Strong Normalizability of Calculus of Explicit Substitutions with Composition

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Abstract. We develop a novel method for proving the strong normalizability of simply typed λx with a composition rule. Bloo and Geuvers [2] proved the strong normalizability of λx with a composition rule, but our composition rule is a new one: $t\langle y{:=}r\rangle\langle x{:=}s\rangle \mapsto t\langle x{:=}s\rangle\langle y{:=}r\langle x{:=}s\rangle\rangle$ if $x \in \mathrm{FV}(r)$. In fact, we prove the stronger result: Suppose we have a reduction sequence that consists of rules of λx, the above composition rule, and the permutation rule $t\langle y{:=}r\rangle\langle x{:=}s\rangle \mapsto t\langle x{:=}s\rangle\langle y{:=}r\rangle$ if $x \notin \mathrm{FV}(r)$, where the successive application of permutation rules is finite. Then, the reduction sequence is finite. This implies that the meta-substitution is admissible in our calculus.

1 Introduction

$\lambda\sigma$ [1] is designed to formalize the implementation of $\lambda\beta$. So it was natural to expect that $\lambda\sigma$ inherits all the good properties of $\lambda\beta$. But Melliès [10] has shown that there is a typed $\lambda\sigma$ term that is not strongly normalizing. In his counterexample, the substitution composition rule plays the essential role. So, since then, a quest for an appropriate composition rule has begun. λ_d and λ_{dn} [8, 7] are variants of $\lambda\sigma$ whose substitution composition rules are more restrictive than that of $\lambda\sigma$. λ_{ws} [4, 6, 5] is a calculus of explicit substitutions whose terms are decorated with 'labels' that correspond to weakenings and it has a full composition rule that is controlled by the information attached to the term. In the case of λx [3], which is the simplest calculus of explicit substitution, terms do not have such extra information. So, it is difficult to control the application of composition rule. Up to now, the best result is the one proved by Bloo and Geuvers [2]. They proved the strong normalizability of λx with the following composition rule:

$$t\langle y{:=}r\rangle\langle x{:=}s\rangle \mapsto t\langle y{:=}r\langle x{:=}s\rangle\rangle \quad \text{if } y \in \mathrm{FV}(\mathsf{x}(t)) \text{ and } x \notin \mathrm{FV}(\mathsf{x}(t)){-}\{y\}$$

where $\mathsf{x}(t)$ stands for the substitution-normal-form of t. This form of composition has been considered as 'on the edge', but our composition rule is a full composition rule controlled by a simple condition, that is,

$$t\langle y{:=}r\rangle\langle x{:=}s\rangle \mapsto t\langle x{:=}s\rangle\langle y{:=}r\langle x{:=}s\rangle\rangle \quad \text{if } x \in \mathrm{FV}(r).$$

In this paper, we prove the strong normalizability of λx with this composition rule. In fact, we prove the stronger result: Suppose we have a reduction sequence that consists of rules of λx, the above composition rule, and the permutation rule

$$t\langle y{:=}r\rangle\langle x{:=}s\rangle \mapsto t\langle x{:=}s\rangle\langle y{:=}r\rangle \quad \text{if } x \notin \mathrm{FV}(r),$$

where the successive application of permutation rules is finite. Then, the reduction sequence is finite. This make the meta-substitution admissible in our calculus.

Usually, when we try to prove the strong normalizability of λx or its variant, it is necessary to show the lemma "If $t\langle x{:=}s\rangle\langle y{:=}r\rangle$ is SN, then $((\lambda x.\, t)s)\langle y{:=}r\rangle$ is SN." Then, we are tempted to prove this lemma by converting a reduction sequence starting from $((\lambda x.\, t)s)\langle y{:=}r\rangle$ to a reduction sequence starting from $t\langle x{:=}s\rangle\langle y{:=}r\rangle$. But if we try to convert a reduction sequence $((\lambda x.\, t)s)\langle y{:=}r\rangle \xrightarrow{*} (\lambda x.\, t\langle y{:=}r\rangle)(s\langle y{:=}r\rangle) \to t\langle y{:=}r\rangle\langle x{:=}s\langle y{:=}r\rangle\rangle$, we get stuck. This is why the proof of strong normalizability of λx is more difficult than that of $\lambda\beta$. Various techniques are introduced to overcome this difficulty, but we present a novel technique. We extend the notion of term by adding a new kind of substitution. Our substitution has a distinctive feature that it can be composed with other substitutions unconditionally. This leads to non-termination, but we can deal with it by modifying the notion of SN. (We call the modified notion semi SN.) In this setting, we can naturally prove the lemma (Lemma 3) that corresponds to the lemma mentioned above, but the drawback is that the proof of void lemma (weakening lemma) becomes complicated.

Unlike the ordinary style, we prove PSN property using intersection type system. Furthermore, we can characterize semi SN using intersection type system, though we omit the proof here.

This paper is organized as follows. In section 2, we describe our calculus. Section 3 is the most complicated part of this paper, which is devoted to prove void lemma. In section 4, we prove the reducibility theorem. In section 5, we briefly note the relation with intersection type system.

2 Explicit Substitution

We define the calculus of explicit substitution λx [3]. We assume a countably infinite set V of variables and define the syntax of *(untyped)* λx-*term* by the following grammar

$$t, s ::= x \mid \lambda x.\, t \mid ts \mid t\langle x{:=}s\rangle$$

where x ranges over V. We omit the formal definition of free and bound variables here. We just remark that in $\lambda x.\, t$ and $t\langle x{:=}s\rangle$, the variable x in t is bound by the binders $\lambda x.$ and $\langle x{:=}s\rangle$. We call x in $\lambda x.$ and $\langle x{:=}s\rangle$ a *binding variable*, $\langle x{:=}s\rangle$ a *substitution*, and s (resp. t) in $t\langle x{:=}s\rangle$ the *body* (resp. *target*) of the substitution. We identify α-convertible terms and adopt the conventions that bound variables are different from free variables and different binders have different binding variables. We write t^y_x for a term obtained by renaming free occurrence of x in t to y.

A *position* is a string over L, F, A, T, B. (An empty string is denoted by Λ.) For a term t and a position π, we define a term t/π as follows.

1. $t/\Lambda \triangleq t$
2. $\lambda x.\, t/\mathtt{L}\pi \triangleq t/\pi$
3. $ts/\mathtt{F}\pi \triangleq t/\pi \quad ts/\mathtt{A}\pi \triangleq s/\pi$
4. $t\langle x{:=}s\rangle/\mathtt{T}\pi \triangleq t/\pi \quad t\langle x{:=}s\rangle/\mathtt{B}\pi \triangleq s/\pi$

Note that t/π may be undefined, but we will use the notation t/π only when it is defined and say π *is a position in* t. Then, we say t/π *is in (position* π *of)* t, or t/π *is a subterm of* t. If $t/\pi \equiv r\langle x{:=}s\rangle$ for some r, we say $\langle x{:=}s\rangle$ *is in (position* π *of)* t.

Let π and σ be positions. We write $\pi \sqsubseteq \sigma$ if $\sigma \equiv \pi\pi'$ for some π', and write $\pi \sqsubset \sigma$ if $\pi' \not\equiv \Lambda$. If π is a position in t, then $t_\pi[s]$ stands for the term which is obtained from t by textually replacing its subterm t/π at π with s.

The conversion rules $\mapsto$ of λx are defined as follows.

$$\begin{array}{rl}
(\boldsymbol{\lambda}) & (\lambda x.\, t)s \mapsto t\langle x{:=}s\rangle \\
(\mathsf{var}) & x\langle x{:=}s\rangle \mapsto s \\
(\mathsf{gc}) & t\langle x{:=}s\rangle \mapsto t \quad \text{if } x \notin \mathrm{FV}(t) \\
(\mathsf{abs}) & (\lambda y.\, t)\langle x{:=}s\rangle \mapsto \lambda y.\, t\langle x{:=}s\rangle \quad \text{if } y \notin \{x\} \cup \mathrm{FV}(s) \\
(\mathsf{app}) & (tr)\langle x{:=}s\rangle \mapsto (t\langle x{:=}s\rangle)(r\langle x{:=}s\rangle)
\end{array}$$

We refer to a set of these rules as λx and, moreover, we have the following composition rule.

$$(\mathsf{comp}) \quad t\langle y{:=}r\rangle\langle x{:=}s\rangle \mapsto t\langle x{:=}s\rangle\langle y{:=}r\langle x{:=}s\rangle\rangle \quad \text{if } x \in \mathrm{FV}(r)$$

We write $(R|\pi) : t \to t'$ if $t/\pi \mapsto s$ by rule (R) and $t' \equiv t_\pi[s]$, and say $(R|\pi)$ *is applied to* t, (R) *is applied at* π *(in* t*)*, or π *is a reduction position of* (R). We just write $t \to t'$ or $(R) : t \to t'$ when (R) and/or π need not be specified. To specify a reduction sequence, we write $\gamma_1, \ldots, \gamma_n, \ldots : t_1 \to \cdots t_n \to \cdots$ when γ_i is applied to t_i.

We will extend the notion of term in the following, so we will refer to the term defined above as an *original* λx*-term*. Now, we prepare another countably infinite set D of variables that is disjoint from V and introduce a new kind of substitution $\{x{:=}s\}$, which we call *definitional substitution* (or *d-substitution* for short). Then, we say t *is a term* or t *satisfies the term formation condition*, if $t : \mathsf{term}$ is derivable using the following rules.

$$\frac{x \in \mathsf{V} \cup \mathsf{D}}{x : \mathsf{term}} \quad \frac{t : \mathsf{term} \quad s : \mathsf{term}}{ts : \mathsf{term}} \quad \frac{t : \mathsf{term} \quad x \in \mathsf{V}}{\lambda x.\, t : \mathsf{term}} \quad \frac{t : \mathsf{term} \quad s : \mathsf{term} \quad x \in \mathsf{V}}{t\langle x{:=}s\rangle : \mathsf{term}}$$

$$\frac{t : \mathsf{term} \quad s : \mathsf{term} \quad x \in \mathsf{D} \quad \mathrm{FV}(s) \subseteq \mathsf{D}}{t\langle x{:=}s\rangle : \mathsf{term}} \quad \frac{t : \mathsf{term} \quad s : \mathsf{term} \quad x \in \mathsf{D} \quad \mathrm{FV}(s) \subseteq \mathsf{D}}{t\{x{:=}s\} : \mathsf{term}}$$

We also assume that when a bound variable is renamed, a variable in V (resp. D) should be renamed to a variable in V (resp. D). A term is *ds-free* if it does not have a subterm of the form $t\{x{:=}s\}$. As for the notion of position, d-substitution is treated in the same way as substitution, that is, $t\{x{:=}s\}/\mathtt{T}\pi \triangleq t/\pi$, $t\{x{:=}s\}/\mathtt{B}\pi \triangleq s/\pi$.

The conversion rules for d-substitution are given by replacing $\langle x{:=}s\rangle$ in (var), (gc), (abs), (app) by $\{x{:=}s\}$ and these rules are referred to by the same names. Furthermore, the following rules are introduced.

$$
\begin{array}{rl}
(\mathsf{d{\circ}x}) & t\langle y{:=}r\rangle\{x{:=}s\} \mapsto t\{x{:=}s\}\langle y{:=}r\{x{:=}s\}\rangle \\
(\mathsf{d{\circ}d}) & t\{y{:=}r\}\{x{:=}s\} \mapsto t\{x{:=}s\}\{y{:=}r\{x{:=}s\}\} \\
(\mathsf{d2x}) & t\{x{:=}s\} \mapsto t\langle x{:=}s\rangle \\
(\mathsf{perm}) & t\langle y{:=}r\rangle\langle x{:=}s\rangle \mapsto t\langle x{:=}s\rangle\langle y{:=}r\rangle \quad \text{if } x \notin \mathrm{FV}(r)
\end{array}
$$

Note that the term formation condition is not violated by the reduction. Though non-termination is easily caused by these rules, we can deal with it by relaxing the notion of SN. We will define it after we introduce another form of term in the next paragraph.

A *definition* is an expression of the form $x := s$ where $x \in \mathsf{D}$ and s is a term such that $\mathrm{FV}(s) \subseteq \mathsf{D}$, and a *def-term* is an expression of the form $\Delta \shortmid t$ where Δ is a sequence of definitions and t is a term. When Δ is an empty sequence, we identify $\Delta \shortmid t$ and t. Let $\Delta \equiv d_1, \ldots, d_n$ be a definition sequence. We write Δ_k for d_k and $\Delta|_k$ for $d_1, \ldots, d_k$. For def-terms, we have the following reduction rules.

$$
\begin{array}{rl}
(R) & \Delta \shortmid t \to \Delta \shortmid t' \quad \text{if } (R) : t \to t' \text{ where } (R) \text{ is one of the above rules} \\
(\mathsf{def}) & \Delta, x := s \shortmid t \to \Delta \shortmid t\{x{:=}s\}
\end{array}
$$

A definition is essentially a d-substitution, but this syntactic distinction is necessary when we define reducibility. Before that, we can think of $x_1{:=}s_1, \ldots, x_n{:=}s_n \shortmid t$ as $t\{x_n{:=}s_n\}\cdots\{x_1{:=}s_1\}$.

