$\lambda\rho$-Calculus: A Natural Deduction for Classical Logic

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In this note, we will pose a new system, named $\lambda\rho$–calculus. While the type assignment system TA_λ gives a natural deduction for implicational intuitionistic logic, the type assignment system $\mathrm{TA}_{\lambda\rho}$ gives a natural deduction for classical implicational logic. Moreover for any classical implicational theorem α there exists a proof of α in $\mathrm{TA}_{\lambda\rho}$ enjoying the subformula property.

1 The type free $\lambda\rho$-calculus

1.1 Definition ($\lambda\rho$-terms). An infinite sequence of ***λ-variables*** is assumed to be given, and an infinite sequence of ***ρ-variables*** is assumed to be given. Then linguistic expressions called ***$\lambda\rho$-terms*** are defined thus:

1. each λ-variable is a $\lambda\rho$-terms, called an ***atom*** or ***atomic term***;
2. if M and N are $\lambda\rho$-terms then (MN) is a $\lambda\rho$-term called an ***application***;
3. if M is a $\lambda\rho$-term and a is a ρ-variable then aM is a $\lambda\rho$-term called an ***absurd***;
4. if f is a λ-variable or a ρ-variable and M is a $\lambda\rho$-term then $(\lambda f.M)$ is a $\lambda\rho$-term called an ***abstract***.

1.1.1 Notation. *λ-variables* are denoted by "u","v","w","x", "y","z", with or without number-subscripts. *ρ-variables* are denoted by "a","b","c","d", with or without number-subscripts. *A term-variable* means a λ-variable or a ρ-variable. *Term variables* are denoted by "f","g","h", with or without number-subscripts. Distinct letters denotes distinct variables unless otherwise stated.

A term $\lambda a.M$ is sometimes denoted by $\rho a.M$ if the variable a is a ρ-variable.

Arbitrary $\lambda\rho$-terms are denoted by "L","M","N","P","Q", "R","S","T", with or without number-subscripts. For "*$\lambda\rho$-term*" we shall say just "*term*".

1.2 Definition (Free, bound). A non-binding variable occurence $\underline{f}$ in a term M is said to be ***bound in*** M iff it is in the scope of an occurrence of λf in M, otherwise it is ***free in*** M.

A variable f is said to be ***bound in*** M iff M contains an occurence of λf; and f is said to be ***free in*** M iff M contains a free occurence of f. The set of all variables free in M is called

$$FV(M).$$

1.3 Definition ($\rho\beta$-reduction). Any $\lambda\rho$-term of form aMN, $(\lambda x.M)N$, $(\lambda a.M)N$ is called a *$\rho\beta$-redex*. *$\rho\beta$-contracting an occurrence of a redex in a term U* means replacing one occurence of

$$\begin{array}{lll} aMN & \text{by} & aM\ , \\ (\lambda x.M)N & \text{by} & [N/x]M\ , \\ (\lambda a.M)N & \text{by} & \lambda a.([\lambda x.a(xN)/a]M)N\ . \end{array}$$

If this changes U to V, we say U *$\rho\beta$-contracts to V*, or

$$U \rhd_{1\rho\beta} V.$$

We say U *$\rho\beta$-reduces to* V, or

$$U \rhd_{\rho\beta} V,$$

iff V is obtained from U by a finite (perhaps empty) series of $\rho\beta$-contractions and changes of bound variables.

1.4 Theorem ((Church-Rosser Theorem for $\rho\beta$)). *If $M \rhd_{\rho\beta} P$ and $M \rhd_{\rho\beta} Q$ then there exist T such that*

$$P \rhd_{\rho\beta} T, \quad Q \rhd_{\rho\beta} T.$$

2 Assigning types to terms

Types are defined as usual.

2.1 Definition. A ***type-assignment*** is any expression

$$M : \tau$$

where M is a $\lambda\rho$-term or a ρ-variable and τ is a type; we call M its ***subject*** and τ its ***predicate***.

2.2 Definition. A ***type-context*** Γ is any finite, perhaps empty, set of type-assignments

$$\Gamma = \{f_1 : \rho_1,\ \ldots,\ f_m : \rho_m\}$$

whose subjects are term-variables which is ***consistent*** in the sense that no term-variable is the subject of more than one assignment. For any such Γ define

$$Subjects(\Gamma) = \{f_1, \ldots, f_m\}.$$

2.2.1 Notation. The result of removing from Γ the assignment whose subject is f (if Γ has one) is called

$$\Gamma - f.$$

(If $f \notin Subjects(\Gamma)$ we define $\Gamma - f = \Gamma$.)

Γ_λ and Γ_ρ denote the subsets of Γ of all assignments whose subjects are λ-variables and ρ-variables respectively.

2.2.2 Note. A type-context Γ is a set, not a sequence. Hence it does not change when its members are permuted or repeated.

2.3 Definition. We say Γ_1 ***is consistent with*** Γ_2 iff $\Gamma_1 \cup \Gamma_2$ is consistent; and $\Gamma_1, \ldots, \Gamma_n$ are ***mutually consistent*** if their union is consistent.

2.4 Definition ($\mathbf{TA}_{\lambda\rho}$-formulae). For any type-context Γ and any type-assignment $M : \tau$, the pair $\langle \Gamma; M : \tau \rangle$ is called ***$TA_{\lambda\rho}$-formula*** and is written as

$$\Gamma_\lambda \mapsto \Gamma_\rho,\ M : \tau$$

(or just $\mapsto M : \tau$ when Γ is empty). We shall call M the ***subject*** of this formula and τ its ***predicate***.

2.4.1 Notation. The following abbreviations will often be used:

$$f_1 : \sigma_1, \ldots, f_n : \sigma_n \quad for \quad \{f_1 : \sigma_1, \ldots, f_n : \sigma_n\},$$

$$\Gamma, f_1 : \sigma_1, \ldots, f_n : \sigma_n \quad for \quad \Gamma \cup \{f_1 : \sigma_1, \ldots, f_n : \sigma_n\}.$$

2.5 Definition (The system $\mathbf{TA}_{\lambda\rho}$). $\mathrm{TA}_{\lambda\rho}$ has an infinite set of axioms and four deduction-rules (called ($\to$E) or $\to$-*elimination*, ($\to$I) or $\to$-*introduction*, (Absurd) and (Rati)), as follows.

Axioms of TA$_{\lambda\rho}$: for every term-variable x and every type τ, $\mathrm{TA}_{\lambda\rho}$ has an axiom

$$x : \tau \mapsto x : \tau.$$

Deduction-rules of TA$_{\lambda\rho}$:

$$\frac{\langle \Gamma_1; P : \sigma \to \tau\rangle \quad \langle \Gamma_2; Q : \sigma\rangle}{\langle \Gamma_1 \cup \Gamma_2; PQ : \tau\rangle}\ (\to E) \qquad [if\ \Gamma_1 \cup \Gamma_2\ is\ consistent],$$

$$\frac{\langle \Gamma; P : \tau\rangle}{\langle \Gamma - x; \lambda x.P : \sigma \to \tau\rangle}\ (\to I) \qquad [if\ \Gamma\ is\ consistent\ with\ x : \sigma],$$

$$\frac{\langle \Gamma; P : \tau\rangle}{\langle \Gamma, a : \tau; aP : \sigma\rangle}\ (Absurd) \qquad [if\ \Gamma\ is\ consistent\ with\ a : \tau],$$

$$\frac{\langle \Gamma; P : \tau\rangle}{\langle \Gamma - a; \lambda a.P : \tau\rangle}\ (Rati) \qquad [if\ \Gamma\ is\ consistent\ with\ a : \tau].$$

2.5.1 Example $(((\alpha \to \beta) \to \alpha) \to \alpha)$.

$$\dfrac{\dfrac{y : (\alpha \to \beta) \to \alpha \mapsto y : (\alpha \to \beta) \to \alpha \quad \dfrac{\dfrac{x : \alpha \mapsto x : \alpha}{x : \alpha \mapsto a : \alpha, ax : \beta}\ (Absurd)}{\mapsto a : \alpha, \lambda x.ax : \alpha \to \beta}\ (\to I)}{y : (\alpha \to \beta) \to \alpha \mapsto a : \alpha, y(\lambda x.ax) : \alpha}\ (\to E)}{\dfrac{y : (\alpha \to \beta) \to \alpha \mapsto \rho a.y(\lambda x.ax) : \alpha}{\mapsto \lambda y.\rho a.y(\lambda x.ax) : ((\alpha \to \beta) \to \alpha) \to \alpha}\ (\to I)}\ (Rati)$$

The above $\mathrm{TA}_{\lambda\rho}$-deduction is often written as the below.

$$\dfrac{\dfrac{y : (\alpha \to \beta) \to \alpha \quad \dfrac{\dfrac{x : \alpha}{ax : \beta}\ (Absurd)}{\lambda x.ax : \alpha \to \beta}\ (\to I)}{y(\lambda x.ax) : \alpha}\ (\to E)}{\rho a.y(\lambda x.ax) : \alpha}\ (Rati)}{\lambda y.\rho a.y(\lambda x.ax) : ((\alpha \to \beta) \to \alpha) \to \alpha}\ (\to I)$$

2.6 Definition. Let Γ be a type-context. Iff there is a $\mathrm{TA}_{\lambda\rho}$-deduction of a formula $\langle \Gamma'; M : \tau\rangle$ for some $\Gamma' \subseteq \Gamma$ we shall say

$$\Gamma_\lambda \vdash_{\lambda\rho} \Gamma_\rho, M : \tau.$$

2.7 Theorem (A $\lambda\rho$-term is a proof of classical implicational logic). *α is a classical implicational theorem if and only if there exists a closed $\lambda\rho$-term M such that $\vdash_{\lambda\rho} M : \alpha$.*

2.8 Theorem (Normalization Theorem for $\mathbf{TA}_{\lambda\rho}$). *Every $TA_{\lambda\rho}$-typable term has a $\rho\beta$-normal form.*

2.9 Theorem (Subformula Property for $\mathbf{TA}_{\lambda\rho}$). *Let $\lambda\rho$-term M be a $\rho\beta$-normal form and Δ a $TA_{\lambda\rho}$-deduction of $\langle \Gamma; M : \alpha\rangle$. Then every type in Δ has an occurence in α or in a type in Γ.*

2.10 Theorem (Subject-reduction Theorem). *If $\Gamma \vdash_{\lambda\rho} \Theta,\ P : \tau$ and $P \rhd_{\rho\beta} Q$ then*

$$\Gamma \vdash_{\lambda\rho} \Theta,\ Q : \tau.$$

References

[1] J. Roger Hindley. *Basic Simple Type Theory*, Vol. 42 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1997.

