\prod_{x \in \mathrm{X}_2(T_2)} W_f^*(x; C) = f(a_+, a_+, a_-) \prod_{x \in \mathrm{X}_2(T_1)} W_f^*(x; C)$$
$$= \prod_{x \in \mathrm{X}_2(T_1)} W_f^*(x; C).$$

Moreover, since T_2 and T_3 are related by Reidemeister moves with boundary fixed, $\Phi_f^*(T)$ does not change under these moves. □

Since $\Phi_f^*(T)$ is an invariant of K by Proposition 4.1, we denote it by $\Phi_f^*(K)$. Note that $\Phi_f^*(K)$ is similar to the shadow cocycle invariant defined in [**3**]; the difference is that the invariant in [**3**] is taken for all the shadow Q-colorings, which may not satisfy condition (iii). (See also [**13**].)

For each $C \in \mathrm{Col}_Q(T)$ and an integer $k \geqslant 0$, we denote by $C * a_-^k \in \mathrm{Col}_Q(T)$ the composite of C and $(*a_-)^k$, where a_- is the color of the terminal arc of T.

LEMMA 4.2. $\prod_{x \in \mathrm{X}_2(T)} W_f^*(x; C * a_-^k) = \prod_{x \in \mathrm{X}_2(T)} W_f^*(x; C)$.

Proof. If T is colored by C, then the k-twisting as in Figure 2 induces the coloring $C * a_-^k$ of T. Since the motion is realized by Reidemeister moves, the products of Boltzmann weights for C and $C * a_-^k$ do not change. □

5. *Colorings of twist-spun knots*

Let T be a tangle diagram of a classical knot K. We study a relationship between the sets $\mathrm{Col}_Q(T)$ and $\mathrm{Col}_Q(\mathrm{D}^r T)$. Let $\iota : T \longrightarrow \mathrm{D}^r T$ be the inclusion by identifying T with $T \times \{P_0\} \subset T \times L \subset \mathrm{D}^r T$ at the base point P_0 of L, and let $\iota^* : \mathrm{Col}_Q(\mathrm{D}^r T) \longrightarrow \mathrm{Col}_Q(T)$ be the induced map. Put $\mathrm{Col}_Q^r(T) = \mathrm{Im}(\iota^*) \subset \mathrm{Col}_Q(T)$.

LEMMA 5.1. *The map ι^* is injective. Moreover, $C \in \mathrm{Col}_Q(T)$ belongs to $\mathrm{Col}_Q^r(T)$ if and only if $C * a_-^r = C$ holds, where a_- is the color of the terminal arc of T.*

Proof. We try to extend $C \in \mathrm{Col}_Q(T)$ to the entire diagram $\mathrm{D}^r T$ as follows.

1. Let a_+ be the color of the initial arc of T. Then the sheets of $\mathrm{D}^r T$, say H_+ and H_-, containing positive and negative branch points, respectively, are colored by a_+ and a_-.

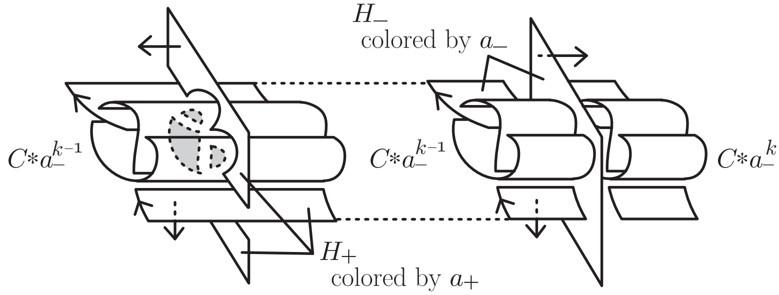

FIGURE 6.

2. The part $T \times L$ is separated by H_- transversely near $T \times \{P_k^-\}$ for $1 \leqslant k \leqslant r$. Since $T \times \{P_0\}$ is colored by C, the tangle after passing through $T \times \{P_k^-\}$ is colored by $C * a_-^k$. See the right-hand side of Figure 6.