A reduction sequence is *non-permutative* if the length of every successive application of (perm)s is finite. A reduction sequence starting from $\Delta \shortmid t$ is *non-trivial* if it is non-permutative and there exists its initial part $\Delta \shortmid t \xrightarrow{*} t'$ such that t' is ds-free. A def-term $\Delta \shortmid t$ is *semi SN* if any non-trivial reduction sequence starting from $\Delta \shortmid t$ terminates, and a sequence of definitions Δ is *semi SN* if $\Delta|_{k-1} \shortmid s_k$ is semi SN for each $\Delta_k \equiv x{:=}s_k$. (Therefore, a ds-free term t is semi SN iff any non-permutative reduction sequence starting from t terminates.) For a ds-free term t, we write $\nu(t)$ for the length of the longest non-permutative reduction sequence starting from t ignoring the number of (perm)s. Note that if $(\mathsf{perm}) : t \to t'$ and $\nu(t)$ is defined then $\nu(t) = \nu(t')$.

A position π is *n-applicative*, if $\pi \equiv \mathtt{T}^{k_1}\mathtt{F}\cdots\mathtt{T}^{k_{n-1}}\mathtt{F}\mathtt{T}^{k_n}$ for some $k_1, \ldots, k_n \geq 0$, and is *$n{\geq}$-applicative* if m-applicative for some $m \leq n$. This notion is used to specify a position in a term of the form $(\cdots(t_0\overline{\theta_0}\ t_1)\overline{\theta_1}\cdots t_n)\overline{\theta_n}$ where $\overline{\theta_i} \equiv \{x_{i1}{:=}s_{i1}\}\cdots\{x_{im_i}{:=}s_{im_i}\}$ and $\{x_{ij}{:=}s_{ij}\}$ stands for either a substitution or a d-substitution. In this term, $\{x_{ij}{:=}s_{ij}\}$ $(0 \leq i \leq n)$ and $(t_{i-1}\overline{\theta_{i-1}}\ t_i)$ $(1 \leq i \leq n)$ are in $n{-}i{+}1$-applicative position and t_0 is in $n{+}1$-applicative position.

Lemma 1. *If there is an infinite non-trivial reduction sequence starting from $\Delta \shortmid t_0 t_1 \cdots t_n$ that has no ($\boldsymbol{\lambda}$) at n-applicative position, at least one of $\Delta \shortmid t_0$, $\Delta \shortmid t_1$, ..., $\Delta \shortmid t_n$ is not semi SN.*

Proof. Let $\overline{\gamma}$ be the reduction sequence. Since all terms in $\overline{\gamma}$ are of the form $(\cdots(t'_0\overline{\theta_0}\ t'_1)\overline{\theta_1}\cdots t'_n)\overline{\theta_n}$ and $\overline{\gamma}$ is infinite, we have three cases. (i) There is an i-applicative position π ($1 \leq i \leq n$) such that there are infinitely many reduction positions of the form $\pi\mathtt{A}\pi'$ in $\overline{\gamma}$. (ii) There is an n-applicative position π such that there are infinitely many reduction positions of the form $\pi\mathtt{F}\pi'$ in $\overline{\gamma}$. (iii) There is an i-applicative position π ($1 \leq i \leq n$) such that there are infinitely many reduction positions of the form $\pi\mathtt{B}\pi'$ in $\overline{\gamma}$.

In the case (i), we construct a reduction sequence starting from $\Delta \shortmid t_{n-i+1}$ by modifying $\overline{\gamma}$, that is, we change or skip each reduction γ in $\overline{\gamma}$ as follows: (In the following, $\tilde{\pi}$ denotes a position obtained by removing all $\mathtt{F}$ from π.)

1. $\gamma \equiv (R|\pi\mathtt{A}\pi')$ where π is an i-applicative position: changed to $(R|\tilde{\pi}\pi')$.
2. $\gamma \equiv (R|\pi\mathtt{B}\pi')$ where π is an $i{\geq}$-applicative position: changed to $(R|\tilde{\pi}\mathtt{B}\pi')$.
3. $\gamma \equiv (R|\pi)$ where $(R) \equiv (\mathsf{comp})$, $(\mathsf{d{\circ}x})$, $(\mathsf{d{\circ}d})$, or (gc) and π is an $i{\geq}$-applicative position: changed to $(R|\tilde{\pi})$.
4. otherwise: skipped.

In the case (ii) (resp. (iii)), we can similarly construct an infinite reduction sequence starting from $\Delta \shortmid t_0$ (resp. $\Delta \shortmid t_{n-i+1}$). □

Lemma 2. *If $\Delta \shortmid t$ is not semi SN, then $\Delta \shortmid ts$, $\Delta \shortmid st$, $\Delta \shortmid t\langle x{:=}s\rangle$, and $\Delta \shortmid s\langle x{:=}t\rangle$ are not semi SN.*

Proof. We prove the case $\Delta \shortmid t\langle x{:=}s\rangle$. Other cases are similar. Suppose we have an infinite non-trivial reduction sequence $\overline{\gamma}$ starting from $\Delta \shortmid t$. By changing (def) to $(\mathsf{def}), (\mathsf{d{\circ}x}|\Lambda)$ and changing $(R|\sigma)$ to $(R|\mathtt{T}\sigma)$, we have an infinite non-trivial reduction sequence starting from $\Delta \shortmid t\langle x{:=}s\rangle$. □

The following lemma is the one that we mentioned in the introduction. By examining how this kind of lemma is used in proving reducibility theorem, we find that we need to prove this lemma only in the case $\langle y{:=}r\rangle$ is a d-substitution. We explain the idea of the proof using a simple example. Suppose we have a reduction sequence $y{:=}r \shortmid (\lambda x.\, t)s \rightarrow ((\lambda x.\, t)s)\{y{:=}r\} \rightarrow ((\lambda x.\, t)s)\langle y{:=}r\rangle \xrightarrow{*} (\lambda x.\, t\langle y{:=}r\rangle)(s\langle y{:=}r\rangle) \rightarrow t\langle y{:=}r\rangle\langle x{:=}s\langle y{:=}r\rangle\rangle$. Then, we have the following sequence $y{:=}r \shortmid t\langle x{:=}s\rangle \rightarrow t\langle x{:=}s\rangle\{y{:=}r\} \rightarrow t\{y{:=}r\}\langle x{:=}s\{y{:=}r\}\rangle \xrightarrow{*} t\langle y{:=}r\rangle \langle x{:=}s\langle y{:=}r\rangle\rangle$ by delaying $(\mathsf{d2x})$.

Lemma 3. *If $\Delta \shortmid (t\langle x{:=}s\rangle)s_1 \cdots s_n$ is semi SN, then $\Delta \shortmid (\lambda x.\, t)ss_1 \cdots s_n$ is semi SN.*

Proof. Suppose $\Delta \shortmid (\lambda x.\, t)ss_1 \cdots s_n$ is not semi SN. By Lemma 1, we have two cases.

1. one of $\Delta \shortmid \lambda x.\, t$, $\Delta \shortmid s$, $\Delta \shortmid s_1$, ..., $\Delta \shortmid s_n$ is not semi SN: If $\Delta \shortmid \lambda x.\, t$ is not semi SN, then $\Delta \shortmid t$ is not semi SN. So, by Lemma 2, $\Delta \shortmid (t\langle x{:=}s\rangle)s_1 \cdots s_n$ is not semi SN.

2. there is an infinite non-trivial reduction sequence $\overline{\gamma}, (\boldsymbol{\lambda}|\sigma), \ldots$ where $(\boldsymbol{\lambda}|\sigma)$ is the first $(\boldsymbol{\lambda})$ applied at $n+1$-applicative position: Let $\overline{\gamma}, (\boldsymbol{\lambda}|\sigma) : \Delta \shortmid (\lambda x.\, t)s\, s_1 \cdots s_n \xrightarrow{*} \Delta' \shortmid r_1 \to \Delta' \shortmid r_2$. (Note that r_1 is of the form $(\cdots(((\lambda x.\, t')s')\overline{\theta_0}\, s'_1)\overline{\theta_1} \cdots s'_n)\overline{\theta_n}$ and r_2 is $(\cdots((t'\langle x{:=}s'\rangle)\overline{\theta_0}\ s'_1)\overline{\theta_1} \cdots s'_n)\overline{\theta_n}$.) We construct a reduction sequence $\Delta \shortmid (t\langle x{:=}s\rangle)s_1 \cdots s_n \xrightarrow{*} \Delta' \shortmid r_2$ by modifying $\overline{\gamma}$ as follows. First, we remove all $n+1\geq$-applicative (d2x)s in $\overline{\gamma}$ and restore the substitutions in the reduction sequence to the d-substitutions accordingly. We trace the d-substitutions and apply (d2x) when they leave $n+1\geq$-applicative position (that is, when they come to the $n+2$-applicative position, the $\pi\mathtt{A}$ position, or the $\pi\mathtt{B}$ position where π is the $n+1\geq$-applicative position). We also change $n+1\geq$-applicative (comp) to (d∘d) and $n+1\geq$-applicative (perm) to (d∘d) and (gc). (Note that substitution $\langle y{:=}r\rangle$ in an $n+1\geq$-applicative position of r_1 comes from Δ.) Then, we have a reduction sequence $\Delta \shortmid (\lambda x.\, t)ss_1 \cdots s_n \xrightarrow{*} \Delta' \shortmid r'_1$ such that some of $n+1\geq$-applicative $\langle y{:=}r\rangle$ in r_1 are $\{y{:=}r\}$ in r'_1. By applying (d2x) to those $\{y{:=}r\}$, we have a new reduction sequence $\Delta \shortmid (\lambda x.\, t)ss_1 \cdots s_n \xrightarrow{*} \Delta' \shortmid r_1$.
   Next, we change or skip each reduction γ in the new sequence as follows, so that we have a reduction sequence $\Delta \shortmid (t\langle x{:=}s\rangle)s_1 \cdots s_n \xrightarrow{*} \Delta' \shortmid r_2$.
   (a) $\gamma \equiv (R|\pi\mathtt{F}\pi')$ where π is an $n+1$-applicative position: (i) $\pi' \equiv \mathtt{T}^k$: skipped if (R) = (abs), and changed to $(R|\pi\mathtt{T}\pi')$ otherwise. (ii) $\pi' \equiv \mathtt{T}^k\mathtt{L}\pi''$: changed to $(R|\pi\mathtt{T}\mathtt{T}^k\pi'')$.
   (b) $\gamma \equiv (R|\pi\mathtt{A}\pi')$ where π is an $n+1$-applicative position: changed to $(R|\pi\mathtt{B}\pi')$.
   (c) otherwise: unchanged.
   Using this modified reduction sequence, we have an infinite non-trivial reduction sequence starting from $\Delta \shortmid (t\langle x{:=}s\rangle)s_1 \cdots s_n$. □

3 Void Lemma

To prove Void Lemma (Lemma 14), we introduce the notion of weight, but we need to define several auxiliary notions to define it.

Let t be a term. We write $dsfr(t)$ for a ds-free term obtained by replacing every subterm $s\{x{:=}r\}$ of t by $s\langle x{:=}r\rangle$. Let $\Delta \shortmid t$ be a def-term where $\Delta \equiv x_1{:=}s_1, \ldots, x_n{:=}s_n$. We write $dsfr(\Delta \shortmid t)$ for a ds-free term $dsfr(t\langle x_n{:=}s_n\rangle \cdots \langle x_1{:=}s_1\rangle)$. For a ds-free term t, $core(t)$ is a term that is obtained by removing 'irrelevant' substitutions from t as follows.

1. $core(x) \triangleq x$
2. $core(\lambda x.\, t) \triangleq \lambda x.\, core(t)$
3. $core(ts) \triangleq core(t)core(s)$
4. $core(t\langle x{:=}s\rangle) \triangleq core(t)$ if $x \in \mathsf{D}$ and $x \notin \mathrm{FV}(core(t))$
5. $core(t\langle x{:=}s\rangle) \triangleq core(t)\langle x{:=}core(s)\rangle$ if $x \in \mathsf{V}$ or $x \in \mathrm{FV}(core(t))$

Note that $t \xrightarrow{*} core(t)$ by (gc)s and $\mathrm{FV}(t) \cap \mathsf{V} = \mathrm{FV}(core(t)) \cap \mathsf{V}$.

Given a term $s\langle x{:=}r\rangle$, we write $sv(s\langle x{:=}r\rangle)$ for x and write $sb(s\langle x{:=}r\rangle)$ for $\langle x{:=}r\rangle$. Let t be a ds-free term and π be a position in t. Let $\rho_1, \ldots, \rho_n$ be an increasing sequence of all positions in t such that $\rho_i \mathtt{T} \sqsubseteq \pi$ $(1 \leq i \leq n)$. We write $CL(\pi, t)$ for a term $(t/\pi)\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle$ where $\langle y_i{:=}r_i\rangle \equiv sb(t/\rho_i)$. This means that $\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle$ is a sequence of substitutions that contain position π in their scopes. We define a set of positions $\{\Lambda\} \cup \{\pi \mathtt{B} \mid sv(t/\pi) \in \mathsf{D}\}$ as $\mathcal{S}(t)$ and say t *is locally semi SN* if $core(CL(\pi, t))$ is semi SN for every $\pi \in \mathcal{S}(t)$.