[2] J. Roger Hindley and Jonathan P. Seldin. *Introduction to combinators and λ–calculus*, Vol. 1 of *London Mathematical Society Student Texts*. Cambridge University Press, 1986.

Quantum machanics and Quantum logic

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Introduction

We define a propositional system realizing a physical system. The propositional system is an orthomodular lattice $P(H)$of projections on Hilbert space H, whose non-distributivity is explained by the experiment of electron spin. We consider reals and propositions on the $P(H)$-valued universe $V^{P(H)}$.

1 Propositional system and Hilbert space

In the physical system, the set $\mathcal{L}$ of propositons satisfys the following axoims (C), (P), (A) .

(C) For each proposition $c \in \mathcal{L}$ there exists a unique orthocomplement $c^\perp \in \mathcal{L}$ such that

- (C_1) $c^{\perp\perp} = c$
- (C_2) $c \vee c^\perp = 1$, and $c \wedge c^\perp = 0$
- (C_3) $b < c \Longrightarrow c^\perp < b^\perp$ for $\forall b \in \mathcal{L}$

(P) If $b < c$ are propositions of $\mathcal{L}$, then the sublattice generated by $\{b, b^\perp, c, c^\perp\}$ is a classical propositional system, that is to say, a distributive lattice.

(A) If $b \neq c$ and $b < c$, one say that c *covers* b when

$$b < x < c \implies x = b \text{ or } x = c.$$

An element which covers 0 is called an *atom*.

- (A_1) If b is a proposition different from 0, there exists an atom $p < b$.
- (A_2) If p is an atom and if $p \wedge b = 0$, then $p \vee b$ covers b.

Then, we define follwing difinition.

Definition 1 *A complete lattice satisfying axioms (C),(P),(A) is called a propositional sytem.*

Definition 2 *In a propositional system $\mathcal{L}$, two propositions b and c are said to be compatible if the sublattice generated by $\{b, b^\perp, c, c^\perp\}$ is distributive, and denoted by $b \downharpoonleft_\circ c$.*

Let $P(H)$ be the set of projections of Hilbert space H, which is a propositional system . If $P \in P(H)$, then the eigenvalues of P is 1 or 0. Eigenvalue 1 corresponds to true and 0 to false. If $P_1, P_2 \in P(H)$ and $K = \{x \mid P_1 x = x\}, L = \{x \mid P_2 x = x\}$, we define

1. $P_1 < P_2$ if and only if $K \subset L$
2. $P_1 \vee P_2$ is the projection of H onto $K \vee L$. ($K \vee L$ is a subspace spanned by $K \cup L$.)
3. $P_1 \wedge P_2$ is the projection of H onto $K \cap L$.
4. $P_1^\perp$ is the projection of H onto $K^\perp$. ($P_1^\perp = (I - P_1)$)

Then, $P(H)$ is a complete lattice satisfying axioms (C),(P),(A). In $P(H)$, commutativity is completely equivalent to compatibility.

2 Quantum Logic for electron spin

Now, we consider the electron spin in the logic of $P(H)$. When we measure deflection produced by a strong magnetic feild in a direction orthogonal to the direction of the path of the electron, we have exactly two outcomes. There is a variable called the "spin" variable that is associated with how the electron is deflected by magnets. (We call two outcomes spin-up and spin-down, respectively.) We know that the components of spin can't be determined in two direction simultaneously. For this reason, we must abandon the distributivity axiom which we postulated for classical propositional system.

Next, we shall state the Hilbert space corresponding to the propositional system. Let x-y plane be the plane which is orthogonal to the path of the electron and x axis is the positive horizontal axis, y axis is the vertical axis. For each axis w, let $\theta_{\rm W}$ be the angle measured counterclockwise between the x axis and the w axis. Let $K_{\rm W}$ be a subspace which is corresponding to "w-spin-up" and $K_{\rm W}^{\perp}$ to "w-spin-down".

The 2 dimensional Hilbert space H provides an description of propositional syatem for electron spin as follows; $K_{\rm W}$ is the one-dimensional subspace of H oriented at angle $\theta_w/2$ from the positive horizontal axis and $K_{\rm W}^{\perp}$ is the orthogonal complement of $K_{\rm W}$. The spin state of an electron can be represented by a unit vector ψ in the Hilbert space H. The probability ω of obtaining the outcome "w-spin-up" is the square of the length of the projection of ψ onto the subspace $K_{\rm w}$, and the probability ω of obtaining the outcome "w-spin-down" is the square of the length of the projection of ψ onto the subspace $K_{\rm w}^{\perp}$.

$$\omega(\text{w-spin-up}) = \langle P_{K_{\rm w}}\psi, \psi\rangle, \quad \omega(\text{w-spin-down}) = \langle P_{K_{\rm w}^{\perp}}\psi, \psi\rangle$$

Then, we have the set of projections on H,

$$P(H) = \{I, 0, P_{K_{\rm W}}, P_{K_{\rm W}}^{\perp} \mid 0 \le \theta_{\rm W} < \pi\}.$$

3 On $P(H)$-valued universe $V^{P(H)}$

Let $P(H)$ be a propositional system. $P(H)$-valued universe $V^{P(H)}$ is constructed by induction.

$$V_\alpha^{P(H)} = \big\{u \mid \exists\beta{<}\alpha\, \exists \mathcal{D}u {\subset} V_\beta^{P(H)} (u : \mathcal{D}u \to P(H))\big\}, \quad V^{P(H)} = \bigcup_{\alpha\in On} V_\alpha^{P(H)}.$$

Implication $\underset{q}{\to}$ is defined by

$$\varphi \underset{q}{\to} \psi \overset{\text{def}}{\Longleftrightarrow} \varphi^{\perp} \vee (\varphi \wedge \psi)$$

For $u, v \in V^{P(H)}$, $[\![u \underset{q}{=} v]\!]$ and $[\![u \underset{q}{\in} v]\!]$ are defined by

$$[\![u \underset{q}{=} v]\!] = \bigwedge_{x\in\mathcal{D}u} (u(x) \underset{q}{\to} [\![x \underset{q}{\in} v]\!]) \wedge \bigwedge_{x\in\mathcal{D}v} (v(x) \underset{q}{\to} [\![x \underset{q}{\in} u]\!]), \quad [\![u \underset{q}{\in} v]\!] = \bigvee_{x\in\mathcal{D}v} [\![u \underset{q}{=} x]\!] \wedge v(x),$$

where $[\![A]\!]$ is an element of $P(H)$.

Next, we turn to real numbers in $V^{P(H)}$.

Definition 3 *u is called a real number if (1)∧(2)∧(3), where*

1. *$u \subset \check{\mathbb{Q}}$, that is, $\forall x(x \in u \underset{q}{\to} x \in \check{\mathbb{Q}})$*
2. *$\exists r \in \check{\mathbb{Q}}(r \in u) \wedge \exists r \in \check{\mathbb{Q}}(r \in u)^{\perp}$*
3. *$\forall r \in \check{\mathbb{Q}}((r \in u) \underset{q}{\leftrightarrow} \forall \in \check{\mathbb{Q}}(r < s \underset{q}{\to} s \in u))$*

Theorem 4 (cf. [3]) *If u is a real in $V^{P(H)}$, then $E_u : \mathbb{R} \to P(H)$ defined by $E_u(\lambda) = \bigwedge_{\lambda<r} [\![\check{r} \in u]\!]$ is a resolution of the identity, i.e. E_u satisfies $\bigwedge_{\lambda\in\mathbb{R}} E_u(\lambda) = 0$ and $\bigvee_{\lambda\in\mathbb{R}} E_u(\lambda) = 1, E_u(\lambda) = \bigwedge_{\lambda<\mu} E_u(\mu)$, and gives the spectral representation of the corresponding self-adjoint operator $A = \int \lambda dE_u(\lambda)$ on the Hilbert space H.*

Conversely, if A is a self-adjoint operator $A = \int \lambda dE(\lambda)$ on H, we define u by $\mathcal{D}u = \{\check{r} \mid r \in \mathbb{Q}\}, u(\check{r}) = \bigwedge_{r<s} E(s)$. Then u is a real in $V^{P(H)}$, and $A = \int \lambda dE_u(\lambda)$.

4 $P(H)$-valued universe $V^{P(H)}$ for electron spin

We consider x-spin and y-spin, that is, propositional system $P(H) = \{I, 0, P_{K_{\mathrm{X}}}, P_{K_{\mathrm{X}}}^{\perp}, P_{K_{\mathrm{Y}}}, P_{K_{\mathrm{Y}}}^{\perp}\}$.

This lattice includes Boolean subalgebras B_1, B_2, where $B_1 = \{I, 0, P_{K_{\mathrm{X}}}, P_{K_{\mathrm{X}}}^{\perp}\}, B_2 = \{I, 0, P_{K_{\mathrm{Y}}}, P_{K_{\mathrm{Y}}}^{\perp}\}$. Thus, $V^{P(H)}$ includes V^{B_1} and V^{B_2}.

Unless otherwise noted, H is the same Hilbert space of dimension 2 as in section 2. Hence, if A is a self-adjoint operator on H, then

$$A \quad = \quad \lambda_1 P + \lambda_2 P^{\perp}.$$

4.1 real numbers

Let u be a real in $V^{P(H)}$ and A_u a self-adjont operator corresponding to u.
For $\check{\lambda}$, we have

$$A_{\check{\lambda}} = \lambda I.$$

If $[\![u \text{ is real}]\!] = I$, $[\![u \underset{q}{=} \check{\lambda_1}]\!] = P_{K_{\mathrm{X}}}$ and $[\![u \underset{q}{=} \check{\lambda_2}]\!] = P_{K_{\mathrm{X}}}^{\perp}$, we have

$$A_u = \lambda_1 P_{K_{\mathrm{X}}} + \lambda_2 P_{K_{\mathrm{X}}}^{\perp}.$$

Especially, if A_u is a proposition "x-spin-up"in $P(H)$, that is, $A_u = P_{K_{\mathrm{X}}}$, then,

$$[\![u \underset{q}{=} \check{1}]\!] = [\![u \underset{q}{=} \check{0}]\!]^{\perp} = P_{K_{\mathrm{X}}}.$$

Then, u is a real satisfying $(u \underset{q}{=} \check{1}) \leftrightarrow (u \underset{q}{=} \check{0})^{\perp}$ in $V^{P(H)}$. $\check{1}(\check{s})$ is 0 or 1, and hence $u(\check{r})$ is compatible with $\check{1}(\check{s})$, $\check{0}(\check{s})$.