3. The small sheets near $T \times \{P_k^+\}$ surrounded by $T \times L$ are uniquely colored by the shadow coloring $C * a_-^{k-1}$ of T; see the left-hand side of Figure 6.

Hence C extends to $\mathrm{D}^r T$ if and only if $C * a_-^r = C$. Since such a Q-coloring of $\mathrm{D}^r T$ is uniquely determined by C as above, the map ι^* is injective. □

We identify $\mathrm{Col}_Q(\mathrm{D}^r T)$ with $\mathrm{Col}_Q^r(T)$ under the map ι^*. The cocycle invariant of the r-twist-spin $\tau^r K$ is translated in terms of a tangle diagram T of K as follows.

PROPOSITION 5.2. *The cocycle invariant of $\tau^r K$ associated with a 3-cocycle $f \in Z^3(Q; G)$ is given by*

$$\Phi_f(\tau^r K) = \sum_{C \in \mathrm{Col}_Q^r(T)} \left[\left\{ \prod_{x \in \mathrm{X}_2(T)} W_f^*(x; C) \right\}^r \times \left\{ \prod_{k=0}^{r-1} \prod_{x \in \mathrm{X}_2(T)} W_f^{\#}(x; C * a_-^k) \right\}^{-1} \right],$$

where $W_f^*(x; C)$ is the weight defined in Section 4, a_- is the color of the terminal arc of T, and $W_f^{\#}(x; C) = f(a, b, a_-)^{\varepsilon(x)}$ for $C(x) = (a, b)$.

Proof. For each $C \in \mathrm{Col}_Q^r(T)$, the triple points $t_k^{\pm}(x) \in \mathrm{X}_3(\mathrm{D}^r T)$ corresponding to $x \in \mathrm{X}_2(T)$ satisfy $C(t_k^+(x)) = (s, a, b) * a_-^{k-1}$ and $C(t_k^-(x)) = (a, b, a_-) * a_-^{k-1}$ for $1 \leqslant k \leqslant r$, where s is the color of the region being distinguished, as shown in Figure 4. Hence, we have

$$W_f(t_k^+(x); C) = W_f^*(x; C * a_-^{k-1}) \qquad \text{and} \qquad W_f(t_k^-(x); C) = W_f^{\#}(x; C * a_-^{k-1})^{-1}.$$

The proof is complete, by Lemma 4.2. □

Let R_p be the dihedral quandle of order p, where p is an odd prime (Example 2.1). We study R_p-colorings of T and $\mathrm{D}^r T$, and we denote the sets of these colorings by $\mathrm{Col}_p(T)$ and $\mathrm{Col}_p(\mathrm{D}^r T)$, respectively. Also we refer to R_p-colorings as *p-colorings*, in the style of Fox [**5**, **6**]. A p-coloring C is *trivial* if the image of C consists of a single element.

LEMMA 5.3. (i) *If r is odd, then $\mathrm{Col}_p(\mathrm{D}^r T)$ consists of trivial p-colorings.*
(ii) *If r is even, then we have $\mathrm{Col}_p(\mathrm{D}^r T) = \mathrm{Col}_p(T)$ under the map ι^*.*

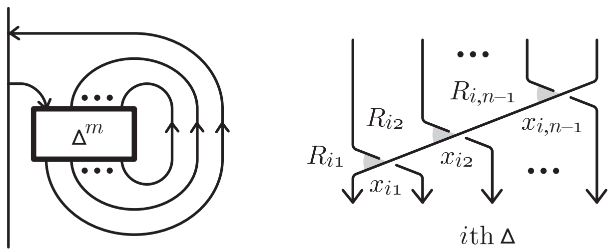

FIGURE 7.