Lemma 4. *Let t be a ds-free term and π, τ be positions in t. Suppose $t/\tau \equiv r\langle x{:=}s\rangle$, $\tau\mathtt{T} \sqsubseteq \pi$, and there is no τ' such that $\tau\mathtt{T} \sqsubset \tau'\mathtt{T} \sqsubseteq \pi$. Then, if $x \in \mathrm{FV}(t/\pi)$, there is a reduction sequence $t/\tau \xrightarrow{*} t'$ such that $t'/\rho \equiv (t/\pi)\langle x{:=}s\rangle$ for some ρ, and if $x \notin \mathrm{FV}(t/\pi)$, there is a reduction sequence $t/\tau \xrightarrow{*} t'$ such that $t'/\rho \equiv t/\pi$ for some ρ and t'/ρ is not in the scope of $\langle x{:=}s\rangle$.*

Proof. By induction on (length of π) − (length of $\tau\mathtt{T}$). In the base case (that is, $r \equiv t/\pi$), take t/τ for t' if $x \in \mathrm{FV}(t/\pi)$, and take r for t' (since $(\mathsf{gc}) : t/\tau \to r$) if $x \notin \mathrm{FV}(t/\pi)$. In the case $r \equiv r_2\langle y{:=}r_1\rangle$, t/π is a subterm of r_1, because there is no τ' such that $\tau\mathtt{T} \sqsubset \tau'\mathtt{T} \sqsubseteq \pi$. If $x \in \mathrm{FV}(t/\pi)$, we have $(\mathsf{comp}) : r_2\langle y{:=}r_1\rangle\langle x{:=}s\rangle \to r_2\langle x{:=}s\rangle\langle y{:=}r_1\langle x{:=}s\rangle\rangle$. By induction hypothesis, $r_1\langle x{:=}s\rangle \xrightarrow{*} t''$ where $t''/\rho \equiv (t/\pi)\langle x{:=}s\rangle$. Therefore, $r_2\langle y{:=}r_1\rangle\langle x{:=}s\rangle \xrightarrow{*} r_2\langle x{:=}s\rangle\langle y{:=}t''\rangle$. If $x \notin \mathrm{FV}(t/\pi)$, we have $(\mathsf{perm}) : r_2\langle y{:=}r_1\rangle\langle x{:=}s\rangle \to r_2\langle x{:=}s\rangle\langle y{:=}r_1\rangle$. Therefore, r_1 is not in the scope of $\langle x{:=}s\rangle$. For other cases of r, straightforward by induction hypothesis. □

Lemma 5. *If $\Delta \shortmid t$ is semi SN, then $dsfr(\Delta \shortmid t)$ is locally semi SN.*

Proof. Put $r \equiv dsfr(\Delta \shortmid t)$. By (def) and $(\mathsf{d2x})$, we have $\Delta \shortmid t \xrightarrow{*} r$. So, r is semi SN. Take $\pi \in \mathcal{S}(r)$ and suppose $CL(\pi, r) \equiv (r/\pi)\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle$. Since $r/\pi \xrightarrow{*} core(r/\pi)$ by (gc), we have $r \xrightarrow{*} r_\pi[core(r/\pi)]$. If $y_n \in \mathrm{FV}(core(r/\pi))$, we have $r_\pi[core(r/\pi)] \xrightarrow{*} r'$ where $r'/\rho \equiv core(r/\pi)\langle y_n{:=}core(r_n)\rangle$ for some ρ by Lemma 4 and the property of $core$. If $y_n \notin \mathrm{FV}(core(r/\pi))$, we can get rid of $\langle y_n{:=}r_n\rangle$ by Lemma 4. By repeating this process, we have $r \xrightarrow{*} r'_{\pi'}[core(CL(\pi, r))]$ for some r' and π'. Therefore, $core(CL(\pi, r))$ is semi SN. □

Lemma 6. *If $\Delta_1 \shortmid t_1 \to \Delta_2 \shortmid t_2$ and $dsfr(\Delta_1 \shortmid t_1)$ is locally semi SN, then $dsfr(\Delta_2 \shortmid t_2)$ is locally semi SN.*

Proof. By case analysis of the reduction rule. □

We define totally ordered sets W_n, F_n $(n = 0, 1, 2, \ldots)$ as follows. In the following, the orders on pairs or ordered sequences are lexicographic and N denotes the set of natural numbers with the ordinary order.

$$\mathsf{W}_0 \triangleq \emptyset \qquad \mathsf{F}_0 \triangleq \{\varepsilon\,(\text{empty sequence})\}$$

$$\mathsf{W}_n \triangleq \mathsf{W}_{n-1} \cup \{\langle k, f\rangle \mid k \in \mathsf{N},\ f \in \mathsf{F}_{n-1}\}$$

$$\mathsf{F}_n \triangleq \{\langle w_1, h_1\rangle, \ldots, \langle w_m, h_m\rangle \mid w_i \in \mathsf{W}_n,\ h_i \in \mathsf{N},\ \langle w_i, h_i\rangle \geq \langle w_{i+1}, h_{i+1}\rangle\}$$

(Note that $\mathsf{F}_{n-1} \subseteq \mathsf{F}_n$.) For $f_1, f_2 \in \mathsf{F}_n$, $f_1 \# f_2$ denotes the merge of two ordered sequences f_1 and f_2 (that is, f_1 and f_2 are concatenated and then sorted). Note that W_n is well-founded. (But $\bigcup_{n=0}^{\infty} \mathsf{W}_n$ is not well-founded.)

We define *height* $h_y(t)$ of a term t with respect to a variable y inductively as follows.

1. $h_y(x) \triangleq 2$
2. $h_y(\lambda x.\, t) \triangleq h_y(t) + 1$
3. $h_y(ts) \triangleq \max(h_y(t), h_y(s)) + 1$
4. $h_y(t\langle x{:=}s\rangle) \triangleq h_y(t) \quad$ if $x \in \mathsf{D}$, or $x \in \mathsf{V}$ and $y \notin \mathrm{FV}(s)$
5. $h_y(t\langle x{:=}s\rangle) \triangleq h_y(t) \cdot h_y(s) \quad$ if $x \in \mathsf{V}$ and $y \in \mathrm{FV}(s)$

Let $\Sigma \equiv \langle x_1{:=}s_1\rangle, \ldots, \langle x_m{:=}s_m\rangle$ be a sequence of substitutions. We write $|\Sigma|$ for m. For a term t, we write Σt for a term $t\langle x_m{:=}s_m\rangle \cdots \langle x_1{:=}s_1\rangle$ and for a set of variables V, we define $\Sigma|V$ as follows.

1. $\varepsilon|V \triangleq \varepsilon$
2. $\Sigma, \langle x{:=}s\rangle|V \triangleq \Sigma|(\mathrm{FV}(s) \cup (V - \{x\})), \langle x{:=}s\rangle \quad$ if $x \in V$
3. $\Sigma, \langle x{:=}s\rangle|V \triangleq \Sigma|V \quad$ if $x \notin V$

For a ds-free term t, we define its *weight* $w(t)$, which is an element of W_n for some n, as follows. (In the following, Σ be a sequence of substitutions.)

1. $w(t) \triangleq w_\varepsilon(t)$
2. $w_\Sigma(t) \triangleq \langle \nu(core(\Sigma t)) + |\Sigma|,\ f_\Sigma(t)\rangle$
3. $f_\Sigma(x) \triangleq \varepsilon$ (empty sequence)
4. $f_\Sigma(\lambda x.\, t) \triangleq f_\Sigma(t)$
5. $f_\Sigma(ts) \triangleq f_\Sigma(t) \mathbin{\#} f_\Sigma(s)$
6. $f_\Sigma(t\langle x{:=}s\rangle) \triangleq \langle w_{\Sigma|\mathrm{FV}(s)}(s), h_x(t)\rangle \mathbin{\#} f_{\Sigma,\langle x:=s\rangle}(t) \quad$ if $x \in \mathsf{D}$
7. $f_\Sigma(t\langle x{:=}s\rangle) \triangleq f_\Sigma(s) \mathbin{\#} f_\Sigma(t) \quad$ if $x \in \mathsf{V}$

We will give some examples of how to calculate $w(t)$. In the following examples, we assume $x, y, u, w \in \mathsf{D}$ and $z \in \mathsf{V}$ and put $k = \nu((xy)\langle x{:=}\lambda z.\, z\rangle)$. We have:

$$
\begin{aligned}
&w((xy)\langle x{:=}\lambda z.\, z\rangle)\\
&= \langle \nu((xy)\langle x{:=}\lambda z.\, z\rangle),\ f_\varepsilon((xy)\langle x{:=}\lambda z.\, z\rangle)\rangle\\
&= \langle k,\ \langle w_\varepsilon(\lambda z.\, z), h_x(xy)\rangle \mathbin{\#} f_{\langle x:=\lambda z.\, z\rangle}(xy)\rangle\\
&= \langle k,\ \langle\langle 0,\varepsilon\rangle, 2\rangle \mathbin{\#} f_{\langle x:=\lambda z.\, z\rangle}(x) \mathbin{\#} f_{\langle x:=\lambda z.\, z\rangle}(y)\rangle = \langle k,\ \langle\langle 0,\varepsilon\rangle, 2\rangle\rangle
\end{aligned}
$$

Another example is:

$$
\begin{aligned}
&w((xw)\langle y{:=}xu\rangle\langle x{:=}\lambda z.\, z\rangle)\\
&= \langle \nu((xw)\langle x{:=}\lambda z.\, z\rangle),\ f_\varepsilon((xw)\langle y{:=}xu\rangle\langle x{:=}\lambda z.\, z\rangle)\rangle\\
&= \langle k,\ \langle w_\varepsilon(\lambda z.\, z), h_x((xw)\langle y{:=}xu\rangle)\rangle \mathbin{\#} f_{\langle x:=\lambda z.\, z\rangle}((xw)\langle y{:=}xu\rangle)\rangle\\
&= \langle k,\ \langle\langle 0,\varepsilon\rangle, 2\rangle \mathbin{\#} \langle w_{\langle x:=\lambda z.\, z\rangle}(xu),\ h_y(xw)\rangle \mathbin{\#} f_{\langle x:=\lambda z.\, z\rangle,\langle y:=xu\rangle}(xw)\rangle\\
&= \langle k,\ \langle\langle 0,\varepsilon\rangle, 2\rangle \mathbin{\#} \langle\langle \nu((xu)\langle x{:=}\lambda z.\, z\rangle){+}1,\ f_{\langle x:=\lambda z.\, z\rangle}(xu)\rangle,\ 2\rangle \mathbin{\#} \varepsilon\rangle\\
&= \langle k,\ (\langle\langle k{+}1,\varepsilon\rangle, 2\rangle, \langle\langle 0,\varepsilon\rangle, 2\rangle)\rangle
\end{aligned}
$$

Though we omit the intermediate calculation steps, we have:

$$w((xw)\langle x{:=}\lambda z.\,z\rangle\langle y{:=}(xu)\langle x{:=}\lambda z.\,z\rangle\rangle) = \langle k,\, (\langle\langle k,\, \langle\langle 0, \varepsilon\rangle, 2\rangle\rangle, 2\rangle, \langle\langle 0, \varepsilon\rangle, 2\rangle)\rangle$$

Note that $w((xw)\langle y{:=}xu\rangle\langle x{:=}\lambda z.\,z\rangle) > w((xw)\langle x{:=}\lambda z.\,z\rangle\langle y{:=}(xu)\langle x{:=}\lambda z.\,z\rangle\rangle)$ in W_3.

Lemma 7. *If t is locally semi SN, then $w(t)$ is defined.*

Proof. For each $\langle x{:=}s\rangle$ in t such that $x \in \mathsf{D}$, $w_{\Sigma|\mathrm{FV}(s)}(s)$ is calculated for some Σ in the calculation process of $w(t)$. Suppose $\langle x{:=}s\rangle$ is in position π and $CL(\pi\mathsf{B}, t) \equiv (t/\pi\mathsf{B})\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle$. Then, $s \equiv t/\pi\mathsf{B}$ and Σ is a subsequence of $\langle y_1{:=}r_1\rangle, \ldots, \langle y_n{:=}r_n\rangle$. Due to the term formation condition, $\mathrm{FV}(s'\langle y{:=}r'\rangle) \subseteq \mathsf{D}$ if $\mathrm{FV}(s') \subseteq \mathsf{D}$ and $y \in \mathsf{D}$. Since $\mathrm{FV}(s) \subseteq \mathsf{D}$, $\langle y_i{:=}r_i\rangle$ such that $y_i \in \mathsf{V}$ can be removed by (gc). Therefore, $core(CL(\pi\mathsf{B}, t)) \xrightarrow{*} core((\Sigma|\mathrm{FV}(s))s)$. Since $core(CL(\pi\mathsf{B}, t))$ is semi SN, $\nu(core((\Sigma|\mathrm{FV}(s))s))$ is defined. □

Lemma 8. *Let r_1, r_2, r_3 be ds-free terms.*
1. $w_{\Sigma,\langle z:=r_1\rangle}(r_2) > w_{\Sigma}(r_2\langle z{:=}r_1\rangle)$
2. $f_{\Sigma,\langle z:=r_1\rangle,\langle y:=r_2\rangle}(r_3) \geq f_{\Sigma,\langle y:=r_2\langle z:=r_1\rangle\rangle,\langle z:=r_1\rangle}(r_3)$ *if* $z \in \mathrm{FV}(r_2)$
3. $f_{\Sigma,\langle z:=r_1\rangle,\langle y:=r_2\rangle}(r_3) = f_{\Sigma,\langle y:=r_2\rangle,\langle z:=r_1\rangle}(r_3)$ *if* $z \notin \mathrm{FV}(r_2)$
4. $f_{\Sigma}(r_1) = f_{\Sigma|\mathrm{FV}(r_1)}(r_1)$