Thus, we define the following definition.

Definition 5 *Proposition u in $V^{P(H)}$ is a real number satisfying $(u \underset{q}{=} \check{1}) \leftrightarrow (u \underset{q}{=} \check{0})^{\perp}$.*

References

[1] C.Piron, Fundations of Quantum Pysics, W.A.Benjamin, Inc. 1976

[2] David W. Cohen, An Introduction to Hilbert space and Quantum Logic, Springer-Verlag New York Inc. 1989

[3] G.Takeuti, Two Applications of Logic to Mathematics, Iwanami and Princeton University Press, Tokyo and Princeton (1978).

[4] G.Takeuti, Quantum Set Theory, Current Issues in Quantum Logic, eds. E.Beltrametti and B.C.van Frassen, Plenum,New York (1981) pp.303-322

[5] 竹内外史 線形代数と量子力学 裳華房 1981

[6] S.Titani, Lattice Valued Set Theory, Arch. Math. Logic 38-6(1999) pp.395-421

[7] R.I.G. ヒューズ　量子論理

Representation of quantum propositions in the quantale valued universe

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1 Introduction

The lattice $P(H)$ consisting of all closed subspaces of a Hilbert space H is a complete orthomodular lattice. $P(H)$ is atomic and has the covering property. A complete orthomodular lattice which is atomic and has covering property is called a *proposition system*, whose elements are called propositions. A pure state is represented as a subspace of dimension 1, and

"Proposition p is certainly true in state x" means $x \leq p$.

Quantum phenomenon is described as results of observations. For a propositional system $P(H)$ the set of observations that we can perform on it has the struture of a unital quantale whose multiplication means "and then", whose unit is the null observation and whose joins are disjunctions. A proposition system and quantale on it form a quantum system.

Let $Q = Q(P(H))$ be a quantale on $P(H)$, consisting of sup-preserving mapping $P(H) \to P(H)$. $P(H)$ is embedded in the quantale Q as the sub-quantale $L(Q)$ of left-sided elements, which is a right Q-module. Then, propositions, pure states and symmetries are all represented as elements of Q. We consider the quantale Q as logic, and Q-valued set theory based on the logic. It is known that for any complete lattice $\mathcal{L}$ we can construct $\mathcal{L}$-vaued universe $V^{\mathcal{L}}$ and develope a $\mathcal{L}$-valued set theory on it (cf. [3]). Observables are represented as real numbers defined in $V^{P(H)}$ (cf. [2]). It follows that propositions and states are represented as real numbers, and symmetries are represented as complex numbers in the quantale valued universe V^Q.

2 A quantum system $(P(H), Q)$

We assume through this paper that H is a separable complex Hilbert space, $P(H)$ is the complete orthomodular lattice consisting of all closed subspaces of H, or equivalently, of all projections on H. A pure state of the physical system is a projection $\in P(H)$ of rank 1. An observable is defined as a correspondence between the propositions associated with the measuring apparatus and those associated with the physical system. Thus, an *observable* is a mapping of a complete Boolean algebra consisting of Borel sets of real numbers, into the proposition sytem, which is considered as a self-adjoint operator on H.

Definition 2.1 *A sup-lattice (complete lattice) Q with an associative binary operation $\&$ is called a* quantale, *if the operation $\&$ is distributive over arbitrary joins:*

$$a\& \bigvee S = \bigvee\{a\&s \mid s \in S\}, \qquad (\bigvee S)\&a = \bigvee\{s\&a \mid s \in S\}, \qquad \textit{for every } a \in Q \textit{ and } S \subset Q.$$

Let $Q = Q(P(H))$ be the set of all $\bigvee$-preserving mapping $P(H) \to P(H)$. Q forms a quantale, where

$$\begin{aligned} \varphi \leqslant \psi \quad &\overset{\text{def}}{\iff} \quad \varphi(x) \leqslant \psi(x) \quad \text{for all } x \in P(H) \\ (\bigvee_\alpha \varphi_\alpha)(x) \quad &\overset{\text{def}}{=} \quad \bigvee_\alpha \varphi_\alpha(x) \quad \text{for all } x \in P(H) \\ (\varphi\&\psi)(x) \quad &\overset{\text{def}}{=} \quad \psi(\varphi(x)) \quad \text{for all } x \in P(H) \end{aligned}$$

the pair $(P(H), Q)$ is called a *quantum system*.

An element a of a quantale Q is said to be *left-sided* if $1\&a \leq a$. The set $L(Q)$ of left-sided elements of Q, $L(Q) = \{a \in Q \mid 1\&a \leq a\}$, is a subquantale of Q.

Theorem 2.1 ([1]) *For each $x \in P(H)$ we define κ_x by $\kappa_x(y) = \begin{cases} 0, & \text{if } y = 0, \\ x, & \text{otherwise.} \end{cases}$. κ_x is in $L(Q)$, and $P(H)$ is embedded in the quantale Q as $L(Q)$, by $x \mapsto \kappa_x$. We consider $P(H)$ as $L(Q)$. That is, $P(H) = L(Q) \subset Q$. $P(H)$ is a Q-module.*

Lemma 2.1 *For $x \in P(H)$ and $a \in Q$, $\kappa_{a(x)} = \kappa_x \& a$.*

3 Q-valued universe V^Q

Let $(P(H), Q)$ be a quantum system. Q-valued universe V^Q is constructed by induction.

$$V^Q_\alpha = \{u \mid \exists \beta < \alpha \, \exists \mathcal{D}u \subset V^Q_\beta (u : \mathcal{D}u \to Q)\} \; ; \quad V^Q = \bigcup_{\alpha \in On} V^Q_\alpha$$

Similarly, $V^{P(H)} \subset V^Q$ is constructed. The formulas on the universe $V^{P(H)}$ are constructed from atomic formulas of the form $u = v$, or $u \in v$, by logical operators $\wedge$, $\vee$, $\neg$, $\&$, *, $^\perp$, $\forall$, $\exists$, and by implications $\to$ interpreted as : $(a \to b) = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$. The formulas on the universe V^Q are constructed from atomic formulas of the form $u = v$, or $u \in v$, and formulas on $V^{P(H)}$ by $\wedge$, $\vee$, $\neg$, $\&$, *, $\forall$, $\exists$, $\to$. We call the set theory on the Q-valued universe V^Q *quantale valued set theory*. We also define an implication $\underset{q}{\to}$ called quantum implication on $P(H)$. If φ and ψ are formulas on $V^{P(H)}$, then $\varphi \underset{q}{\to} \psi$ is defined as

$$\varphi \underset{q}{\to} \psi \overset{\text{def}}{\iff} \varphi^\perp \vee (\varphi \wedge \psi).$$

Truth value of a sentence φ is denoted by $[\![\varphi]\!]$. Atomic formulas $u = v$ and $u \in v$ where $u, v \in V^Q$ are interpreted by induction on the rank of u, v as follows :

$$[\![u = v]\!] = \bigwedge_{x \in \mathcal{D}u} (u(x) \to [\![x \in v]\!]) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \to [\![x \in u]\!]) \; ; \quad [\![u \in v]\!] = \bigvee_{x \in \mathcal{D}v} v(x) \wedge [\![u = x]\!]$$

V^Q is a model of lattice valued set theory (cf. [3]). The equality $\underset{q}{=}$ and the membership relation $\underset{q}{\in}$ corresponding to the quantum implication $\underset{q}{\to}$ is defined by

$$u \underset{q}{=} v \overset{\text{def}}{\iff} \forall x (x \underset{q}{\in} u \underset{q}{\to} x \underset{q}{\in} v) \wedge \forall x (x \underset{q}{\in} v \underset{q}{\to} x \underset{q}{\in} u) \; ; \; u \underset{q}{\in} v \overset{\text{def}}{\iff} \exists x (x \underset{q}{=} u \wedge x \in v).$$

3.1 Real and complex numbers in V^Q

We denote the set of all rational numbers, real numbers, and complex numbers by $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$, respectively. It is known that the set of rational numbers defined in V^Q coincides with the check set $\check{\mathbf{Q}}$ corresponding to $\mathbf{Q}$. We denote the check set corresponding to a set x by $\check{x}$.

A *real number* in V^Q is defined as an element of $V^{P(H)}$ which is an upper part u of Dedekind cut of rational numbers. That is, $u \in V^Q$ is called a *real number* if $(0) \wedge (1) \wedge (2) \wedge (3)$, where

$$\begin{aligned}
&(0) \quad \forall x \in \mathbf{Q} (1\&(x \in u) \to (x \in u)) \\
&(1) \quad u \subset \check{\mathbf{Q}}, \text{ that is, } \forall x (x \in u \to x \in \check{\mathbf{Q}}), \\
&(2) \quad \exists r \in \check{\mathbf{Q}} (r \in u) \wedge \exists r \in \check{\mathbf{Q}} (r \in u)^\perp, \\
&(3) \quad \forall r \in \check{\mathbf{Q}} \left((r \in u) \longleftrightarrow \forall s \in \check{\mathbf{Q}} (r < s \underset{q}{\to} s \in u) \right).
\end{aligned}$$

Let $u, v \in V^Q$ are real number in V^Q, that is, $[\![u, v \text{ are reals }]\!] = 1$. u, v are said to be *compatible*, if there exists a Boolean sub-algebra B of $P(H)$ such that $u, v \in V^B$. If u, v are compatible real numbers in V^Q, then $u \cdot v$ and $u + v$ are defined ; and $u + iv$ is called a *complex* , where $i^2 = -\check{1}$.

Lemma 3.1 *For each $p \in P(H)$, we define $\hat{p} \in V^Q$ such that $\hat{p} \subset \check{\mathbf{Q}}$ by* $\hat{p}(\check{r}) = \begin{cases} 0, & \text{if } r < 0 \\ p^\perp, & \text{if } 0 \le r < 1 \\ 1, & \text{if } 1 \le r \end{cases}$.
Then $[\![\hat{p} \underset{q}{=} \check{1}]\!] = [\![\hat{p} \underset{q}{=} \check{0}]\!]^\perp = p$. Conversely, if $u \in V^Q$ satisfies $[\![u \subset \check{\mathbf{Q}}]\!] = 1$ and $[\![u \underset{q}{=} \check{1}]\!] = [\![u \underset{q}{=} \check{0}]\!]^\perp = p$, then $[\![u = \hat{p}]\!] = 1$.