Proof. The lemma follows immediately from Lemma 5.1 and the property of R_p that $(a*b)*b=a$ for any a and $b\in R_p$. □

Let $\theta_p\in Z^3(R_p;\mathbb{Z}_p)$ be the 3-cocycle given in Example 2.1. By taking the coefficient group $\mathbb{Z}_p=\langle t\mid t^p=1\rangle$, we identify the group ring $\mathbb{Z}[\mathbb{Z}_p]$ with the Laurent polynomial ring $\mathbb{Z}[t^{\pm1}]/(t^p-1)$. Let $\rho^r:\mathbb{Z}[t^{\pm1}]/(t^p-1)\longrightarrow\mathbb{Z}[t^{\pm1}]/(t^p-1)$ be the map induced by $t\mapsto t^r$. We abbreviate 'θ_p' in subscripts to 'p'.

THEOREM 5.4. (i) *If r is odd, then we have $\Phi_p(\tau^rK)=p$.*
(ii) *If r is even, then we have $\Phi_p(\tau^rK)=\rho^r\Phi_p^*(K)$.*

Proof. (i) Since τ^rK admits only trivial p-colorings, we have $\Phi_p(\tau^rK)=p$, by definition.

(ii) Since θ_p satisfies $\theta_p(a*c,b*c,c)=\theta_p(a,b,c)^{-1}$ for any $a,b,c\in R_p$, we have $W_p^\#(x;C*a_-)=W_p^\#(x;C)^{-1}$. Hence we have

$$\Phi_p(\tau^rK)=\sum_C\Big\{\prod_x W_p^*(x;C)\Big\}^r=\rho^r\Phi_p^*(K),$$

by Proposition 5.2 and Lemma 5.3(ii). □

6. *Concrete calculations*

Let $T(m,n)$ denote the torus-knot of type (m,n), where $m,n>1$ are relatively prime integers. Take a tangle diagram T of $T(m,n)$ as the closure of the word Δ^m in the n-braid group B_n, where $\Delta=\sigma_{n-1}\sigma_{n-2}\dots\sigma_1$ with $\sigma_1,\dots,\sigma_{n-1}$ the standard generators of B_n; see Figure 7. In the ith Δ from the top $(1\leqslant i\leqslant m)$, let $x_{i1},\dots,x_{i,n-1}$ be the crossings, and let $R_{i1},\dots,R_{i,n-1}$ be the specified regions as shown on the right of the figure.

Let $\alpha_1,\dots,\alpha_n$ be the top arcs of Δ^m. Given a vector $\boldsymbol{a}=(a_1,\dots,a_n)\in(R_p)^n$, a p-coloring C of Δ^m is uniquely determined such that $C(\alpha_j)=a_j$ for $1\leqslant j\leqslant n$. Since $a*b\equiv 2b-a\pmod p$, the colors of the bottom arcs of Δ^m are given by the vector $\boldsymbol{a}P^m$, where P is the following $m\times m$ matrix.

$$P=\left(\begin{array}{c|cccc} 0 & -1 & & & \\ \vdots & & \ddots & & \text{\Large 0} \\ 0 & \text{\Large 0} & & & -1 \\ \hline 1 & 2 & \dots & & 2 \end{array}\right).$$

Any p-coloring of any tangle diagram satisfies the requirement that the initial and terminal arcs have the same color (see [**12**]), since $a_+ * a_- = a * a_+$ in the proof of Proposition 4.1 implies that $a_+ = a_-$ in R_p. Hence C extends to the above diagram T if and only if $\boldsymbol{a}P^m = \boldsymbol{a}$. By solving this equation, we obtain the following two lemmas. The proofs are straightforward, and we omit them here.

We remark that since m and n are relatively prime and $T(m,n)$ and $T(n,m)$ are the same knot, we may assume without loss of generality that m is odd.