Proof. 1. We have $w_{\Sigma,\langle z:=r_1\rangle}(r_2) = \langle \nu(core((\Sigma, \langle z{:=}r_1\rangle)r_2)) + |\Sigma, \langle z{:=}r_1\rangle|,\, f_1\rangle$ and $w_{\Sigma}(r_2\langle z{:=}r_1\rangle) = \langle \nu(core(\Sigma(r_2\langle z{:=}r_1\rangle))) + |\Sigma|,\, f_2\rangle$ for some f_1, f_2. Since $core((\Sigma, \langle z{:=}r_1\rangle)r_2) \equiv core(\Sigma(r_2\langle z{:=}r_1\rangle))$ and $|\Sigma, \langle z{:=}r_1\rangle| > |\Sigma|$, we have the result.
2. Since $f_{\Sigma,\ldots}(r_3)$ is calculated recursively on r_3, for each s such that $\langle x{:=}s\rangle$ is in r_3 and $x \in \mathsf{D}$, we need to compare $\nu(core(\Sigma_1 s)) + |\Sigma_1|$ and $\nu(core(\Sigma_2 s)) + |\Sigma_2|$ where $\Sigma_1 \equiv \Sigma, \langle z{:=}r_1\rangle, \langle y{:=}r_2\rangle, \Sigma'|\mathrm{FV}(s)$ and $\Sigma_2 \equiv \Sigma, \langle y{:=}r_2\langle z{:=}r_1\rangle\rangle, \langle z{:=}r_1\rangle, \Sigma'|\mathrm{FV}(s)$. Since $core(\Sigma_1 s) \xrightarrow{*} core(\Sigma_2 s)$ and $|\Sigma_1| \geq |\Sigma_2|$, we have the result.
3. Similar to the above case.
4. For any substitution $\langle x{:=}s\rangle$ in r_1 and any Σ', we have $\Sigma'|\mathrm{FV}(s) = (\Sigma'|\mathrm{FV}(r_1))|\mathrm{FV}(s)$, because $\mathrm{FV}(s) \subseteq \mathrm{FV}(r_1)$. □

Lemma 9. *Let t_1 be a ds-free term. If $(R) : t_1 \to t_2$ and t_1 is locally semi SN, then $w(t_1) > w(t_2)$ if $(R) \neq (\mathsf{perm})$ and $w(t_1) = w(t_2)$ if $(R) = (\mathsf{perm})$.*

Proof. Let $(R|\pi) : t_1 \to t_2$ be the reduction and ρ be a position in t_1 such that

1. $\rho \sqsubseteq \pi$
2. $\rho \equiv \Lambda$, or there exists ρ' such that $\rho \equiv \rho'\mathsf{B}$ and $sv(t/\rho') \in \mathsf{D}$
3. for any ρ'' such that $\rho \sqsubset \rho''\mathsf{B} \sqsubseteq \pi$, $sv(t/\rho'') \in \mathsf{V}$

By comparing the calculation processes of $w(t_1)$ and $w(t_2)$, we find following three points where differences may occur. (Note that $w(t_2)$ is defined by Lemma 6.)

A. The main difference comes from $w_{\Sigma_1}(t_1/\rho)$ and $w_{\Sigma_2}(t_2/\rho)$ where $\Sigma_1 = \Sigma_2$ if $(R) \neq (\mathsf{gc})$ and $\Sigma_1|\mathrm{FV}(t_2/\rho) = \Sigma_2$ if $(R) = (\mathsf{gc})$. (Note that only (gc) can erase free variables.) Since $w_{\Sigma_i}(t_i/\rho) = \langle \nu(\mathit{core}(\Sigma_i(t_i/\rho))) + |\Sigma_i|,\ f_{\Sigma_i}(t_i/\rho)\rangle$ and $\mathit{core}(t_i/\pi)$ is a subterm of $\mathit{core}(t_i/\rho)$ $(i = 1, 2)$, we will compare them following the outline given below.
   - If $(R) \neq (\mathsf{perm})$ and $(R) \neq (\mathsf{gc})$, we will show that $\nu(\mathit{core}(t_1/\pi)) > \nu(\mathit{core}(t_2/\pi))$, or $\nu(\mathit{core}(t_1/\pi)) = \nu(\mathit{core}(t_2/\pi))$ and $f_\Sigma(t_1/\pi) > f_\Sigma(t_2/\pi)$.
   - If $(R) = (\mathsf{gc})$, let $\nu_i \equiv \nu(\mathit{core}(\Sigma_i(t_i/\rho))) + |\Sigma_i|$ $(i = 1, 2)$. We will show that $\nu_1 > \nu_2$, or $\nu_1 \geq \nu_2$ and $f_{\Sigma_1}(t_1/\pi) > f_{\Sigma_2}(t_2/\pi)$.
   - If $(R) = (\mathsf{perm})$, we will show that $\nu(\mathit{core}(t_1/\pi)) = \nu(\mathit{core}(t_2/\pi))$ and $f_\Sigma(t_1/\pi) = f_\Sigma(t_2/\pi)$.

B. Suppose $\rho \equiv \rho'\mathsf{B}$. Other than the point A, there are two cases t_1/ρ may contribute to the value of $\nu(\mathit{core}(\Sigma_1 s_1))$ that is a part of $w_{\Sigma_1}(s_1)$. (If so, there is a corresponding calculation $\nu(\mathit{core}(\Sigma_2 s_2))$ that t_2/ρ may contribute to.)
   - t_1/ρ is a subterm of s_1: In this case, t_2/ρ is a subterm of s_2 and $\Sigma_1|\mathrm{FV}(s_2) \equiv \Sigma_2$.
   - $\Sigma_1 \equiv \Sigma_1', \langle y{:=}r_1\rangle, \Sigma$ and t_1/ρ is a subterm of r_1: In this case, $s_1 \equiv s_2$, $\Sigma_2 \equiv \Sigma_2', \langle y{:=}r_2\rangle, \Sigma$, and t_2/ρ is a subterm of r_2.

   In both cases, we have $\nu(\mathit{core}(\Sigma_1 s_1) + |\Sigma_1| \geq \nu(\mathit{core}(\Sigma_2 s_2)) + |\Sigma_2|$ if $(R) \neq (\mathsf{perm})$, and $\nu(\mathit{core}(\Sigma_1 s_1) + |\Sigma_1| = \nu(\mathit{core}(\Sigma_2 s_2)) + |\Sigma_2|$ if $(R) = (\mathsf{perm})$. This follows from the proof of the point A.

C. There may be a subterm $r_1\langle y{:=}s\rangle$ of t_1 such that t_1/π is a subterm of r_1 and $y \in \mathsf{D}$. Suppose $r_1 \to r_2$ by this reduction. Then, we may have $h_y(r_1) < h_y(r_2)$ when $(R) = (\boldsymbol{\lambda})$ or $(R) = (\mathsf{comp})$ in the case 11 of the following case analysis, if $r_1\langle y{:=}s\rangle$ is a subterm of t_1/ρ. (If not, t_1/ρ does not contribute to $h_y(r_1)$, because $sv(t_1/\rho) \in \mathsf{D}$.) This implies $\langle w_\Sigma(s), h_y(r_1)\rangle < \langle w_\Sigma(s), h_y(r_2)\rangle$, but in these cases we can verify $\nu(\mathit{core}(t_1/\rho)) > \nu(\mathit{core}(t_2/\rho))$. Note also that $h_y(r_1) = h_y(r_2)$ in the case $(R) = (\mathsf{perm})$.

Now, we will do case analysis for the point A, but here we examine only a few cases. (In the following, we write r_i' for $\mathit{core}(r_i)$ $(i = 1, 2, 3)$.)

1. $(R) = (\boldsymbol{\lambda})$ and $t_1/\pi \equiv (\lambda y.\, r_1)r_2$: In this case, $t_2/\pi \equiv r_1\langle y{:=}r_2\rangle$, $\mathit{core}(t_1/\pi) \equiv (\lambda y.\, r_1')r_2'$, and $\mathit{core}(t_2/\pi) \equiv r_1'\langle y{:=}r_2'\rangle$. So, we have $(\boldsymbol{\lambda}) : \mathit{core}(t_1/\pi) \to \mathit{core}(t_2/\pi)$. Therefore, $\nu(\mathit{core}(t_1/\pi)) > \nu(\mathit{core}(t_2/\pi))$.
2. $(R) = (\mathsf{var})$ and $t_1/\pi \equiv z\langle z{:=}r_1\rangle$: In this case, $t_2/\pi \equiv r_1$, $\mathit{core}(t_1/\pi) \equiv z\langle z{:=}r_1'\rangle$, and $\mathit{core}(t_2/\pi) \equiv r_1'$. So, we have $(\mathsf{var}) : \mathit{core}(t_1/\pi) \to \mathit{core}(t_2/\pi)$. Therefore, $\nu(\mathit{core}(t_1/\pi)) > \nu(\mathit{core}(t_2/\pi))$.
3. $(R) = (\mathsf{app})$, $t_1/\pi \equiv (r_2 r_3)\langle z{:=}r_1\rangle$, and $z \in \mathsf{D}$: In this case, $t_2/\pi \equiv (r_2\langle z{:=}r_1\rangle)(r_3\langle z{:=}r_1\rangle)$.
   (a) $z \in \mathrm{FV}(r_1')$ or $z \in \mathrm{FV}(r_2')$: In this case, $\mathit{core}(t_1/\pi) \equiv (r_2' r_3')\langle z{:=}r_1'\rangle$ and $\mathit{core}(t_2/\pi) \equiv (r_2'\langle z{:=}r_1'\rangle)(r_3'\langle z{:=}r_1'\rangle)$ or $(r_2'\langle z{:=}r_1'\rangle)r_3'$ or $r_2'(r_3'\langle z{:=}r_1'\rangle)$. So, we have $\mathit{core}(t_1/\pi) \xrightarrow{+} \mathit{core}(t_2/\pi)$ by (app) and 0 or 1 (gc). Therefore, $\nu(\mathit{core}(t_1/\pi)) > \nu(\mathit{core}(t_2/\pi))$.

(b) $z \notin \mathrm{FV}(r_1')$ and $z \notin \mathrm{FV}(r_2')$: In this case, $core(t_1/\pi) \equiv core(t_2/\pi) \equiv r_1'r_2'$. So, we will show $f_\Sigma(t_1/\pi) > f_\Sigma(t_2/\pi)$.

$$
\begin{aligned}
& f_\Sigma((r_2r_3)\langle z{:=}r_1\rangle) \\
& = \langle w_1, h_x(r_2r_3)\rangle \# f_{\Sigma,\langle z:=r_1\rangle}(r_2r_3) \\
& = \langle w_1, h_x(r_2r_3)\rangle \# f_{\Sigma,\langle z:=r_1\rangle}(r_2) \# f_{\Sigma,\langle z:=r_1\rangle}(r_3) \\
& f_\Sigma((r_2\langle z{:=}r_1\rangle)(r_3\langle z{:=}r_1\rangle)) \\
& = f_\Sigma(r_2\langle z{:=}r_1\rangle) \# f_\Sigma(r_3\langle z{:=}r_1\rangle) \\
& = \langle w_1, h_x(r_2)\rangle \# f_{\Sigma,\langle z:=r_1\rangle}(r_2) \# \langle w_1, h_x(r_3)\rangle \# f_{\Sigma,\langle z:=r_1\rangle}(r_3)
\end{aligned}
$$

where $w_1 \equiv w_{\Sigma|\mathrm{FV}(r_1)}(r_1)$. Since $h_x(r_2r_3) > h_x(r_2)$ and $h_x(r_2r_3) > h_x(r_3)$, we have $f_\Sigma((r_2r_3)\langle z{:=}r_1\rangle) > f_\Sigma((r_2\langle z{:=}r_1\rangle)(r_3\langle z{:=}r_1\rangle))$.

4. $(R) = (\mathsf{app})$, $t_1/\pi \equiv (r_2r_3)\langle z{:=}r_1\rangle$, and $z \in \mathsf{V}$: In this case, $t_2/\pi \equiv (r_2\langle z{:=}r_1\rangle)(r_3\langle z{:=}r_1\rangle)$. Since $y \in \mathsf{V}$, we have $core(t_1/\pi) \equiv (r_2'r_3')\langle z{:=}r_1\rangle$ and $core(t_2/\pi) \equiv (r_2'\langle z{:=}r_1'\rangle)(r_3'\langle z{:=}r_1'\rangle)$. So, we have $(\mathsf{app}) : core(t_1/\pi) \to core(t_2/\pi)$. Therefore, $\nu(core(t_1/\pi)) > \nu(core(t_2/\pi))$.
5. $(R) = (\mathsf{abs})$ and $t_1/\pi \equiv (\lambda z.\, r_2)\langle z{:=}r_1\rangle$: Similar to the case 3 and 4.
6. $(R) = (\mathsf{gc})$, $t_1/\pi \equiv r_2\langle z{:=}r_1\rangle$, and $z \in \mathsf{D}$: In this case, $t_2/\pi \equiv r_2$ and $z \notin \mathrm{FV}(r_2)$. So, we have $core(\Sigma_1(t_1/\rho)) \equiv core(\Sigma_2(t_2/\rho))$ and $|\Sigma_1| \geq |\Sigma_2|$. Since $f_{\Sigma_1}(r_2\langle z{:=}r_1\rangle) = \langle w_{\Sigma|\mathrm{FV}(r_1)}(r_1), h_x(r_2)\rangle \# f_{\Sigma_1,\langle z:=r_1\rangle}(r_2)$, we have $f_{\Sigma_1}(t_1/\pi) > f_{\Sigma_2}(t_2/\pi)$.
7. $(R) = (\mathsf{gc})$, $t_1/\pi \equiv r_2\langle z{:=}r_1\rangle$, and $z \in \mathsf{V}$: We have $core(t_1/\pi) \equiv r_2'\langle z{:=}r_1'\rangle$. So, we have $core(\Sigma_1(t_1/\rho)) \xrightarrow{+} core(\Sigma_2(t_2/\rho))$ by (gc)s. Since $|\Sigma_1| \geq |\Sigma_2|$, we have $\nu(core(\Sigma_1(t_1/\rho))) + |\Sigma_1| > \nu(core(\Sigma_2(t_2/\rho))) + |\Sigma_2|$.
8. $(R) = (\mathsf{comp})$, $t_1/\pi \equiv r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle$ and $z, y \in \mathsf{D}$: In this case, $t_2/\pi \equiv r_3\langle z{:=}r_1\rangle\langle y{:=}r_2\langle z{:=}r_1\rangle\rangle$ and $z \in \mathrm{FV}(r_2)$.