Thus, we define *proposition* in V^Q as a real number u satisfying $(u \underset{q}{=} \check{1}) \leftrightarrow (u \underset{q}{=} \check{0})^\perp$. A proposition p is called a *pure state* if $\forall r \in \mathbf{Prop}(0 < r \le p \to r = p)$. Propositions u, v are said to be *orthogonal* if $(u \underset{q}{=} \check{1}) \leftrightarrow (v \underset{q}{=} \check{1})^\perp$. A positive real ρ is said to be a *state*, if there exist mutually orthogonal pure states $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3, \cdots$, and positive check reals μ_1, μ_2, $\mu_3, \cdots$ with sum equal to $\check{1}$ such that $\rho = \sum_{n=1}^{\infty} \mu_n \hat{p}_n$. The set of all states is denoted by $\mathbf{St}$.

It is known that if we carry out a measurement corresponding to the proposition b in initial state ρ, then the probability of obtaining the answer "yes" is the trace of linear operator $b\,\rho$ on H. It follows that there exists a function $w_\rho : \mathbf{Prop} \to [\check{0}, \check{1}] \subset \check{\mathbf{R}}$, which associates the probability to $\hat{b}$ such that

1. $w_\rho(\hat{1}) = \check{1}$,
2. If $\{\hat{b}_n\}$ $(n = 1, 2, \cdots)$ is a set of mutually orthogonal projections, then $w_\rho(\bigvee_{n=1}^{\infty} \hat{b}_n) = \sum_{n=1}^{\infty} w_\rho(\hat{b}_n)$
3. If $w_\rho(\hat{b}) = w_\rho(\hat{c}) = 0$ then $w_\rho(\hat{b} \vee \hat{c}) = 0$.

If we obtain the answer "yes", then the state of the system immediately after the experiment is $(\rho \vee b^\perp) \wedge b$, which is a pure state. We denote $\big((\rho \vee b^\perp) \wedge b\big)\hat{}$ by $\varphi_b(\hat{\rho})$. Now we can define the product $\hat{\rho} \cdot \hat{b}$ of reals $\hat{\rho}$ and $\hat{b}$ as $w_\rho(\hat{b}) \cdot \varphi_b(\hat{\rho})$. $w_\rho(\hat{b})\varphi_b(\hat{\rho}) + w_\rho(\hat{b}^\perp)\varphi_{b^\perp}(\hat{\rho})$ is a state, and we consider that the measurement correspondng to the proposition b is the mapping on $\mathbf{St}$ given by

$$\hat{\rho} \mapsto w_\rho(b)\varphi_b(\hat{\rho}) + w_\rho(\hat{b}^\perp)\varphi_{b^\perp}(\hat{\rho}).$$

A symmetry is an automorphism of the orthomodular lattice of propositions, which is induced by a unitery operator on H. Let u be a unitary operator on H. u is a normal operator, and is represented as a complex in V^Q. Let $\tilde{u}$ be mapping on $P(H)$ defined by $\tilde{u} : p \mapsto upu^*$. Since $\tilde{u}$ is an automorphism on $P(H)$, $\tilde{u}$ is an element of Q, and induces an automorphism σ_u on $\mathbf{St}$. We call σ_u the *symmetry* associated with complex number u in V^Q.

References

[1] C.J.Mulvey and J.Wick Pelletier, Quantization of the Calculus of Relations, Canadian Mathematical Society, Conference Proceeding, Vol. 13, 1992.

[2] G.Takeuti, Two Applications of Logic to Mathematics, Iwanami and Princeton University Press, Tokyo and Princeton (1978).

[3] S.Titani, Lattice Valued Set Theory, Arch. Math. Logic 38-6(1999) pp.395-421

Completions of a metric space without choice

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The definition of a completion of a metric space (Definition 2) seems queer, which is adopted in many books. The notion of dense is defined not merely by metric but by topology. Pure completion (Definition 2) seems more natural. We show that completion is not always unique upto isomorphism without the axiom of choice. In familiar method to have a completion, quotient of the space of the Cauchy sequences, we don't have a complete metric space without choice (Example 2). Now, we don't suppose the axiom of choice.

Definition 1. Let $\langle X, d\rangle$ be a metric space. a real valued function f on X is a *location* (see [1]) in X if the following conditions satisfy.

1) $f(x) \geq |f(y) - d(x, y)|$ for all $x,\ y \in X$.

2) $\inf_{x\in X} f(x) = 0$.

We denote the set of the locations in X by $\widehat{X}$. For $a \in X$, we define the non-negative real valued function by $d_a(x) = d(x, a)$. Define $\widehat{d}\colon \widehat{X} \times \widehat{X} \to \mathbb{R}$ by $\widehat{d}(f, g) = \inf_{x\in X} \big(f(x) + g(x)\big)$.

Lemma 1. *[1]*

1) Any location f in X has at most one zero. If $a \in X$ is a zero of f then $f = d_a$.

2) $\widehat{d}(f, g) = \sup_{x\in X} |f(x) - g(x)|$.

3) $\widehat{d}$ is a metric on $\widehat{X}$.

4) $\widehat{d}(f, d_a) = f(a)$.

5) The correspondence $x \mapsto d_x$ is an isometric embedding from X into $\widehat{X}$. That is $d(x, y) = \widehat{d}(d_x, d_y)$ for $x,\ y \in X$.

We can identify $a \in X$ and $d_a \in \widehat{X}$ by the lemma above. From now on, we admit $X \subseteq \widehat{X}$ in this way.

Theorem 2. *The metric space $(\widehat{X}, \widehat{d}\,)$ is complete.*

Definition 2. A metric space Y is a *completion* of a metric space X, if Y is a complete metric space and X is a dense subspace of Y.

A metric space X is *strongly complete* if X itself is the only completion of X.

A metric space Y is a *complete closure* of a metric space X, if Y is a maximal completion of X, or Y is the only completion of X containing Y.

A metric space Y is a *pure completion* of a metric space X, if Y is a minimal complete metric superspace of X.

Lemma 3. *Let X be a dense subspace of a metric space $\langle Y, d\rangle$. Let φ be a map from Y defined by $\varphi(y) = d_y \restriction X$. Then φ is an isometric embedding into $\widehat{X}$ and $\varphi \restriction X$ is the identity.*

Theorem 4 (Existence of complete closure). *$\widehat{X}$ is a complete closure of X.*

Theorem 5 (Uniqueness of complete closure). *There is a unique complete closure of an metric space $\langle X, d\rangle$ upto isomorphism.*

Theorem 6 (Unique existence of pure completion). *We denote*

$$X^\natural = \bigcap \{Y \mid X \subseteq Y \subseteq \widehat{X},\ Y \text{ is complete}\}$$

Then $X^\natural$ is a pure completion of X. Pure completion is unique upto isomorphism.

For a subset $A \subseteq \widehat{X}$, We denote

$$\widetilde{A} = \left\{a \in \widehat{X} \;\middle|\; \text{there is a Cauchy sequence in } A \text{ which converges to } a\right\}$$

We see that $X \subseteq \widetilde{X} \subseteq X^\natural \subseteq \widehat{X}$. In ZF (without choice), we cannot prove any equality of the relation above by the following examples.

Example 1. Let D_1 be a Dedekind-finite infinite subset of interval $(0, 1]$ with $\inf D_1 = 0$. Then $0 \notin D_1 = \widetilde{D_1}$ and $0 \in \widehat{D_1}$. In this case $D_1 = \widetilde{D_1} = {D_1}^\natural \neq \widehat{D_1}$.

Example 2. For $x \in D_1$, define $S_x \subseteq \mathbb{R}^2$ by

$$S_x = \left\{ \left(xy + \frac{1-y}{n+1}, y\right) \;\middle|\; 0 < y \leq 1 \right\} \qquad \text{if } n \in \mathbf{N},\ \frac{1}{n+2} < x \leq \frac{1}{n+1}.$$

Put $D_2 = \bigcup \{S_x \mid x \in D_1\}$. Then $(0,0) \notin \widetilde{D_2}$, $(0,0) \in \widetilde{\widetilde{D_2}} \subseteq {D_2}^\natural$, $(0,1) \notin {D_2}^\natural$ and $(0,1) \in \widehat{D_2}$. In this case $D_2 \neq \widetilde{D_2} \neq {D_2}^\natural \neq \widehat{D_2}$ and $\widetilde{D_2}$ is not complete.

Proposition 7. *Define inductively the function F from the ordinals by $F(\zeta) = X \cup \bigcup_{\xi<\zeta} \widetilde{F(\xi)}$. Then there is an ordinal κ such that $X^\natural = F(\kappa)$.*

References

[1] F. Richman, *The fundamental theorem of algebra: a constructive development without choice*, ***Pacific Journal of math.***, 196 No.1 (2000), 213–230.

[2] Thomas J. Jech, ***The axiom of choice***, North-Holland, Amsterdam, (1973).

Representation of Preference Orders by Hyperreal Numbers

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Game theory is now a basic framework of microeconomics in analyzing economic behavior of consumers and producers in a market. One of the assumptions in this theory is that *each agent behaves so as to maximize his or her own payoff*, and this assumption is used to explain a lot of competitive or cooperative behavior of the agents in a market. Such payoffs are usually given in the analysis of a game as integers, sometimes as real numbers, in any case they are given by a numerical function, which is to be interpreted concretely as an agent's preference for goods or social welfare in the real world. Our aim is to study the validity of such numerical representation, in other words, we shall examine what kind of conditions are needed to represent an agent's preference by means of numerical values.

To put it more precisely, suppose we are in a situation to play a game, and let S be the set of items, one of which we will gain after playing the game. Because some of the items may be preferable rather than others, we shall have to calculate the best strategy to obtain as desirable item as possible. According to the methodology of game theory, this calculation will generally be carried out as follows: we assign a real-valued function $U : S \to \mathbf{R}$ which somehow reflect our preference,[†] and choose the best strategy that maximize the expected value of the function. Now, our question is: How can we assign such a numerical function $U : S \to \mathbf{R}$?

von Neumann–Morgenstern Utility Functions

One of the answer is the axiomatic formulation given by von Neumann and Morgenstern in their pioneering work [5], which is the most generally accepted formulation ensuring the existence of numerical utilities. Let us review and summarize some basic results of expected utility theory from the work of von Neumann and Morgenstern. The following description is based on Fishburn [2].