Lemma 6.1. *Assume that m is odd.*

(i) *If m is not divisible by p, or n is odd, then $\mathrm{Col}_p(T)$ consists of trivial p-colorings.*

(ii) *If m is divisible by p and n is even, then we have $\mathrm{Col}_p(T) = \{C_{ab} \mid a, b \in R_p\}$, where C_{ab} is associated with the vector $\boldsymbol{a} = (a, b, \ldots, a, b)$.*

Lemma 6.2. *Let $C_{ab} \in \mathrm{Col}_p(T)$ be the p-coloring in Lemma 6.1, and put $\delta = a - b$.*

(i) *$C_{ab}(x_{ij}) = (a - (i-1)\delta, a - i\delta)$ and $C_{ab}(R_{ij}) = a - (j-1)\delta$ for odd j.*

(ii) *$C_{ab}(x_{ij}) = (a - i\delta, a - i\delta)$ and $C_{ab}(R_{ij}) = a - (2i-j)\delta$ for even j.*

In the following theorem, the conditions under which $T(m,n)$ and $\tau^r T(m,n)$ admit a non-trivial p-coloring are given in Lemmas 5.3 and 6.1.

Theorem 6.3. (i) *If $T(m,n)$ admits a non-trivial p-coloring, then we have*

$$\Phi_p^*(T(m,n)) = p\left(\sum_{i=0}^{p-1} t^{-(mn/2p)i^2}\right) \in \frac{\mathbb{Z}[t^{\pm 1}]}{(t^p - 1)};$$

otherwise, we have $\Phi_p^*\big(T(m,n)\big) = p$.

(ii) *If $\tau^r T(m,n)$ admits a non-trivial p-coloring, then we have*

$$\Phi_p(\tau^r T(m,n)) = p\left(\sum_{i=0}^{p-1} t^{-(mnr/2p)i^2}\right) \in \frac{\mathbb{Z}[t^{\pm 1}]}{(t^p - 1)};$$

otherwise, we have $\Phi_p(\tau^r T(m,n)) = p$.

Proof. (i) It is sufficient to prove the case where $m = p$ and n is even; in fact, we have

$$\Phi_p^*(T(m,n)) = \rho^{m/p}\Phi_p^*(T(p,n))$$

if m is divisible by p. We first write the coefficient group $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ additively, and use equality instead of the congruence.

We prove that

$$\sum_{i,j} W_p^*(x_{ij}; C_{ab}) = -(n/2)\delta^2 \in \mathbb{Z}_p$$

for a fixed $C_{ab} \in \mathrm{Col}_p(T)$. If $\delta = a - b = 0$, the equation clearly holds. Assume that $\delta \neq 0$. If j is even, then $W_p^*(x_{ij}; C_{ab}) = \theta_p(a - i\delta, a - i\delta, a - (2i-j)\delta) = 0$. Hence we also assume that j is odd.

Then we have

$$\begin{aligned}\sum_{i=1}^{p} W_p^*(x_{ij};C_{ab}) &= \sum_{i=1}^{p} \theta_p\big(a-(i-1)\delta, a-i\delta, a-(j-1)\delta\big)\\ &= \delta\sum_{i=1}^{p}\frac{i(a-(i+1)\delta)^p + i(a-(i-1)\delta)^p - 2i(a-i\delta)^p}{p}\\ &\quad -j\delta\sum_{i=1}^{p}\frac{(a-(i+1)\delta)^p + (a-(i-1)\delta)^p - 2(a-i\delta)^p}{p}.\end{aligned}$$

On the other hand, it is not difficult to see that if $x = y \pmod p$, then $x^p = y^p \pmod{p^2}$. Hence, by taking the above numerators modulo p^2, we have

$$\begin{aligned}&\sum_{i=1}^{p}\Big\{i(a-(i+1)\delta)^p + i(a-(i-1)\delta)^p - 2i(a-i\delta)^p\Big\}\\ &\quad= \sum_{i=2}^{p+1}(i-1)(a-i\delta)^p + \sum_{i=0}^{p-1}(i+1)(a-i\delta)^p - 2\sum_{i=1}^{p} i(a-i\delta)^p\\ &\quad= p(b^p - a^p),\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^{p}\Big\{(a-(i+1)\delta)^p + (a-(i-1)\delta)^p - 2(a-i\delta)^p\Big\}\\ &\quad= \sum_{i=1}^{p}(a-(i+1)\delta)^p + \sum_{i=1}^{p}(a-(i-1)\delta)^p - 2\sum_{i=1}^{p}(a-i\delta)^p\\ &\quad= 0.\end{aligned}$$