   (a) $y \in \mathrm{FV}(r_3')$, $z \in \mathrm{FV}(r_2')$: If $z \in \mathrm{FV}(r_3')$, then $core(t_1/\pi) \equiv r_3'\langle y{:=}r_2'\rangle\langle z{:=}r_1'\rangle$ and $core(t_2/\pi) \equiv r_3'\langle z{:=}r_1'\rangle\langle y{:=}r_2'\langle z{:=}r_1'\rangle\rangle$. So, we have $(\mathsf{comp}) : core(t_1/\pi) \to core(t_2/\pi)$. If $z \notin \mathrm{FV}(r_3')$, then $core(t_1/\pi) \equiv r_3'\langle y{:=}r_2'\rangle\langle z{:=}r_1'\rangle$ and $core(t_2/\pi) \equiv r_3'\langle y{:=}r_2'\langle z{:=}r_1'\rangle\rangle$. So, we have $(\mathsf{comp}), (\mathsf{gc}) : core(t_1/\pi) \to r_3'\langle z{:=}r_1'\rangle\langle y{:=}r_2'\langle z{:=}r_1'\rangle\rangle \to core(t_2/\pi)$. In both cases, we have $\nu(core(t_1/\pi)) > \nu(core(t_2/\pi))$.

   (b) otherwise: If $y \in \mathrm{FV}(r_3')$, $z \notin \mathrm{FV}(r_2')$, and $z \in \mathrm{FV}(r_3')$, then $core(t_1/\pi) \equiv r_3'\langle y{:=}r_2'\rangle\langle z{:=}r_1'\rangle$ and $core(t_2/\pi) \equiv r_3'\langle z{:=}r_1'\rangle\langle y{:=}r_2'\rangle$, otherwise, $core(t_1/\pi) \equiv core(t_2/\pi)$. Since $(\mathsf{perm}) : core(t_1/\pi) \to core(t_2/\pi)$ in the former case, we have $\nu(core(t_1/\pi)) = \nu(core(t_2/\pi))$ in both cases. So, we will show $f_\Sigma(t_1/\pi) > f_\Sigma(t_2/\pi)$.

$$
\begin{aligned}
& f_\Sigma(r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle) \\
& = \langle w_{\Sigma|\mathrm{FV}(r_1)}(r_1), h_z(r_3\langle y{:=}r_2\rangle)\rangle \# f_{\Sigma,\langle z:=r_1\rangle}(r_3\langle y{:=}r_2\rangle) \\
& = \langle w_{\Sigma|\mathrm{FV}(r_1)}(r_1), h_z(r_3)\rangle \# \langle w_{\Sigma,\langle z:=r_1\rangle|\mathrm{FV}(r_2)}(r_2), h_y(r_3)\rangle \# \\
& \qquad f_{\Sigma,\langle z:=r_1\rangle,\langle y:=r_2\rangle}(r_3)
\end{aligned}
$$

$$
\begin{aligned}
&f_\Sigma(r_3\langle z{:=}r_1\rangle\langle y{:=}r_2\langle z{:=}r_1\rangle\rangle)\\
&= \langle w_{\Sigma_1}(r_2\langle z{:=}r_1\rangle), h_y(r_3\langle z{:=}r_1\rangle)\rangle \,\#\, f_{\Sigma,\langle y:=r_2\langle z:=r_1\rangle\rangle}(r_3\langle z{:=}r_1\rangle)\\
&= \langle w_{\Sigma_1}(r_2\langle z{:=}r_1\rangle), h_y(r_3)\rangle \,\#\, \langle w_{\Sigma_2}(r_1), h_z(r_3)\rangle \#\\
&\qquad f_{\Sigma,\langle y:=r_2\langle z:=r_1\rangle\rangle,\langle z:=r_1\rangle}(r_3)
\end{aligned}
$$

where $\Sigma_1 \equiv \Sigma|\mathrm{FV}(r_2\langle z{:=}r_1\rangle)$ and $\Sigma_2 \equiv \Sigma, \langle y{:=}r_2\langle z{:=}r_1\rangle\rangle|\mathrm{FV}(r_1)$. We have $\Sigma, \langle z{:=}r_1\rangle|\mathrm{FV}(r_2) \equiv \Sigma|(\mathrm{FV}(r_1) \cup (\mathrm{FV}(r_2) - \{z\})), \langle z{:=}r_1\rangle \equiv \Sigma_1, \langle z{:=}r_1\rangle$. We also have $\Sigma_2 \equiv \Sigma|\mathrm{FV}(r_1)$, since $y \notin \mathrm{FV}(r_1)$. Therefore, by Lemma 8, we have $f_\Sigma(r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle) > f_\Sigma(r_3\langle z{:=}r_1\rangle\langle y{:=}r_2\langle z{:=}r_1\rangle\rangle)$.

9. $(R) =$ (comp), $t_1/\pi \equiv r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle$, $z \in \mathsf{D}$, and $y \in \mathsf{V}$: This case is similar to the above, except that $core(r_3\langle y{:=}r_2\rangle) \equiv r_3'\langle y{:=}r_2'\rangle$ since $y \in \mathsf{V}$, and that $f_\Sigma(t_1/\pi) > f_\Sigma(t_2/\pi)$ is proved as follows.

$$
\begin{aligned}
&f_\Sigma(r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle)\\
&= \langle w_{\Sigma|\mathrm{FV}(r_1)}(r_1), h_z(r_3\langle y{:=}r_2\rangle)\rangle \,\#\, f_{\Sigma,\langle z:=r_1\rangle}(r_3\langle y{:=}r_2\rangle)\\
&= \langle w_{\Sigma|\mathrm{FV}(r_1)}(r_1), h_z(r_3\langle y{:=}r_2\rangle)\rangle \,\#\, f_{\Sigma,\langle z:=r_1\rangle}(r_2) \,\#\, f_{\Sigma,\langle z:=r_1\rangle}(r_3)\\
&f_\Sigma(r_3\langle z{:=}r_1\rangle\langle y{:=}r_2\langle z{:=}r_1\rangle\rangle)\\
&= f_\Sigma(r_2\langle z{:=}r_1\rangle) \,\#\, f_\Sigma(r_3\langle z{:=}r_1\rangle)\\
&= \langle w_{\Sigma|\mathrm{FV}(r_1)}(r_1), h_z(r_2)\rangle \,\#\, f_{\Sigma,\langle z:=r_1\rangle}(r_2) \,\#\, \langle w_{\Sigma|\mathrm{FV}(r_1)}(r_1), h_z(r_3)\rangle \#\\
&\qquad f_{\Sigma,\langle z:=r_1\rangle}(r_3)
\end{aligned}
$$

Since $z \in \mathrm{FV}(r_2)$ and $y \in \mathsf{V}$, we have $h_z(r_3\langle y{:=}r_2\rangle) > h_z(r_2)$ and $h_z(r_3\langle y{:=}r_2\rangle) > h_z(r_3)$. Therefore, $f_\Sigma(r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle) > f_\Sigma(r_3\langle z{:=}r_1\rangle\langle y{:=}r_2\langle z{:=}r_1\rangle\rangle)$.

10. $(R) =$ (comp), $t_1/\pi \equiv r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle$, $z \in \mathsf{V}$, and $y \in \mathsf{D}$: This case is void, because $\mathrm{FV}(r_2) \subseteq \mathsf{D}$.

11. $(R) =$ (comp), $t_1/\pi \equiv r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle$ and $z, y \in \mathsf{V}$: By the remark after the definition of *core*, we have $z \in \mathrm{FV}(r_2')$. We also have $core(r_3\langle y{:=}r_2\rangle) \equiv r_3'\langle y{:=}r_2'\rangle$ since $y \in \mathsf{V}$. Therefore, this can be proved similarly to the case 8a.

12. $(R) =$ (perm), $t_1/\pi \equiv r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle$ and $z, y \in \mathsf{D}$: In this case, $t_2/\pi \equiv r_3\langle z{:=}r_1\rangle\langle y{:=}r_2\rangle$ and $z \notin \mathrm{FV}(r_2)$. If $y \in \mathrm{FV}(r_3')$ and $z \in \mathrm{FV}(r_3')$, then (perm) : $core(t_1/\pi) \to core(t_2/\pi)$, otherwise, $core(t_1/\pi) \equiv core(t_2/\pi)$. In both cases, we have $\nu(core(t_1/\pi)) = \nu(core(t_2/\pi))$. So, we will show $f_\Sigma(t_1/\pi) = f_\Sigma(t_2/\pi)$.

$$
\begin{aligned}
&f_\Sigma(r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle)\\
&= \langle w_{\Sigma|\mathrm{FV}(r_1)}(r_1), h_z(r_3\langle y{:=}r_2\rangle)\rangle \,\#\, f_{\Sigma,\langle z:=r_1\rangle}(r_3\langle y{:=}r_2\rangle)\\
&= \langle w_{\Sigma|\mathrm{FV}(r_1)}(r_1), h_z(r_3)\rangle \,\#\, \langle w_{\Sigma,\langle z:=r_1\rangle|\mathrm{FV}(r_2)}(r_2), h_y(r_3)\rangle \#\\
&\qquad f_{\Sigma,\langle z:=r_1\rangle,\langle y:=r_2\rangle}(r_3)\\
&f_\Sigma(r_3\langle z{:=}r_1\rangle\langle y{:=}r_2\rangle)\\
&= \langle w_{\Sigma_1}(r_2), h_y(r_3\langle z{:=}r_1\rangle)\rangle \,\#\, f_{\Sigma,\langle y:=r_2\rangle}(r_3\langle z{:=}r_1\rangle)\\
&= \langle w_{\Sigma_1}(r_2), h_y(r_3)\rangle \,\#\, \langle w_{\Sigma_2}(r_1), h_z(r_3)\rangle \,\#\, f_{\Sigma,\langle y:=r_2\rangle,\langle z:=r_1\rangle}(r_3)
\end{aligned}
$$

where $\Sigma_1 \equiv \Sigma|\mathrm{FV}(r_2)$ and $\Sigma_2 \equiv \Sigma, \langle y{:=}r_2\rangle|\mathrm{FV}(r_1)$. We have $\Sigma, \langle z{:=}r_1\rangle|\mathrm{FV}(r_2) \equiv \Sigma|\mathrm{FV}(r_2) \equiv \Sigma_1$, since $z \notin \mathrm{FV}(r_2)$. We also have $\Sigma_2 \equiv \Sigma|\mathrm{FV}(r_1)$, since $y \notin \mathrm{FV}(r_1)$. By Lemma 8, we have the result.

13. $(R) = (\mathsf{perm})$, $t_1/\pi \equiv r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle$, $z \in \mathsf{D}$, and $y \in \mathsf{V}$: In this case, $t_2/\pi \equiv r_3\langle z{:=}r_1\rangle\langle y{:=}r_2\rangle$ and $z \notin \mathrm{FV}(r_2)$. If $z \in \mathrm{FV}(r_3')$, then $(\mathsf{perm}) : \mathit{core}(t_1/\pi) \to \mathit{core}(t_2/\pi)$, otherwise, $\mathit{core}(t_1/\pi) \equiv \mathit{core}(t_2/\pi)$. In both cases, we have $\nu(\mathit{core}(t_1/\pi)) = \nu(\mathit{core}(t_2/\pi))$. So, we will show $f_\Sigma(t_1/\pi) = f_\Sigma(t_2/\pi)$.

$$\begin{aligned}
&f_\Sigma(r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle)\\
&= \langle w_{\Sigma|\mathrm{FV}(r_1)}(r_1), h_z(r_3\langle y{:=}r_2\rangle)\rangle \# f_{\Sigma,\langle z:=r_1\rangle}(r_3\langle y{:=}r_2\rangle)\\
&= \langle w_{\Sigma|\mathrm{FV}(r_1)}(r_1), h_z(r_3)\rangle \# f_{\Sigma,\langle z:=r_1\rangle}(r_2) \# f_{\Sigma,\langle z:=r_1\rangle}(r_3)\\
&f_\Sigma(r_3\langle z{:=}r_1\rangle\langle y{:=}r_2\rangle)\\
&= f_\Sigma(r_2) \# f_\Sigma(r_3\langle z{:=}r_1\rangle)\\
&= f_\Sigma(r_2) \# \langle w_{\Sigma|\mathrm{FV}(r_1)}(r_1), h_z(r_3)\rangle \# f_{\Sigma,\langle z:=r_1\rangle}(r_3)
\end{aligned}$$

Since $z \notin \mathrm{FV}(r_2)$, we have $f_{\Sigma,\langle z:=r_1\rangle}(r_2) = f_\Sigma(r_2)$ by Lemma 8. So, we have $f_\Sigma(r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle) = f_\Sigma(r_3\langle z{:=}r_1\rangle\langle y{:=}r_2\rangle)$.
14. $(R) = (\mathsf{perm})$, $t_1/\pi \equiv r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle$, $z \in \mathsf{V}$, and $y \in \mathsf{D}$: Symmetric to the above case.
15. $(R) = (\mathsf{perm})$, $t_1/\pi \equiv r_3\langle y{:=}r_2\rangle\langle z{:=}r_1\rangle$ and $z, y \in \mathsf{V}$: Easy. □

By this lemma, we have shown that weights of terms in a non-permutative reduction sequence decreases, but to conclude that the sequence is finite, it is necessary to show that all weights are in W_n for some n.