A binary relation $\preceq$ on a set D is a *weak order* if it satisfies the following conditions:

(i) for all $x, y, z \in D$, $\quad x \preceq y$ and $y \preceq z$ imply $x \preceq z$,

(ii) for all $x, y \in D$, $\quad x \preceq y$ or $y \preceq x$.

The strict part $\prec$ and indifference part $\sim$ are defined from $\preceq$ by

$x \prec y$ if and only if $\quad x \preceq y$ and not $(y \preceq x)$,

$x \sim y$ if and only if $\quad x \preceq y$ and $y \preceq x$.

Let S denote a set of items. A *simple lottery* is a mapping $\lambda : S \to [0, 1]$ with the following properties:

(i) $\displaystyle\sum_{s \in S} \lambda(s) = 1$, (ii) there is a finite $S' \subseteq S$ such that $\lambda(s) = 0$ for all $s \in S \setminus S'$.

Let $\Delta(S)$ denote the set of all simple lotteries. Given $\lambda, \mu \in \Delta(S)$ and $\theta \in [0, 1]$, define the *convex combination* $\theta\lambda + (1 - \theta)\mu$ by

$$[\theta\lambda + (1 - \theta)\mu](s) \;=\; \theta\lambda(s) + (1 - \theta)\mu(s) \quad \text{for all } s \in S.$$

It can easily be verified that the convex combination $\theta\lambda + (1 - \theta)\mu$ also belongs to $\Delta(S)$.

[†]Such a function $U : S \to \mathbf{R}$ is referred to as a *utility function*, and the value $U(s)$ for $s \in S$ is called the *utility* of the item s.

Now we consider the following axioms for a preference relation $\preceq$ on $\Delta(S)$: For all x, y, z in $\Delta(S)$,

B1 $\preceq$ is a weak order on $\Delta(S)$,

B2 if $x \prec y$ then $\theta x + (1-\theta)z \prec \theta y + (1-\theta)z$ for all $\theta \in (0,1)$,

B3 if $x \prec y$ and $y \prec z$, then there are $\theta, \xi \in (0,1)$ such that

$$\theta x + (1-\theta)z \prec y \quad \text{and} \quad y \prec \xi x + (1-\xi)z.$$

Theorem 1 (see Fishburn [2])
A preference order $\preceq$ on $\Delta(S)$ satisfies B1–B3 if and only if there is a utility function $U : S \to \mathbf{R}$ such that

$$\lambda \prec \mu \iff \sum_{s \in S} \lambda(s)U(s) < \sum_{s \in S} \mu(s)U(s).$$

Also, U is unique up to a positive affine transformation, i.e. if $V : S \to \mathbf{R}$ is another function with the same property, then there exist real constants $a > 0$ and b such that

$$V(s) = a\,U(s) + b \quad \text{for all } s \in S.$$

Additive Utilities

The above formulation by von Neumann and Morgenstern is based on the abstract notion of 'preference on the lotteries', so that the meaning of B1–B3 is not quite clear. To avoid such abstraction, we propose an alternative system of axioms which is based on more concrete notion of 'preference on the consumption plans'. The advantage of our formulation is that a utility function is ensured to be additive, i.e. utilities can be added together.

Let $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a set of n-types of indivisible items for some $n \geq 2$, which may include a diamond, a car, a grain of sesame, and so on. The set $\Omega(S)$ of *consumption plans* of these items can be defined as

$$\Omega(S) = \{\, k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n \mid k_1, k_2 \cdots, k_n \in \mathbf{N} \,\}^\dagger,$$

where $k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n$ simply denotes "k_1 of α_1, k_2 of α_2, $\cdots$, and k_n of α_n". The additive operation $+$ is naturally applied to consumption plans:

$$(k_1\alpha_1 + \cdots + k_n\alpha_n) + (l_1\alpha_1 + \cdots + l_n\alpha_n) = (k_1 + l_1)\alpha_1 + \cdots + (k_n + l_n)\alpha_n.$$

Let us assume that there exists a preference relation $\preceq$ on $\Omega(S)$ satisfying the following conditions:

A1 $\preceq$ is a weak order on $\Omega(S)$.

A2 For all $x, y, z \in \Omega(S)$, $x \prec y$ implies $x + z \prec y + z$.

A3 For all $x, y, z \in \Omega(S)$, $x \sim y$ implies $x + z \sim y + z$.

The ordering condition **A1** is an assumption that a preference relation of a rational person should be at least a weak order. The monotonicity condition **A2** is an assertion that the strict part of a preference relation should be preserved by adding the same items to both sides of an inequality. For example, consider the case of two items $(\alpha_1, \alpha_2) =$ (cookie , cake) and suppose, in the first place, we have a preference

$$2 \times \text{ cookie } \prec 1 \times \text{ cake },$$

which means that we like a cake better than two cookies. Then, by **A2**, the following preferences are inferred:

$$\begin{aligned} &\Rightarrow && 3 \times \text{ cookie } &&\prec\ 1 \times \text{ cake } + 1 \times \text{ cookie }, \\ &\Rightarrow && 1 \times \text{ cake } + 3 \times \text{ cookie } &&\prec\ 2 \times \text{ cake } + 1 \times \text{ cookie }, \\ &\Rightarrow && \cdots \end{aligned}$$

by adding a cookie to both sides, then adding a cake, and so on. The meaning of **A3** can be described in a similar way.

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† $\mathbf{N}$ denotes the set of all natural numbers including zero.

The following theorem states that it is enough for us to assume these simple scheme of conditions **A1**–**A3** to represent a preference relation in terms of hyperreal numbers. Let ε be an infinitely small number, i.e. $0 < \varepsilon$ and $\varepsilon < 1/k$ for $k = 1, 2, 3, \cdots$.

Theorem 2 (Ikeda [4]) Let $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a set of n-types of items for some $n \geq 2$.
A preference relation $\preceq$ on $\Omega(S)$ satisfies **A1**–**A3** if and only if there are hyperreal numbers $u_1, \cdots, u_n$ such that for all $k_1, \cdots, k_n, l_1, \cdots, l_n \in \mathbf{N}$,

$$\sum_{i=1}^{n} k_i \alpha_i \preceq \sum_{i=1}^{n} l_i \alpha_i \quad \Longleftrightarrow \quad \sum_{i=1}^{n} k_i u_i \leq \sum_{i=1}^{n} l_i u_i$$

in which $u_1, \cdots, u_n$ are of the form:

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ r_{21} & \cdots & r_{2n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ \varepsilon \\ \vdots \\ \varepsilon^{n-1} \end{pmatrix}$$

where $\{r_{ij}\}_{1 \leq i,j \leq n}$ is an $n \times n$ matrix of real numbers satisfying either $r_{ij} = 0$ for all i, j, or that the first column vector is nonzero, i.e. $(r_{11}, r_{12}, \cdots, r_{n1}) \neq (0, 0, \cdots, 0)$.
Also, the first column vector is unique up to a positive real multiple, i.e. if $\{s_{ij}\}_{1 \leq i,j \leq n}$ is another matrix with the same property, then there exists a positive real number $a > 0$ such that

$$\begin{pmatrix} s_{11} \\ \vdots \\ s_{n1} \end{pmatrix} = a \begin{pmatrix} r_{11} \\ \vdots \\ r_{n1} \end{pmatrix}$$

In other words, the 'real parts' of the hyperreal numbers $u_1, \cdots, u_n$ are uniquely determined up to a positive real multiple.

Example 3 Let us give an example illustrating why *hyperreal* numbers get into our consideration. Let $(\alpha_1, \alpha_2) = (\text{ sesame }, \text{ diamond })$, and suppose we have a preference

$$0 \prec \alpha_1 \quad \text{and} \quad k\alpha_1 \prec \alpha_2 \quad \text{for all } k > 0.$$

That is to say, a grain of sesame is better than nothing, and a diamond is better than any finite grains of sesame. This is consistent with the axioms **A1**–**A3**, but cannot be represented additively by real numbers because there is no pair of real numbers a_1, a_2 such that $0 < ka_1 < a_2$ for all $k > 0$. Thus the utilities must be expressed as $(u_1, u_2) = (\varepsilon, 1)$. In other words, a grain of sesame has infinitely small utility in comparison with a diamond.

References

[1] Brams, S. J. and Fishburn, P. C. (2000). Fair Division of Indivisible Items between Two People with Identical Preferences: Envy-freeness, Pareto-optimality, and Equity. *Social Choice and Welfare*, **17**, 247–267.

[2] Fishburn, P. C. (1970). *Utility Theory for Decision Making*. Wiley, New York.

[3] Goldblatt, R. (1998). *Lectures on the Hyperreals: An Introduction to Nonstandard Analysis*. Springer, New York.

[4] Ikeda, K. (2001). *Non-Archimedean Additive Utilities for Discrete Items*. manuscript.

[5] von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press. second edition, 1947; third edition, 1953.

Kripke incompleteness of predicate extentions of Gabbay-de Jongh's logic of the finite binary trees.

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1 Introduction.

In the previous papers [4, 5], the author gave several completeness and incompleteness results on some predicate extensions with the constant domain of intermediate and modal propositional logics by means of the theory of canonical formulas (cf. [1]).

However, these results are on subframe and cofinal subframe logics, and little is known for non cofinal subframe logics.

In this note, we show the Kripke incompleteness of the intermediate predicate logic $\mathbf{H}_* + Tr_2 + K + D$, where Tr_2 is the axiom of Gabbay-de Jongh's logic of the finite binary trees [2] $\bigwedge_{i=0}^{2}((p_i \supset \bigvee_{j\neq i} p_j) \supset \bigvee_{j\neq i} p_j) \supset \bigvee_{i=0}^{2} p_i$, K is Kuroda's axiom $\forall x\neg\neg A(x) \supset \neg\neg\forall x A(x)$ and D is the axiom of constant domain $\forall x(A(x) \vee B) \supset (\forall x A(x) \vee B)$.

Our result is the following.

Theorem 1 *Let* $\mathbf{L}$ *be an intermediate logic with the following conditions:*

- $\mathbf{L}$ *is Kripke complete.*
- $\mathbf{L}$ *is contained in* $\mathbf{H}_* + Tr_2 + K + D$.