Here we use $\{a-(i+k)\delta \mid i = 1, 2, \dots, p\} = \{i \mid i = 1, 2, \dots, p\} \pmod p$ for $k = 1, 0, -1$. Then we have

$$\sum_{i=1}^{p} W_p^*(x_{ij};C_{ab}) = \delta\cdot(b^p - a^p) - j\delta\cdot 0 = (b-a)\delta = -\delta^2,$$

and

$$\sum_{i,j} W_p^*(x_{ij};C_{ab}) = -(n/2)\delta^2$$

by taking the sum for odd $1 \leqslant j \leqslant n-1$. By rewriting $\mathbb{Z}_p = \langle t \mid t^p = 1\rangle$ multiplicatively, we have

$$\Phi_p^*(T(p,n)) = \sum_{a,b\in\mathbb{Z}_p} t^{-(n/2)(a-b)^2} = p\Bigg(\sum_{i=0}^{p-1} t^{-(n/2)i^2}\Bigg).$$

(ii) This follows immediately from (i) and Theorem 5.4. □

For a knotted surface F in $\mathbb{R}^4$, we denote by $-F$ the same surface with the orientation reversed. We say that F is *non-invertible* if F is not ambient isotopic to $-F$ in $\mathbb{R}^4$. Gordon [**7**] proved that any twist-spun torus knot $\tau^r T(m,n)$ is non-invertible, where m and n are relatively prime with $m, n, r \geqslant 2$. His argument is based on the fact that $\tau^r K$ is fibered (see [**16**]). However, the corresponding fact is not known for knotted surfaces of higher genus. On the other hand, the cocycle invariants can be used to study the non-invertibility of knotted surfaces, regardless of genus. It is known that for any 3-cocycle $f \in Z^3(Q;G)$, the cocycle invariant $\Phi_f(-F) \in \mathbb{Z}[G]$ is obtained from $\Phi_f(F)$ by replacing g with g^{-1} for any $g \in G$ (see [**2**]).

Proof of Theorem 1.1. We may prove that if m, n, r and p satisfy the conditions (i)–(iv), then $\Phi_p(F_g^r(m,n)) \neq \rho^{-1}\Phi_p(F_g^r(m,n))$. Since a diagram of $F_g^r(m,n)$ is obtained from that of $F_0^r(m,n) = \tau^r T(m,n)$ by attaching g trivial 1-handles without introducing new singularities, we have $\Phi_p(F_g^r(m,n)) = \Phi_p(\tau^r T(m,n))$, by definition. Hence, it is sufficient to prove that $\Phi_p(\tau^r T(m,n)) \neq \rho^{-1}\Phi_p(\tau^r T(m,n))$. This can easily be seen from Theorem 6.3(ii) and the fact that

$$p\left(\sum_{i=0}^{p-1} t^{-Ni^2}\right) \neq p\left(\sum_{i=0}^{p-1} t^{Ni^2}\right) \qquad \text{in } \frac{\mathbb{Z}[t^{\pm 1}]}{(t^p - 1)}$$

if and only if $p \equiv 3 \pmod 4$ and N is not divisible by p.

Among the non-invertible knotted surfaces $F_g^r(m,n)$ satisfying the conditions (i)–(iv), we consider the ones given by $m = p$ and $n = r = 2$, for example. Then we see that the family $\{F_g^2(p,2) : p = 3, 7, 11, 19, \dots\}$ is infinite, for if $p \neq p'$, then

$$\Phi_p(F_g^2(p,2)) = p\left(\sum_{i=0}^{p-1} t^{-2i^2}\right) \qquad \text{and} \qquad \Phi_p\big(F_g^2(p',2)\big) = p$$

by Theorem 6.3(ii), and hence $F_g^2(p,2)$ is not ambient isotopic to $F_g^2(p',2)$. □

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