When $w \notin \mathsf{W}_{n-1}$ and $w \in \mathsf{W}_n$, we define the *level* of w as n and the level of w is denoted by $|w|$. We can calculate the level of a ds-free term t as follows.

1. $lv(x) \triangleq 1$
2. $lv(\lambda x.\, t) \triangleq lv(t)$
3. $lv(ts) \triangleq \max(lv(t), lv(s))$
4. $lv(t\langle x{:=}s\rangle) \triangleq \max(lv(t), lv(s)+1) \quad$ if $x \in \mathsf{D}$
5. $lv(t\langle x{:=}s\rangle) \triangleq \max(lv(t), lv(s)) \quad$ if $x \in \mathsf{V}$

Lemma 10. *Let t be a ds-free term and suppose $w(t)$ is defined. Then, $|w(t)| = lv(t)$.*

Proof. When $f \notin \mathsf{F}_{n-1}$ and $f \in \mathsf{F}_n$, we write $|f|$ for n. Then, by clause 2 of the definition of weight, we have $|w_\Sigma(t)| = |f_\Sigma(t)| + 1$ if $w_\Sigma(t)$ is defined. By induction on t, we prove that $|w_\Sigma(t)| = lv(t)$ if $w_\Sigma(t)$ is defined. Here, we prove only the case $t\langle x{:=}s\rangle$ ($x \in \mathsf{D}$). By clause 6, we have $|f_\Sigma(t\langle x{:=}s\rangle)| = \max(|w_{\Sigma|\mathrm{FV}(s)}(s)|, |f_{\Sigma,\langle x:=s\rangle}(t)|)$. Therefore, $|w_\Sigma(t\langle x{:=}s\rangle)| = \max(|w_{\Sigma|\mathrm{FV}(s)}(s)| + 1, |f_{\Sigma,\langle x:=s\rangle}(t)| + 1) = \max(lv(s)+1, lv(t)) = lv(t\langle x{:=}s\rangle)$. □

Since the level may increase by reduction, we need to estimate the upper bound of the levels in a reduction sequence. We define a *potential level* $plv(t)$ of a ds-free term t as follows. (In the following clauses, x in $\langle x{:=}s\rangle$ satisfies $x \in \mathsf{D}$.)

1. $plv(x) \triangleq 1$
2. $plv(\lambda x.\, t) \triangleq plv(t)$

3. $plv(ts) \triangleq \max(plv(t), plv(s))$
4. $plv(y\langle x{:=}s\rangle) \triangleq plv(s) + 1$
5. $plv((\lambda y.\, t)\langle x{:=}s\rangle) \triangleq plv(t\langle x{:=}s\rangle)$
6. $plv((tr)\langle x{:=}s\rangle) \triangleq \max(plv(t\langle x{:=}s\rangle), plv(r\langle x{:=}s\rangle))$
7. $plv(t\langle y{:=}r\rangle\langle x{:=}s\rangle) \triangleq \max(plv(t\langle y{:=}r\langle x{:=}s\rangle\rangle), plv(t\langle x{:=}s\rangle))$ if $x \in \mathrm{FV}(r)$
8. $plv(t\langle y{:=}r\rangle\langle x{:=}s\rangle) \triangleq \max(plv(t\langle y{:=}r\rangle), plv(t\langle x{:=}s\rangle))$ if $x \notin \mathrm{FV}(r)$
9. $plv(t\langle y{:=}r\rangle) \triangleq \max(plv(t), plv(r))$ if $y \in \mathsf{V}$

Lemma 11. *plv is well-defined.*

Proof. Let t be a ds-free term. When t is of the form $t_0\langle x_n{:=}t_n\rangle \cdots \langle x_1{:=}t_1\rangle$ where t_0 is not of the form $r_1\langle y{:=}r_2\rangle$, we define prefix length of t as n. We can prove the well-definedness of $plv(t)$ by double induction on the size of t and the prefix length of t. ◻

Lemma 12. *Let t_1 be a ds-free term. If $t_1 \to t_2$, then $plv(t_1) \geq plv(t_2)$.*

Proof. By case analysis of the reduction rule. ◻

Lemma 13. *Let t be a ds-free term and suppose $w(t)$ is defined. Then, $|w(t)| \leq plv(t)$.*

Proof. Since $lv(t) \leq plv(t)$ implies this lemma by Lemma 10, we prove it by induction on the definition of $plv(t)$. The critical case is clause 7. By induction hypothesis, we have

$$lv(t\langle y{:=}r\langle x{:=}s\rangle\rangle) \leq plv(t\langle y{:=}r\langle x{:=}s\rangle\rangle), \quad lv(t\langle x{:=}s\rangle) \leq plv(t\langle x{:=}s\rangle)$$

Since $lv(t\langle y{:=}r\rangle) \leq lv(t\langle y{:=}r\langle x{:=}s\rangle\rangle)$ and $lv(t\langle y{:=}r\rangle\langle x{:=}s\rangle) = \max(lv(t\langle y{:=}r\rangle), lv(t\langle x{:=}s\rangle))$, we have $lv(t\langle y{:=}r\rangle\langle x{:=}s\rangle) \leq plv(t\langle y{:=}r\rangle\langle x{:=}s\rangle)$. ◻

Lemma 14 (Void Lemma). *If $\Delta \shortmid s$ and $\Delta \shortmid t$ are semi SN and $x \notin \mathrm{FV}(t)$, then $\Delta, x := s \shortmid t$ is semi SN.*

Proof. Given a non-trivial reduction sequence starting from $\Delta, x := s \shortmid t$, we have a ds-free term t_0 such that $\Delta, x := s \shortmid t \xrightarrow{*} t_0$ and $\overline{\gamma} : t_0 \to t_1 \to t_2 \to \cdots$ is a non-permutative reduction sequence. First, we will show that t_0 is locally semi SN. Let $\Delta \equiv x_1{:=}s_1, \ldots, x_n{:=}s_n$ and define $V(r) \triangleq \{x \in \mathsf{D} \mid r_2\{x{:=}r_1\}$ or $r_2\langle x{:=}r_1\rangle$ is a subterm of $r\}$. Without loss of generality, we can assume $\{x_1, \ldots, x_n\}$, $V(s_1)$, ..., $V(s_n)$, $\{x\}$, $V(s)$, $V(t)$ are mutually disjoint. Put $r \equiv dsfr(\Delta, x := s \shortmid t)$ and take any $\pi \in \mathcal{S}(r)$. If $\pi \equiv \Lambda$ or $sv(r/\pi) \in V(t)$, put $r' \equiv dsfr(\Delta \shortmid t)$. Otherwise, put $r' \equiv dsfr(\Delta \shortmid s)$. Let π' be Λ if $\pi \equiv \Lambda$, and π' be a position such that $sv(r'/\pi') \equiv sv(r/\pi)$ otherwise. Then, $core(CL(\pi, r)) \equiv core(CL(\pi', r'))$, because $x \notin \mathrm{FV}(t)$. Since $\Delta \shortmid s$ and $\Delta \shortmid t$ are semi SN, $core(CL(\pi', r'))$ is semi SN by Lemma 5. Therefore, r is locally semi SN. So, t_0 is locally semi SN by Lemma 6.

By Lemma 13 and Lemma 12, we have $|w(t_i)| \leq plv(t_0)$. Therefore, $\overline{\gamma}$ is finite by Lemma 9. ◻

4 Type System and Reducibility

We recall *simply typed* λx. We define *types* (A, B) of simply typed λx by

$$A, B ::= o \mid A{\to}B$$

where o ranges over atomic types. A *declaration* is an expression of the form $x : A$ where x is a variable and A is a type, and a *typing judgement* is an expression of the form $\Gamma \vdash t : A$ where Γ is a sequence of declarations whose variables are different each other, t is an original λx-term, and A is a type. The inference rules to derive typing judgements of simply typed λx are given as follows and t is said to be a *typable term* if a typing judgement of the form $\Gamma \vdash t : A$ is derivable for some Γ and A.

$$\frac{}{\Gamma_1, x : A, \Gamma_2 \vdash x : A}\ (\mathsf{var})$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.\, t : A{\to}B}\ (\mathsf{abs}) \qquad \frac{\Gamma \vdash t : A{\to}B \quad \Gamma \vdash s : A}{\Gamma \vdash ts : B}\ (\mathsf{app})$$

$$\frac{\Gamma \vdash s : A \quad \Gamma, x : A \vdash t : B}{\Gamma \vdash t\langle x{:=}s\rangle : B}\ (\mathsf{xsub})$$

We have defined typability on original λx-terms, but we will define our reducible set as a set of def-terms $\Delta \shortmid t$ such that $\mathrm{FV}(t) \subseteq \mathsf{D}$.

We introduce the following relation $<$ on def-terms.

$$\Delta \shortmid t \;<\; \Delta, x := s \shortmid t \qquad \text{if } x \notin \mathrm{FV}(t)$$

We write $\leq$ for the reflexive and transitive closure of $<$. We associate a set of def-terms for each type and sequence of definitions as follows.

$$[o]_\Delta \triangleq \{\Delta \shortmid t \mid \mathrm{FV}(t) \subseteq \mathsf{D} \text{ and } \Delta \shortmid t \text{ is semi SN}\}$$

$$[A{\to}B]_\Delta \triangleq \left\{ \Delta \shortmid t \;\middle|\; \begin{array}{l} \mathrm{FV}(t) \subseteq \mathsf{D},\ \Delta \text{ is semi SN, and} \\ \text{for all } \Sigma \text{ such that } \Delta \shortmid t \leq \Sigma \shortmid t \text{ and } \Sigma \text{ is semi SN,} \\ \text{and for all } \Sigma \shortmid s \in [A]_\Sigma, \\ \Sigma \shortmid ts \in [B]_\Sigma \end{array} \right\}$$

We say $\Delta \shortmid t$ is *reducible* if $\Delta \shortmid t \in [A]_\Delta$ for some A.

In the following, we prove that a reducible def-term is semi SN and that a typable term is reducible. First, we prove the former part (Proposition 1).

Proposition 1. *(1) If $\Delta \shortmid t \in [C]_\Delta$, then $\Delta \shortmid t$ is semi SN.*
(2) Let $\Delta \equiv y_1{:=}r_1, \ldots, y_m{:=}r_n$. If $\Delta \shortmid s_1$, …, $\Delta \shortmid s_n$ $(n \geq 0)$, and Δ are semi SN and $x \notin \{y_1, \ldots, y_m\}$, then $\Delta \shortmid xs_1 \cdots s_n \in [C]_\Delta$.

Proof. We prove (1) and (2) simultaneously by induction on C.

1. $C \equiv o$: (1) Clear. (2) Since $\Delta \shortmid xs_1 \cdots s_n$ is semi SN by Lemma 1, $\Delta \shortmid xs_1 \cdots s_n \in [o]_\Delta$.

2. $C \equiv A \to B$: (1) Take a new variable y. Then, $\Delta \mid y \in [A]_\Delta$ by induction hypothesis. So, we have $\Delta \mid ty \in [B]_\Delta$. By induction hypothesis, $\Delta \mid ty$ is semi SN. Since a reduction sequence starting from $\Delta \mid t$ can be lifted to a reduction sequence starting from $\Delta \mid ty$, $\Delta \mid t$ is semi SN. (2) Take any Σ such that $\Delta \mid xs_1 \cdots s_n \leq \Sigma \mid xs_1 \cdots s_n$ and Σ is semi SN, and take any $\Sigma \mid r \in [A]_\Sigma$. By induction hypothesis, $\Sigma \mid r$ is semi SN. By Lemma 14, $\Sigma \mid s_1, \ldots, \Sigma \mid s_n$ are semi SN. Therefore, $\Sigma \mid xs_1 \cdots s_n r \in [B]_\Sigma$ by induction hypothesis. □

Next, we prove that a typable term is reducible (Theorem 1).

Lemma 15. *If $\Delta \mid t \in [C]_\Delta$, $\Delta \mid t \leq \Sigma \mid t$, and Σ is semi SN, then $\Sigma \mid t \in [C]_\Sigma$.*

Proof. By induction on type C, using Lemma 14 in the base case. □

Lemma 16. *If $\Delta \mid t \in [C]_\Delta$ and $t \to t'$, then $\Delta \mid t' \in [C]_\Delta$.*

Proof. By induction on type C. □

Lemma 17. *If $\Delta, x := s \mid ts_1 \cdots s_n \in [C]_{\Delta, x:=s}$ and $x \notin \bigcup_{i=1}^n \mathrm{FV}(s_i)$, then $\Delta \mid (t\{x{:=}s\})s_1 \cdots s_n \in [C]_\Delta$, therefore, $\Delta \mid (t\langle x{:=}s\rangle)s_1 \cdots s_n \in [C]_\Delta$.*

Proof. We prove the first part of the lemma by induction on type C.