Then, the propositional part of $\mathbf{L}$ *is the intuitionistic logic.*

To prove the theorem, we construct a Kripke model $\langle \widehat{\mathbf{T}}, D, \models\rangle$ and a formula A satisfying the following three conditions:

(i). Every instance of Tr_2, K and D is true in $\langle \widehat{\mathbf{T}}, D, \models\rangle$.

(ii). A cannot be separated from $\mathbf{H}_* + Tr_2 + K + D$ by Kripke frames, that is, A is valid in the Kripke frames for $\mathbf{H}_* + Tr_2 + K + D$.

(iii). A is refuted in $\langle \widehat{\mathbf{T}}, D, \models\rangle$.

2 Outline of the proof.

It is known that Gabbay-de Jongh's logic $\mathbf{H} + Tr_2$ is complete but not strongly complete. This fact is implicitely showed in [3]. Another proof is given in [1, Theorem 6.2].

Since the expressive power of predicate logics is stronger than that of propositional logics, it may be possible to transfer the proof of the non-strong completeness [1, Theorem 6.2] to an incompleteness proof of predicate extensions of $\mathbf{H} + Tr_2$.

$2^{\leq\omega}$ denotes the 0-1 sequences of length at most ω and $2^{<\omega}$ denotes the finite 0-1 sequences. $lh(s)$ denotes the length of $s \in 2^{<\omega}$.

We denote $s \subseteq t$ if $s \in 2^{\omega}$ is an initial segment of $t \in 2^{\leq\omega}$. $s \subsetneq t$ is the abbreviation of $s \subseteq t$ and $s \neq t$. For $s \in 2^{<\omega}$ and $t \in 2^{\leq\omega}$, $s{*}t$ denotes the concatenation of s and t. For $s \in 2^{\leq\omega}$ and n, $s|n$ denotes the initial segment of length n of s.

We define partially ordered sets $\mathbf{T}$, $\widehat{\mathbf{T}}$ and $\mathbf{T}_n$ as follows:

$$T = 2^{<\omega}, \mathbf{T} = \langle T, \subseteq \rangle;\ \widehat{T} = 2^{\leq \omega}, \widehat{\mathbf{T}} = \langle \widehat{T}, \subseteq \rangle,\ \mathbf{T}_n = \langle 2^{\leq n}, \subseteq \rangle.$$

The proof of the non-strong completeness [1, Theorem 6.2] used the fact that every instance of Tr_2 is true in the Kripke model for propositional logic $\langle \widehat{\mathbf{T}}, \models \rangle$ defined by $s \models p_t \ \Leftrightarrow\ t \subsetneq s$, where p_t $(t \in 2^{<\omega})$ are propositional variables.

So, we define the Kripke model $\langle \widehat{\mathbf{T}}, D, \models \rangle$ as follows:

- $D = 2^{<\omega}$,
- $s \models p(t) \ \Leftrightarrow\ t \subsetneq s$.

As we have expected, we can prove the following.

Lemma 1 *Quantifier elimination holds in the model $\langle \widehat{\mathbf{T}}, D, \models \rangle$. More precisely, for every formula A, there exist constants $t_1, t_2, dots, t_n \in D$ such that*

$$\langle\rangle \models A \equiv (p(t_1) \vee p(t_2) \vee \cdots \vee p(t_n)).$$

Since the axioms K and D are valid in frame $\langle \widehat{\mathbf{T}}, D \rangle$, the above lemma implies the condition (i).

We define the formula A as follows:

$$P = \forall x((\neg\neg p(x) \supset p(x)) \supset (p(x) \vee \neg p(x)))$$

$$Q_1(x, y, z) = p(x) \equiv (\neg\neg p(y) \vee \neg\neg p(z))$$

$$Q_2(y, z) = \neg(\neg\neg p(y) \wedge \neg\neg p(z))$$

$$Q_3(x, y) = (\neg\neg p(y) \supset p(y)) \supset (\neg\neg p(x) \supset p(x))$$

$$A = (P \wedge \forall x \exists y, z(Q_1(x, y, z) \wedge Q_2(y, z) \wedge Q_3(x, y) \wedge Q_3(x, z))) \supset \forall x(\neg\neg p(x) \supset p(x))$$

It is not difficult to verify the condition (iii).

Lemma 2 *A is refuted at $\langle\rangle$ in $\langle \widehat{\mathbf{T}}, D, \models \rangle$.*

So, the rest is to show the condition (ii).

Suppose A is refuted in a Kripke model $\langle \mathbf{N}, U, \models \rangle$. Without loss of generality, we may assume A is refuted at the least element 0 of $\mathbf{N}$.

By induction on the length of s, we can choose $x_s \in U(0)$ $(s \in T)$ and u_s $(s \in T)$ such that

$$u_s \models \neg\neg p(x_s),\ u_s \not\models p(x_s)$$
$$0 \models Q_1(x_s, x_{s*\langle 0\rangle}, x_{s*\langle 1\rangle}) \wedge Q_2(x_{s*\langle 0\rangle}, x_{s*\langle 1\rangle}) \wedge Q_3(x_s, x_{s*\langle 0\rangle}) \wedge Q_3(x_s, x_{s*\langle 1\rangle}).$$

It can be proved the partial map

$$f(u) = \begin{cases} s & u = u_s \\ \text{undefined} & \text{otherwise} \end{cases}$$

is a subreduction from $\mathbf{N}$ to $\mathbf{T}$.

We can extend f to a total map g from $\mathbf{N}$ to $\widehat{\mathbf{T}}$ with the following conditions.

- g is order preserving.
- The restriction of g to $g^{-1}(T)$ is a p-morphism from $g^{-1}(T)$ to $\mathbf{T}$.

By the above conditions, we can prove that $\mathbf{N}$ characterizes the intuitionistic logic $\mathbf{H}$.

Lemma 3 (cf. Smoryński [6]) *Suppose A is refuted in a Kripke model $\langle \mathbf{N}, U, \models \rangle$. Every finite frame is a p-morphic image of $\mathbf{N}$. So, $\mathbf{N}$ characterizes int$H$.*

3 Remarks.

We can consider several problems.

One interpretation of our theorem is that we have not known a good axiom to complete Gabbay-de Jongh's logic. There is a complete intermediate logic $\mathbf{H}_* + Tr_2 + (p \vee \neg p) \vee \exists x \forall y (A(x) \equiv A(y))$ whose propositional part is $\mathbf{H} + Tr_2$. Since this logic is almost a propositional logic, it is natural to ask the following.

Problem 1 *Find a non-trivial finitely axiomatizable intermediate predicate logic whose propositional part is Gabbay-de Jongh's logic.*

Our incompleteness proof heavily depends on the fact that the propositional part is Gababy-de Jongh's logic.

Problem 2 *Reformulate our result by means of canonical formulas.*

The problem is to change the propositional part.

Problem 3 *Let* $\mathbf{J}$ *be a Kripke complete intermediate logic. Determin* $\mathbf{J}_* + K + D$ *is complete or not. In particular, how about the following cases.*

- $\mathbf{J}$ *is Scott's logic.*
- $\mathbf{J}$ *is Kreisel-Putman's logic.*

Since Scott's logic is not strongly complete, it may be possible to apply the similar technique to Gabbay-de Jongh's logic. However, the problem is much difficult for Kreisel-Putman's logic since Kreisel-Putman's logic is elementary.

References

[1] A. Chagrov, and M. Zakharyaschev, **Modal logic,** Clarendon Press, Oxford (1997)

[2] D.M. Gabbay, and D.H. de Jongh, *Sequences of decidable and finitely axiomatizable intermediate logics with the disjunction property,* **Journal of Symbolic Logic** 39 (1974), pp. 67-79.

[3] V.B. Shehtman *On incomplete propositional logics,* **Doklady Akademii Nauk SSSR** 235 (1977), 542-545. (English translation) **Soviet Mathematics, Doklady** 18 (1977), 985-989.

[4] T. Shimura, *Kripke completeness of some intermediate predicate logics with the axiom of constant domain and a variant of canonical formulas,* **Studia Logica** 52 (1993), pp. 23–40.

[5] T. Shimura, *Kripke incompleteness of predicate extensions of the modal logics axiomatized by a canonical formula for a frame with a nontrivial cluster,* **Studia Logica** 65 (2000), 237–247.

[6] C. Smorynski, *Investigations of Intuitionistic Formal Systems by Means of Kripke Models,* Thesis, University of Illinois.

Products of infinitely many modal logics

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Abstract

The definition of normal products introduced in [3] is based on the standard way of introducing "products" in measure theory and topology. Here, we will show that the same idea is also applicable to infinitely many products.

1 Products of infinitely many modal general frames

We assume the logics are extensions of $\mathbf{K} \oplus \Diamond\top$ in order to define products of infinitely many modal logics.

For modal algebras $\mathcal{P}_\lambda$ on (W_λ, R_λ), define $\bigotimes\limits_{\lambda\in\Lambda} \mathcal{P}_\lambda$ to be the modal algebra on $(\prod\limits_{\lambda\in\Lambda} W_\lambda, \prod\limits_{\lambda\in\Lambda} R_\lambda)$ generated by $\{\pi_\lambda^{-1}(X_\lambda) \mid X_\lambda \in \mathcal{P}_\lambda \text{ and } \lambda \in \Lambda\}$. We say $\pi^{-1}_{\{\lambda_1,\ldots,\lambda_n\}}(X_{\lambda_1} \times \cdots \times X_{\lambda_n})$ to be *cylindric set* of $\{\mathcal{P}_\lambda \mid \lambda \in \Lambda\}$ for a finite subset $\{\lambda_1, \ldots, \lambda_n\}$ of Λ and $X_{\lambda_i} \in \mathcal{P}_{\lambda_i}$. Note that $\pi^{-1}_{\{\lambda_1,\ldots,\lambda_n\}}(X_{\lambda_1} \times \cdots \times X_{\lambda_n}) = \pi^{-1}_{\lambda_1}(X_{\lambda_1}) \cap \cdots \cap \pi^{-1}_{\lambda_1}(X_{\lambda_n})$ and that C is a cylindric set of $\{\mathcal{P}_\lambda \mid \lambda \in \Lambda\}$ iff $C = \prod\limits_{\lambda\in\Lambda} X_\lambda$, where $X_\lambda \in \mathcal{P}_\lambda$ for every $\lambda \in \Lambda$ and $X_\lambda = W_\lambda$ finitely except $\lambda \in \Lambda$.