1. $C \equiv o$: By Proposition 1, $\Delta, x := s \mid ts_1 \cdots s_n$ is semi SN. We have $\Delta, x := s \mid ts_1 \cdots s_n \xrightarrow{*} \Delta \mid (t\{x{:=}s\})(s_1\{x{:=}s\}) \cdots (s_n\{x{:=}s\}) \xrightarrow{*} \Delta \mid (t\{x{:=}s\})s_1 \cdots s_n$, since $x \notin \bigcup_{i=1}^n \mathrm{FV}(s_i)$. Therefore, $\Delta \mid (t\{x{:=}s\})s_1 \cdots s_n$ is semi SN.
2. $C \equiv A \to B$: We put $t' \equiv (t\{x{:=}s\})s_1 \cdots s_n$. Take any Σ such that $\Delta \mid t' \leq \Sigma \mid t'$ and Σ is semi SN, and take any $\Sigma \mid r \in [A]_\Sigma$. We can take $y \in \mathsf{D}$ such that $y \notin \bigcup_{i=1}^n \mathrm{FV}(s_i) \cup \mathrm{FV}(r)$. Then, by Lemma 15, we have $\Sigma, y := s \mid r \in [A]_{\Sigma, y:=s}$. Then, we have $\Sigma, y := s \mid t_x^y s_1 \cdots s_n r \in [B]_{\Sigma, y:=s}$. Therefore, $\Sigma \mid t'r \in [B]_\Sigma$ by induction hypothesis.

Therefore, $\Delta \mid (t\{x{:=}s\})s_1 \cdots s_n \in [C]_\Delta$. By Lemma 16, $\Delta \mid (t\langle x{:=}s\rangle)s_1 \cdots s_n \in [C]_\Delta$. □

Here is a subtle point. We say t *is inner* if t is a ds-free term such that $x \in \mathsf{V}$ for every substitution $\langle x{:=}s\rangle$ in t.

Lemma 18. *Let t be an inner term and suppose $x \in \mathsf{D}$ and $z \in \mathsf{V}$. If $\Delta \mid (t\langle x{:=}s\rangle)s_1 \cdots s_n \in [C]_\Delta$, then $\Delta \mid (t_x^z\langle z{:=}s\rangle)s_1 \cdots s_n \in [C]_\Delta$.*

Proof. Since $t_x^z\langle z{:=}s\rangle$ satisfies the term formation condition and the conditions for the application of reduction rules does not change, we have the result when $C \equiv o$. The rest of the lemma can be proved by induction on type C. □

Lemma 19. *If $\Delta \mid (t\langle x{:=}s\rangle)s_1 \cdots s_n \in [C]_\Delta$, then $\Delta \mid (\lambda x.\, t)ss_1 \cdots s_n \in [C]_\Delta$.*

Proof. By induction on type C, using Lemma 3 in the base case. □

Lemma 20. *Let $\Delta \equiv \Delta_1, x := s, \Delta_2$. If $\Delta \shortmid s_1$, ..., $\Delta \shortmid s_n$ $(n \geq 0)$, Δ are semi SN and $\Delta_1 \shortmid s$ is reducible, then $\Delta \shortmid x s_1 \cdots s_n \in [C]_\Delta$.*

Proof. Since $\Delta_1 \shortmid s$ is semi SN by Proposition 1, it is easy to see that $\Delta_1, x := s \shortmid x$ is semi SN. Therefore, $\Delta \shortmid x$ is semi SN by Lemma 14. Now, suppose $\Delta \shortmid x s_1 \cdots s_n$ is not semi SN. By Lemma 1, we have two cases.

1. one of $\Delta \shortmid x$, $\Delta \shortmid s_1$, ..., $\Delta \shortmid s_n$ is not semi SN: Contradiction.
2. there is an infinite reduction sequence that begins with $\gamma_1, \ldots, \gamma_m, (\boldsymbol{\lambda}|\sigma)$ where $(\boldsymbol{\lambda}|\sigma)$ is the first $(\boldsymbol{\lambda})$ applied at n-applicative position: To apply $(\boldsymbol{\lambda})$ at n-applicative position, x should be instantiated, that is, $\gamma_i : \Delta' \shortmid t_\sigma[x\langle x{:=}s'\rangle] \to \Delta' \shortmid t_\sigma[s']$ for some i where σ is an n-applicative position and $\Delta \shortmid s \xrightarrow{*} \Delta' \shortmid s'$. Since $\Delta_1 \shortmid s$ is reducible, $\Delta \shortmid s'$ is reducible by Lemma 15, 16, and 17. Therefore, $\Delta \shortmid s' s_1 \cdots s_n$ is semi SN. By applying the reduction sequence γ_1, ..., γ_{i-1} to $\Delta \shortmid s' s_1 \cdots s_n$, we have $\Delta' \shortmid t_\sigma[s']$. Contradiction.

Therefore, $\Delta \shortmid x s_1 \cdots s_n$ is semi SN. The rest of the lemma can be proved by induction on type C. □

We can assume an injective map $\iota : \mathsf{V} \to \mathsf{D}$ and write $\widehat{x}$ for $\iota(x)$. We write $\widehat{t}$ for a term obtained by renaming each free occurrence of $x \in \mathsf{V}$ in t by $\widehat{x}$. Let $\Delta \equiv \widehat{x_1}{:=}s_1, \ldots, \widehat{x_n}{:=}s_n$ be a definition sequence and $\Gamma \equiv x_{i_1} : A_{i_1}, \ldots, x_{i_m} : A_{i_m}$ $(1 \leq i_1 < \cdots < i_m \leq n)$ be a declaration sequence. We say Δ is a *reducible instance* of Γ if $\Delta' \shortmid s_{i_j} \in [A_{i_j}]_{\Delta'}$ $(\Delta' \equiv \Delta|_{i_j - 1})$ for each j in $1..m$.

Theorem 1. *Let Δ be a definition sequence and suppose $\Gamma \vdash r : C$ is derivable. If Δ is a reducible instance of Γ, then $\Delta \shortmid \widehat{r} \in [C]_\Delta$.*

Proof. By induction on the derivation of $\Gamma \vdash r : C$.

1. (var): By Proposition 1 or Lemma 20.
2. (abs): In this case, $r \equiv \lambda x.\, t$ and $C \equiv A{\to}B$. Take any Δ' such that $\Delta \shortmid \widehat{\lambda x.\, t} \leq \Delta' \shortmid \widehat{\lambda x.\, t}$ and Δ' is semi SN, and take any $\Delta' \shortmid s \in [A]_{\Delta'}$. Since $\Delta', \widehat{x}{:=}s$ is a reducible instance of $\Gamma, x{:}A$, we have $\Delta', \widehat{x}{:=}s \shortmid \widehat{t} \in [B]_{\Delta', \widehat{x}:=s}$ by induction hypothesis. Since $\widehat{t}$ is inner, we have $\Delta' \shortmid \widehat{(\lambda x.\, t)} s \in [B]_{\Delta'}$ by Lemma 17, 18, and 19. Therefore, $\Delta \shortmid \widehat{\lambda x.\, t} \in [A{\to}B]_\Delta$.
3. (app): In this case, $r \equiv ts$ and $C \equiv A$. By induction hypothesis, $\Delta \shortmid \widehat{t} \in [A{\to}B]_\Delta$ and $\Delta \shortmid \widehat{s} \in [A]_\Delta$. Therefore, $\Delta \shortmid \widehat{ts} \in [B]_\Delta$.
4. (xsub): In this case, $r \equiv t\langle x{:=}s\rangle$ and $C \equiv B$. Since $\Delta \shortmid \widehat{s} \in [A]_\Delta$ by induction hypothesis, $\Delta, \widehat{x}{:=}\widehat{s}$ is a reducible instance of $\Gamma, x{:}A$. So, $\Delta, \widehat{x}{:=}\widehat{s} \shortmid \widehat{t} \in [B]_{\Delta, \widehat{x}:=\widehat{s}}$ by induction hypothesis. Since $\widehat{t}$ is inner, we have $\Delta \shortmid \widehat{t\langle x{:=}s\rangle} \in [B]_\Delta$ by Lemma 17 and 18. □

Corollary 1. *If t is typable in λx, then t is semi SN, therefore, t is SN with respect to λx + (comp).*

5 Intersection Type System

We extend the type system λx by introducing the intersection type $A \cap B$ and adding the following typing rules.

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash t : B}{\Gamma \vdash t : A \cap B} \qquad \frac{\Gamma \vdash t : A \cap B}{\Gamma \vdash t : A} \qquad \frac{\Gamma \vdash t : A \cap B}{\Gamma \vdash t : B}$$

But in this section, we regard Γ in a typing judgement $\Gamma \vdash t : A$ as a finite set of declarations whose variables are different each other. We call this type system $\lambda x \cap$, and call the type system $\lambda x \cap - (\mathsf{xsub})$ (where terms are restricted to pure λ-terms) $\lambda \cap$.

Theorem 2 (PSN). *Let t be a pure λ-term. If t is SN with respect to β-rule, then t is semi SN, therefore, t is SN with respect to $\lambda x + (\mathsf{comp})$.*

Proof. By the well-known result (e.g. [11]), t is typable in $\lambda \cap$. So, t is typable in $\lambda x \cap$. By defining $[A \cap B]_\Delta \triangleq [A]_\Delta \cap [B]_\Delta$, we can extend Corollary 1 to $\lambda x \cap$. Therefore, t is semi SN. □

Lengrand et al. [9] have shown that strong normalizability with respect to λx is characterized by a type system $\lambda x \cap$ extended with a rule called (drop). By adjusting their proof method to our case, we will show that semi SN is characterized by $\lambda x \cap$.

We define preorder $\leq$ on types as the smallest reflexive and transitive relation that satisfies the following.

1. $A \cap B \leq A$
2. $A \cap B \leq B$
3. if $C \leq A$ and $C \leq B$, then $C \leq A \cap B$

We define relation $\sim$ on types as follows: $A \sim B$ iff $A \leq B$ and $B \leq A$. We define relation $\leq$ on finite sets of declarations as follows: $\Gamma \leq \Gamma'$ iff

for each $x{:}A' \in \Gamma'$, there exists A such that $x{:}A \in \Gamma$ and $A \leq A'$.

We define $\Gamma_1 \cap \Gamma_2 \triangleq \{x{:}A_1 \cap A_2 \mid x{:}A_1 \in \Gamma_1,\ x{:}A_2 \in \Gamma_2\} \cup \{x{:}A \mid x{:}A \in \Gamma_1,\ x{:}A \notin \Gamma_2\} \cup \{x{:}A \mid x{:}A \notin \Gamma_1,\ x{:}A \in \Gamma_2\}$.

Lemma 21. *If $\Gamma' \leq \Gamma$, $A \leq A'$, and $\Gamma \vdash t : A$, then $\Gamma' \vdash t : A'$.*

As usual, we can prove the Generation Lemma.

Lemma 22 (Generation Lemma).
1. $\Gamma \vdash x : A$ iff there exists $x{:}A' \in \Gamma$ such that $A' \leq A$.
2. $\Gamma \vdash ts : B$ iff there exist $A_1, \ldots, A_n, B_1, \ldots, B_n$ such that $B \geq B_1 \cap \cdots \cap B_n$, $\Gamma \vdash t : A_i \to B_i$, and $\Gamma \vdash s : A_i$ $(1 \leq i \leq n)$.
3. $\Gamma \vdash \lambda x.t : C$ iff there exist $A_1, \ldots, A_n, B_1, \ldots, B_n$ such that $C \sim (A_1 \to B_1) \cap \cdots \cap (A_n \to B_n)$ and $\Gamma, x{:}A_i \vdash t : B_i$ $(1 \leq i \leq n)$.
4. $\Gamma \vdash t\langle x{:=}s\rangle : B$ iff there exists A such that $\Gamma \vdash s : A$ and $\Gamma, x{:}A \vdash t : B$.

Lemma 23. *If $s_1, \dots, s_n$ are typable, then for any type A there exists Γ such that $\Gamma \vdash xs_1 \cdots s_n : A$.*

Proof. Suppose $\Gamma_1 \vdash s_1 : A_1, \dots, \Gamma_n \vdash s_n : A_n$. Then, we have $(\cap_{i=1}^{n} \Gamma_i) \cap x{:}A_1 \to \cdots \to A_n \to A \vdash xs_1 \cdots s_n : A$. □

Lemma 24. *Let t be a β-normal pure λ-term.*
1. If t is not an abstraction, then for any type A there exists Γ such that $\Gamma \vdash t : A$ in $\lambda\cap$.
2. If t is an abstraction, then t is typable in $\lambda\cap$.

Proof. By induction on t. □

We introduce the following rules.

$$\begin{array}{rl}
(\mathsf{var^c}) & x\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle\langle x{:=}s\rangle \to s\langle y_n{:=}r_n\lceil x{:=}s\rceil\rangle \cdots \langle y_1{:=}r_1\lceil x{:=}s\rceil\rangle \\
(\mathsf{gc^c}) & t\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle\langle x{:=}s\rangle \to t\langle y_n{:=}r_n\lceil x{:=}s\rceil\rangle \cdots \langle y_1{:=}r_1\lceil x{:=}s\rceil\rangle \\
 & \text{if } x \notin \mathrm{FV}(t)
\end{array}$$

where $t\lceil x{:=}s\rceil$ is defined as follows.

1. $t\lceil x{:=}s\rceil \triangleq t\langle x{:=}s\rangle$ if $x \in \mathrm{FV}(t)$
2. $t\lceil x{:=}s\rceil \triangleq t$ if $x \notin \mathrm{FV}(t)$

Then, we define a strategy $\rightsquigarrow$ for $\lambda\mathsf{x} + (\mathsf{var^c}) + (\mathsf{gc^c})$ as follows. (In the following, $\mathcal{C}[\ ]$ and $\mathcal{A}[\ ]$ are contexts of the form $\mathcal{C}[\ ] \equiv ([\ ]\langle x_n{:=}s_n\rangle \cdots \langle x_1{:=}s_1\rangle)t_1 \cdots t_m$ and $\mathcal{A}[\ ] \equiv [\ ]t_1 \cdots t_m$ and $nf(s)$ means that s is normal with respect to $\rightsquigarrow$.)

1. $\lambda x.\, t \rightsquigarrow \lambda x.\, t'$ if $t \rightsquigarrow t'$
2. $\mathcal{A}[xs_1 \cdots s_n t] \rightsquigarrow \mathcal{A}[xs_1 \cdots s_n t']$ if $nf(xs_1 \cdots s_n)$ and $t \rightsquigarrow t'$
3. $\mathcal{A}[(\lambda x.\, t)s] \rightsquigarrow \mathcal{A}[t\langle x{:=}s\rangle]$
4. $\mathcal{C}[(\lambda y.\, t)\langle x{:=}s\rangle] \rightsquigarrow \mathcal{C}[\lambda y.\, t\langle x{:=}s\rangle]$
5. $\mathcal{C}[(tr)\langle x{:=}s\rangle] \rightsquigarrow \mathcal{C}[(t\langle x{:=}s\rangle)(r\langle x{:=}s\rangle)]$
6. $\mathcal{A}[x\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle\langle x{:=}s\rangle] \rightsquigarrow \mathcal{A}[s\langle y_n{:=}r_n\lceil x{:=}s\rceil\rangle \cdots \langle y_1{:=}r_1\lceil x{:=}s\rceil\rangle]$
7. $\mathcal{A}[z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle\langle x{:=}s\rangle] \rightsquigarrow \mathcal{A}[z\langle y_n{:=}r_n\lceil x{:=}s\rceil\rangle \cdots \langle y_1{:=}r_1\lceil x{:=}s\rceil\rangle]$
   if $x \not\equiv z$ and $x \in \mathrm{FV}(z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle)$
8. $\mathcal{A}[z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle\langle x{:=}s\rangle] \rightsquigarrow \mathcal{A}[z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle]$
   if $x \notin \mathrm{FV}(z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle)$ and $nf(s)$
9. $\mathcal{A}[z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle\langle x{:=}s\rangle] \rightsquigarrow \mathcal{A}[z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle\langle x{:=}s'\rangle]$
   if $x \notin \mathrm{FV}(z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle)$ and $s \rightsquigarrow s'$

Lemma 25 (Subject Expansion). *If $t \rightsquigarrow s$ and $\Gamma \vdash s : A$,*

$$\text{then} \begin{cases} \Gamma' \vdash t : A \ \text{ for some } \Gamma' \leq \Gamma & \text{if } t \text{ is not an abstraction} \\ \Gamma' \vdash t : A' \ \text{ for some } \Gamma' \leq \Gamma \text{ and } A' & \text{if } t \text{ is an abstraction} \end{cases}$$

Proof. By considering the derivation of $\mathcal{A}[s]$ or $\mathcal{C}[s]$, we can easily prove the following two claims:

Suppose that, for any Γ and A such that $\Gamma \vdash s : A$, there exists $\Gamma' \leq \Gamma$ such that $\Gamma' \vdash t : A$. If $\Gamma \vdash \mathcal{A}[s] : C$, then there exists $\Gamma' \leq \Gamma$ such that $\Gamma' \vdash \mathcal{A}[t] : C$.

and

Suppose that $\Gamma \vdash s : A$ implies $\Gamma \vdash t : A$ for any Γ and A. If $\Gamma \vdash \mathcal{C}[s] : C$, then $\Gamma \vdash \mathcal{C}[t] : C$.

We prove the lemma by induction on the definition of $\rightsquigarrow$. The base cases are 3–7. First, we prove these base cases when $\mathcal{A}[\,] \equiv \mathcal{C}[\,] \equiv [\,]$.

3. Suppose $\Gamma \vdash t\langle x{:=}s\rangle : B$. By Lemma 22, we have $\Gamma \vdash s : A$ and $\Gamma, x{:}A \vdash t : B$ for some A. Therefore, $\Gamma \vdash (\lambda x.\, t)s : B$
4. Suppose $\Gamma \vdash \lambda y.\, t\langle x{:=}s\rangle : C$. By Lemma 22, there exist $A_1, \ldots, A_n, B_1, \ldots, B_n$ such that $C \sim (A_1{\to}B_1) \cap \cdots \cap (A_n{\to}B_n)$ and $\Gamma, y{:}A_i \vdash t\langle x{:=}s\rangle : B_i$ $(1 \leq i \leq n)$. So, by Lemma 22, there exists C_i such that $\Gamma, y{:}A_i \vdash s : C_i$ and $\Gamma, y{:}A_i, x{:}C_i \vdash t : B_i$. Since $y \notin \mathrm{FV}(s)$, we have $\Gamma \vdash s : C_i$. Hence, $\Gamma \vdash (\lambda y.\, t)\langle x{:=}s\rangle : A_i{\to}B_i$. Therefore, $\Gamma \vdash (\lambda y.\, t)\langle x{:=}s\rangle : C$.
5. Suppose $\Gamma \vdash (t\langle x{:=}s\rangle)(r\langle x{:=}s\rangle) : B$. By Lemma 22, there exist $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ such that $B \geq B_1 \cap \cdots \cap B_n$, $\Gamma \vdash t\langle x{:=}s\rangle : A_i{\to}B_i$, and $\Gamma \vdash r\langle x{:=}s\rangle : A_i$ $(1 \leq i \leq n)$. So, by Lemma 22, there exist C_i, C_i' such that $\Gamma \vdash s : C_i$, $\Gamma, x{:}C_i \vdash t : A_i{\to}B_i$, $\Gamma \vdash s : C_i'$, and $\Gamma, x{:}C_i' \vdash r : A_i$. We put $D_i \equiv C_i \cap C_i'$. Then, by Lemma 21, we have $\Gamma \vdash s : D_i$, $\Gamma, x{:}D_i \vdash t : A_i{\to}B_i$, and $\Gamma, x{:}D_i \vdash r : A_i$. Hence, $\Gamma \vdash (tr)\langle x{:=}s\rangle : B_i$. Therefore, $\Gamma \vdash (tr)\langle x{:=}s\rangle : B$.
6. Suppose $\Gamma \vdash s\langle y_n{:=}r_n\lceil x{:=}s\rceil\rangle \cdots \langle y_1{:=}r_1\lceil x{:=}s\rceil\rangle : B$. By Lemma 22, there exist $A_1, \ldots, A_n$ such that $\Gamma, y_1{:}A_1, \ldots, y_{i-1}{:}A_{i-1} \vdash r_i\lceil x{:=}s\rceil : A_i$ $(1 \leq i \leq n)$ and $\Gamma, y_1{:}A_1, \ldots, y_n{:}A_n \vdash s : B$. We put $I \equiv \{i \mid x \in \mathrm{FV}(r_i)\}$. If $i \in I$, there exists B_i such that $\Gamma, y_1{:}A_1, \ldots, y_{i-1}{:}A_{i-1} \vdash s : B_i$ and $\Gamma, y_1{:}A_1, \ldots, y_{i-1}{:}A_{i-1}, x{:}B_i \vdash r_i : A_i$. Since $\{y_1, \ldots, y_n\} \cap \mathrm{FV}(s) = \emptyset$, we have $\Gamma \vdash s : B$. We put $C \equiv (\cap_{i \in I} B_i) \cap B$ if $I \neq \emptyset$ and $C \equiv B$ otherwise. Then, we have $\Gamma \vdash s : C$. Hence, $\Gamma \vdash x\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle\langle x{:=}s\rangle : C$. Therefore, $\Gamma \vdash x\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle\langle x{:=}s\rangle : B$.
7. Suppose $\Gamma \vdash z\langle y_n{:=}r_n\lceil x{:=}s\rceil\rangle \cdots \langle y_1{:=}r_1\lceil x{:=}s\rceil\rangle : B$. By Lemma 22, there exist $A_1, \ldots, A_n$ such that $\Gamma, y_1{:}A_1, \ldots, y_{i-1}{:}A_{i-1} \vdash r_i\lceil x{:=}s\rceil : A_i$ $(1 \leq i \leq n)$ and $\Gamma, y_1{:}A_1, \ldots, y_n{:}A_n \vdash z : B$. We put $I \equiv \{i \mid x \in \mathrm{FV}(r_i)\}$. Then, $I \neq \emptyset$. If $i \in I$, there exists B_i such that $\Gamma, y_1{:}A_1, \ldots, y_{i-1}{:}A_{i-1} \vdash s : B_i$ and $\Gamma, y_1{:}A_1, \ldots, y_{i-1}{:}A_{i-1}, x{:}B_i \vdash r_i : A_i$. Since $\{y_1, \ldots, y_n\} \cap \mathrm{FV}(s) = \emptyset$, we have $\Gamma \vdash s : B_i$. We put $C \equiv \cap_{i \in I} B_i$. Then, we have $\Gamma \vdash s : C$. Therefore, $\Gamma \vdash z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle\langle x{:=}s\rangle : B$.

By the above two claims, we have the result in the base case. For the induction step, we first prove when $\mathcal{A}[\,] \equiv [\,]$.

1. Suppose $\Gamma \vdash \lambda x.\, t' : C$. By Lemma 22, there exist $A_1, \ldots, A_n, B_1, \ldots, B_n$ such that $C \sim (A_1{\to}B_1) \cap \cdots \cap (A_n{\to}B_n)$ and $\Gamma, x{:}A_i \vdash t' : B_i$ $(1 \leq i \leq n)$. By induction hypothesis, we have $\Gamma', x{:}A_1' \vdash t : B_1'$ for some $\Gamma', x{:}A_1' \leq \Gamma, x{:}A_1$ and B_1'. Therefore, $\Gamma' \vdash \lambda x.\, t : A_1'{\to}B_1'$.

2. Suppose $\Gamma \vdash xs_1 \cdots s_n t' : B$. By Lemma 22, there exist $A'_1, \ldots, A'_n, B_1, \ldots, B_n$ such that $B \geq B_1 \cap \cdots \cap B_n$, $\Gamma \vdash xs_1 \cdots s_n : A'_i \to B_i$, and $\Gamma \vdash t' : A'_i$ $(1 \leq i \leq n)$. By induction hypothesis, we have $\Gamma_i \vdash t : A_i$ for some $\Gamma_i \leq \Gamma$ and A_i. By Lemma 23, there exists Γ'_i such that $\Gamma'_i \vdash xs_1 \cdots s_n : A_i \to B_i$. Hence, $\Gamma_i \cap \Gamma'_i \vdash xs_1 \cdots s_n t : B_i$. Therefore, $\cap_{i=1}^{n}(\Gamma_i \cap \Gamma'_i) \vdash xs_1 \cdots s_n t : B$.
8. Suppose $\Gamma \vdash z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle : B$. Since $nf(s)$, $\Gamma' \vdash s : C$ for some Γ' and C by Lemma 24. Therefore, $\Gamma \cap \Gamma' \vdash z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle\langle x{:=}s\rangle : B$ since $x \notin \mathrm{FV}(z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle)$.
9. Suppose $\Gamma \vdash z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle\langle x{:=}s'\rangle : B$. By Lemma 22, there exists A' such that $\Gamma \vdash s' : A'$ and $\Gamma, x{:}A' \vdash z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle : B$. By induction hypothesis, we have $\Gamma' \vdash s : A$ for some $\Gamma' \leq \Gamma$ and A. Since $x \notin \mathrm{FV}(z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle)$, we have $\Gamma, x{:}A \vdash z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle : B$. Therefore, $\Gamma \cap \Gamma' \vdash z\langle y_n{:=}r_n\rangle \cdots \langle y_1{:=}r_1\rangle\langle x{:=}s\rangle : B$.

We are done by applying the first claim to the above cases other than case 1. □

Theorem 3. *Let t be an original λx-term. Then, the followings are equivalent.*
(1) t is semi SN.
(2) t is SN with respect to $\lambda\mathrm{x} + (\mathsf{var}^\mathsf{c}) + (\mathsf{gc}^\mathsf{c})$.
(3) t is typable in $\lambda\mathrm{x}\cap$.

Proof. (1)⇒(2): Since $(\mathsf{var}^\mathsf{c})$ (resp. (gc^c)) is decomposed into a several applications of (comp) or (perm) followed by (var) (resp. (gc)), a reduction sequence that consists of $\lambda\mathrm{x} + (\mathsf{var}^\mathsf{c}) + (\mathsf{gc}^\mathsf{c})$ can be converted to a non-permutative sequence.
(2)⇒(3): Since $\rightsquigarrow$ is a strategy for $\lambda\mathrm{x} + (\mathsf{var}^\mathsf{c}) + (\mathsf{gc}^\mathsf{c})$, there is a normal form of t with respect to $\rightsquigarrow$. Because the normal form is a pure λ-term, it is typable in $\lambda\cap$ by Lemma 24. Therefore, t is typable in $\lambda\mathrm{x}\cap$ by Lemma 25.
(3)⇒(1): By Theorem 2. □

6 Conclusion

In this paper, we have developed a novel method for proving the strong normalizability of simply typed λx with a composition rule. Using this method, we have proved the new result. Our composition rule is the first full composition rule in λx that is controlled by a very simple condition. The characteristic feature of our calculus is that we can freely push a substitution into a term in our calculus, which is justified by the notion of semi SN.

We believe we can apply our method to other calculi of explicit substitutions and simplify the proof of SN or introduce a composition rule.

We also remark that some part of our idea has evolved from our proof of SN of $\lambda\varepsilon$ [12]. The proof is very complicated, but now we can simplify it using the method developed in this paper.

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