Then we can also define as follows.

$$\bigotimes_{\lambda\in\Lambda} \mathcal{P}_\lambda = \{\bigcup_{i\in I} C_i \mid C_i \text{ is a cylindric set of } \{\mathcal{P}_\lambda \mid \lambda \in \Lambda\} \text{ for a finite } I \ \}.$$

For given modal general frames $\mathcal{F}_\lambda = (W_\lambda, R_\lambda, \mathcal{P}_\lambda)$ $(\lambda \in \Lambda)$, the *normal product* $\bigotimes\limits_{\lambda\in\Lambda} \mathcal{F}_\lambda$ of $\mathcal{F}_\lambda$'s is a modal general frame $(\prod\limits_{\lambda\in\Lambda} W_\lambda, \prod\limits_{\lambda\in\Lambda} R_\lambda, \bigotimes\limits_{\lambda\in\Lambda} \mathcal{P}_\lambda)$.

Theorem 1 *A projection* $\pi_\lambda : \prod\limits_{\lambda\in\Lambda} W_\lambda \to W_\lambda$ *is always a reduction of* $\bigotimes\limits_{\lambda\in\Lambda} \mathcal{F}_\lambda$ *to* $\mathcal{F}_\lambda$.

The next theorem gathers together several essential properties of normal products.

Theorem 2

1. *If* f_λ *is a reduction of* $\mathcal{F}_\lambda$ *to* $\mathcal{G}_\lambda$ *for* $\lambda \in \Lambda$, *then the* product map $\bigotimes\limits_{\lambda\in\Lambda} f_\lambda$ *is a reduction of* $\bigotimes\limits_{\lambda\in\Lambda} \mathcal{F}_\lambda$ *to* $\bigotimes\limits_{\lambda\in\Lambda} \mathcal{G}_\lambda$, *where* $\bigotimes\limits_{\lambda\in\Lambda} f_\lambda$ *is defined by* $\big((\bigotimes\limits_{\lambda\in\Lambda} f_\lambda)(w)\big)(\lambda) = f_\lambda(w_\lambda)$ *for* $w \in \prod\limits_{\lambda\in\Lambda} W_\lambda$.
2. *If* $\mathcal{G}_\lambda$ *is a generated subframe of* $\mathcal{F}_\lambda$ *for* $\lambda \in \Lambda$, *then* $\bigotimes\limits_{\lambda\in\Lambda} \mathcal{G}_\lambda$ *is also a generated subframe of* $\bigotimes\limits_{\lambda\in\Lambda} \mathcal{F}_\lambda$.
3. *The identity map* $id : \sum\limits_{\lambda\in\prod_{\mu\in M}\Lambda_\mu} \prod\limits_{\mu\in M} W^\mu_{\lambda_\mu} \to \prod\limits_{\mu\in M} \sum\limits_{\lambda_\mu\in\Lambda_\mu} W^\mu_{\lambda_\mu}$ *is a reduction of* $\sum\limits_{\lambda\in\prod_{\mu\in M}\Lambda_\mu} \bigotimes\limits_{\mu\in M} \mathcal{F}^\mu_{\lambda_\mu}$ *to* $\bigotimes\limits_{\mu\in M} \sum\limits_{\lambda_\mu\in\Lambda_\mu} \mathcal{F}^\mu_{\lambda_\mu}$.

For modal algebras $\boldsymbol{A_\lambda}$'s, we define their *normal product* $\mathop{\gg}\limits_{\lambda\in\Lambda} \boldsymbol{A_\lambda}$ as $(\mathop{\gg}\limits_{\lambda\in\Lambda} (\boldsymbol{A_\lambda})_+)^+$.

Corollary 3 *For modal general frames $\mathcal{F}_\lambda$ for $\lambda \in \Lambda$,*

$$(\mathop{\gg}_{\lambda\in\Lambda} \mathcal{F}_\lambda)^+ \cong \mathop{\gg}_{\lambda\in\Lambda} (\mathcal{F}_\lambda)^+. \tag{1}$$

Theorem 4 *If modal general frame $\mathcal{F}_\lambda$ is descriptive for $\lambda \in \Lambda$, then their normal product $\mathop{\gg}\limits_{\lambda\in\Lambda} \mathcal{F}_\lambda$ is also descriptive.*

Corollary 5 *For modal algebras $\boldsymbol{A_\lambda}$'s,*

$$(\mathop{\gg}_{\lambda\in\Lambda} \boldsymbol{A_\lambda})_+ \cong \mathop{\gg}_{\lambda\in\Lambda} (\boldsymbol{A_\lambda})_+. \tag{2}$$

We can show the duality of Theorem 2 as follows.

Theorem 6

1. *If $\boldsymbol{B_\lambda}$ is a homomorphic image of $\boldsymbol{A_\lambda}$, then $\mathop{\gg}\limits_{\lambda\in\Lambda} \boldsymbol{B_\lambda}$ is also a homomorphic image of $\mathop{\gg}\limits_{\lambda\in\Lambda} \boldsymbol{A_\lambda}$.*
2. *If $\boldsymbol{B_\lambda}$ is a subalgebra of $\boldsymbol{A_\lambda}$, then $\mathop{\gg}\limits_{\lambda\in\Lambda} \boldsymbol{B_\lambda}$ is isomorphic to a subalgebra of $\mathop{\gg}\limits_{\lambda\in\Lambda} \boldsymbol{A_\lambda}$.*
3. *$\mathop{\gg}\limits_{\mu\in M} \prod\limits_{\lambda_\mu\in\Lambda_\mu} \boldsymbol{A}^\mu_{\lambda_\mu}$ is isomorphic to a subalgebra of $\prod\limits_{\lambda\in\prod_{\mu\in M}\Lambda_\mu} \mathop{\gg}\limits_{\mu\in M} \boldsymbol{A}^\mu_{\lambda_\mu}$.*

By Theorem 6, we have the following.

Theorem 7 *For classes $\mathcal{K}_\lambda$ of modal algebras for $\lambda \in \Lambda$,*

1. $\mathop{\gg}\limits_{\lambda\in\Lambda} \boldsymbol{H}(\mathcal{K}_\lambda) \subseteq \boldsymbol{H}(\mathop{\gg}\limits_{\lambda\in\Lambda} \mathcal{K}_\lambda)$,
2. $\mathop{\gg}\limits_{\lambda\in\Lambda} \boldsymbol{S}(\mathcal{K}_\lambda) \subseteq \boldsymbol{IS}(\mathop{\gg}\limits_{\lambda\in\Lambda} \mathcal{K}_\lambda)$,
3. $\mathop{\gg}\limits_{\lambda\in\Lambda} \boldsymbol{P}(\mathcal{K}_\lambda) \subseteq \boldsymbol{ISP}(\mathop{\gg}\limits_{\lambda\in\Lambda} \mathcal{K}_\lambda)$.

Hence,

4. $\boldsymbol{V}\big(\mathop{\gg}\limits_{\lambda\in\Lambda} \boldsymbol{V}(\mathcal{K}_\lambda)\big) = \boldsymbol{V}(\mathop{\gg}\limits_{\lambda\in\Lambda} \mathcal{K}_\lambda)$.

2 Products of infinitely many modal logics

For modal logics L_λ's, define the *normal product* $\mathop{\gg}\limits_{\lambda\in\Lambda} L_\lambda$ to be $\mathop{\gg}\limits_{\lambda\in\Lambda} \boldsymbol{L}(\boldsymbol{F}(L_\lambda)$. By (1) and (2), $\mathop{\gg}\limits_{\lambda\in\Lambda} L_\lambda = \boldsymbol{L}(\mathop{\gg}\limits_{\lambda\in\Lambda} \boldsymbol{V}(L_\lambda))$ holds. Thus, we may also take the latter as the definition of $\mathop{\gg}\limits_{\lambda\in\Lambda} L_\lambda$. By Theorem 7(4), we have the following.

Theorem 8 *Let $\mathcal{K}_\lambda$ be classes of modal algebras, and let $\mathcal{C}_\lambda$ be classes of modal general frames for $\lambda \in \Lambda$. Then*

1. $\mathop{\gg}\limits_{\lambda\in\Lambda} \boldsymbol{L}(\mathcal{K}_\lambda) = \boldsymbol{L}(\mathop{\gg}\limits_{\lambda\in\Lambda} \mathcal{K}_\lambda)$,
2. $\mathop{\gg}\limits_{\lambda\in\Lambda} \boldsymbol{L}(\mathcal{C}_\lambda) = \boldsymbol{L}(\mathop{\gg}\limits_{\lambda\in\Lambda} \mathcal{C}_\lambda)$.

Theorem 9 *Suppose that L_λ is modal logic for $\lambda \in \Lambda$. Then*

$$\bigcap_{\lambda\in\Lambda} L_\lambda \supseteq \mathop{\gg}_{\lambda\in\Lambda} L_\lambda.$$

Theorem 10 *Let φ be a formula which is equivalent to a conjunction of formulas of the form $\Box^m(\psi \to \chi)$, where $m \geq 0$, χ is constructed by $\wedge, \Box, \Diamond, \top, \bot$ from propositional variables and ψ is obtained from propositional variables applying $\wedge, \vee, \Box, \Diamond, \top, \bot$ in such a way that no subformula of the form either $\psi_1 \vee \psi_2$ or $\Diamond\psi_1$ occurs in the scope of any $\Box$ and that ψ_1 and ψ_2 have no common propositional variable in a subformula of the form $\psi_1 \wedge \psi_2$. Then $\varphi \in \bigcap_{\lambda\in\Lambda} L_\lambda \Rightarrow \varphi \in \lll_{\lambda\in\Lambda} L_\lambda$.*

3 Shifted products of infinitely many modal logics

Suppose that $\boldsymbol{A_\lambda} = (A_\lambda, \wedge, \vee, \to, 0, 1, \Box_\lambda)$ is mono-modal algebra, for $\lambda \in \Lambda$. Define multi-modal algebra $\boldsymbol{A_\lambda}^\circ$ as $\boldsymbol{A_\lambda}^\circ = (A_\lambda, \wedge, \vee, \to, 0, 1, \{\Box_\mu \mid \mu \in \Lambda\})$ with $\Box_\mu a = a$ for all $\mu \neq \lambda$ and all $a \in A$. For modal general frame $\mathcal{F}_\lambda$ for $\lambda \in \Lambda$, define modal general frame $\mathcal{F}_\lambda^\circ$, with binary relations, as $\mathcal{F}_\lambda^\circ = (W_\lambda, \{R_\mu \mid \mu \in \Lambda\}, \mathcal{P})$, where $R_\mu = \Delta$ for all $\mu \neq \lambda$. For a class $\mathcal{K}$ of mono-modal algebras and a class $\mathcal{C}$ of modal general frames, $\mathcal{K}^\circ$ and $\mathcal{C}^\circ$ are defined in a natural way. Now, define the *shifted product* $\bigotimes_{\lambda\in\Lambda} \boldsymbol{A_\lambda}$ of $\boldsymbol{A_\lambda}$'s, (the shifted product $\bigotimes_{\lambda\in\Lambda} \mathcal{F}_\lambda$ of $\mathcal{F}_\lambda$'s) by $\bigotimes_{\lambda\in\Lambda} \boldsymbol{A_\lambda} = \lll_{\lambda\in\Lambda} \boldsymbol{A_\lambda}^\circ$ ($\bigotimes_{\lambda\in\Lambda} \mathcal{F}_\lambda = \lll_{\lambda\in\Lambda} \mathcal{F}_\lambda^\circ$). We can define *shifted product* of classes of modal algebras and those of classes of modal general frames in an obvious way. Then, we define the shifted product $\bigotimes_{\lambda\in\Lambda} L_\lambda$ of modal logics L_λ's by $\bigotimes_{\lambda\in\Lambda} L_\lambda = \boldsymbol{L}\big(\bigotimes_{\lambda\in\Lambda} \boldsymbol{F}(L_\lambda)\big)$.

Theorem 11 *Let $\mathcal{K}_\lambda$ be classes of modal algebras, and let $\mathcal{C}_\lambda$ be classes of modal general frames for $\lambda \in \Lambda$. Then*

1. $\bigotimes_{\lambda\in\Lambda} \boldsymbol{L}(\mathcal{K}_\lambda) = \boldsymbol{L}(\bigotimes_{\lambda\in\Lambda} \mathcal{K}_\lambda)$,
2. $\bigotimes_{\lambda\in\Lambda} \boldsymbol{L}(\mathcal{C}_\lambda) = \boldsymbol{L}(\bigotimes_{\lambda\in\Lambda} \mathcal{C}_\lambda)$.

Suppose that L_λ is a modal logic. Define multi-modal logics $L_\lambda{}^\circ$ as $L_\lambda{}^\circ = \underset{\mu\in\Lambda}{*} L_\mu$, where $L_\mu = \mathbf{K} \oplus \Box p \leftrightarrow p$ for $\mu \neq \lambda$ and $*$ denotes the *fusion* of logics.

Theorem 12 *For modal logics L_λ's,*

$$\bigotimes_{\lambda\in\Lambda} L_\lambda = \lll_{\lambda\in\Lambda} L_\lambda^\circ.$$

We define the multi-modal logic $[L_\lambda \mid \lambda \in \Lambda]$ to be $\underset{\lambda\in\Lambda}{*} L_\lambda \oplus \Box_{\lambda_i}\Box_{\lambda_j} p \leftrightarrow \Box_{\lambda_j}\Box_{\lambda_i} p \oplus \Diamond_{\lambda_i}\Box_{\lambda_j} p \to \Box_{\lambda_j}\Diamond_{\lambda_i} p$, for $i \neq j$. The following theorem lists some properties of $\otimes$.

Theorem 13 *Let L_λ be modal logic for $\lambda \in \Lambda$. Then*

1. *If L_λ's are consistent logics, then $\bigotimes_{\lambda\in\Lambda} L_\lambda$ is a conservative extension of L_λ,*
2. $\bigotimes_{\lambda\in\Lambda} L_\lambda \supseteq [L_\lambda \mid \lambda \in \Lambda]$.

References

[1] V. Shehtman, *Two-dimensional modal logics.* Math. Zametki, vol. 5, pp. 759 – 772, (In Russian), 1978.

[2] D. Gabbay and V. Shehtman, *Products of modal logics, Part I.* L. J. of IGPL, vol. 6 No. 1, pp. 73 – 146, 1998.

[3] Y. Hasimoto, *Normal products of modal logics.* to appear in *Advances in Modal Logic, Volume 3*, CSLI Publications.

Proof Theoretical Studies on Semilattice Relevant Logics

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1 Introduction

Relevant logics which are defined by "semilattice models" are called *semilattice relevant logics*; they have been studied in the literature, e.g., [3, 4, 5, 8, 9]. A typical semilattice relevant logic ${}^{\cup}\mathbf{R}$ is defined to be *the set of formulas that are valid in any* ${}^{\cup}\mathbf{R}$*-models*, where a ${}^{\cup}\mathbf{R}$-model $\langle I, \cdot, \mathbf{e}\rangle$ is a semilattice with identity $\mathbf{e}$ (i.e., $\langle I, \cdot, \mathbf{e}\rangle$ is an idempotent commutative monoid). This structure can be considered to be a "structure of information"—I is a set of pieces of information, $\cdot$ is a binary operator which combines two pieces of information, and $\mathbf{e}$ is an empty piece of information. The notion "$\alpha \models A$" (a formula A holds according to a piece α of information) is inductively defined in a natural way:

$$\alpha \models A{\to}B \iff \forall\beta \in I\ [(\beta \models A) \Rightarrow (\alpha{\cdot}\beta \models B)].$$

$$\alpha \models A{\wedge}B \iff (\alpha \models A) \text{ and } (\alpha \models B).$$

$$\alpha \models A{\vee}B \iff (\alpha \models A) \text{ or } (\alpha \models B).$$

(In this paper, formulas are constructed by the connectives $\to$ (implication), $\wedge$ (conjunction) and $\vee$ (disjunction).) We say that a formula A is *valid in* the model if and only if $\mathbf{e} \models A$.

In addition to ${}^{\cup}\mathbf{R}$, the semilattice relevant logics ${}^{\cup}\mathbf{T}$, ${}^{\cup}\mathbf{RW}$, and ${}^{\cup}\mathbf{TW}$ appear in this paper, where they are obtained by modifying the definition of ${}^{\cup}\mathbf{R}$-model and/or the definition of "$\models$" as follows. The definition of "$\alpha \models A{\to}B$" in ${}^{\cup}\mathbf{T}/{}^{\cup}\mathbf{TW}$-models is described as

$$\forall\beta \succeq \alpha\ [(\beta \models A) \Rightarrow (\alpha{\cdot}\beta \models B)]$$

where $\succeq$ is a binary relation on I, and ${}^{\cup}\mathbf{RW}/{}^{\cup}\mathbf{TW}$-models are commutative monoids without the idempotence postulate: $\alpha{\cdot}\alpha = \alpha$.

A point of semilattice models is that *"$\alpha \models A \Rightarrow \alpha{\cdot}\beta \models A$" is not always true*, which reflects the principle of relevant logics—adding an irrelevant assumption (β) makes a fall of truth value. This excludes "paradoxical" formulas, e.g., $A{\to}(B{\to}A)$, while acceptable formulas such as $A{\to}A$, $(A{\wedge}B){\to}A$, $A{\to}(A{\vee}B)$ and $(A{\wedge}(B{\vee}C)){\to}((A{\wedge}B){\vee}C)$ are valid in any semilattice models. Moreover, there are some marginal formulas that correspond to some postulates on models. For example, the formula $(A{\to}(A{\to}B)){\to}(A{\to}B)$ corresponds to the idempotence postulate: this formula is not in ${}^{\cup}\mathbf{RW}$ or ${}^{\cup}\mathbf{TW}$ while it is in ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$. (Since this formula is called *contraction*, the logics ${}^{\cup}\mathbf{RW}$ and ${}^{\cup}\mathbf{TW}$ are also called *contractionless logics*.) These examples show an advantage of semilattice models: various features of relevant logics are represented by simple devices of models.

The "orthodox" relevant logics (see, e.g., [1, 2]) and the semilattice relevant logics coincide in the $\{\to, \wedge\}$-fragments, but they have a slight difference if the connective $\vee$ exists. For example, it is known

([9]) that $\mathbf{R}_{\to\wedge\vee} \subsetneq {}^{\cup}\mathbf{R}$ where $\mathbf{R}_{\to\wedge\vee}$ is the set of provable formulas in the $\{\to,\wedge,\vee\}$-fragment of the well-known relevant logic $\mathbf{R}$. The orthodox relevant logics are usually defined by their proof systems, and models of them are relatively complicated. On the other hand semilattice relevant logics are defined by simple models; then development of proof theoretical studies is a natural requirement for semilattice relevant logics. This paper is a study in such a direction.

The results of this paper are summarized as follows.

1. Equivalence between the LK-style and LJ-style labelled sequent calculi. For the logic ${}^{\cup}\mathbf{R}$, there is a cut-free labelled sequent calculus, called $\mathbf{LK}^{\cup}\mathbf{R}$, where the labels $\alpha, \beta, \ldots$ are finite sets, and the both sides of a sequent are multisets of labelled formulas.

Axioms of $\mathbf{LK}^{\cup}\mathbf{R}$: $\quad \alpha : A \Rightarrow \alpha : A$

Inference rules of $\mathbf{LK}^{\cup}\mathbf{R}$:

- The weakening/contraction rules for both side. For example:

$$\frac{\Gamma \Rightarrow \Delta}{\alpha : A, \Gamma \Rightarrow \Delta}\ (\text{weakening left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha : A, \alpha : A}{\Gamma \Rightarrow \Delta, \alpha : A}\ (\text{contraction right})$$

- Rules for $\to$:

$$\frac{\Gamma \Rightarrow \Delta, \beta : A \quad \alpha \cup \beta : B, \Pi \Rightarrow \Sigma}{\alpha : A \to B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\to \text{ left}) \qquad \frac{\{a\} : A, \Gamma \Rightarrow \Delta, \alpha \cup \{a\} : B}{\Gamma \Rightarrow \Delta, \alpha : A \to B}\ (\to \text{ right})$$

  where a does not appear in the lower sequent of "$\to$ right".

- Rules for $\wedge$ and $\vee$. For example,

$$\frac{\alpha : A, \Gamma \Rightarrow \Delta \quad \alpha : B, \Gamma \Rightarrow \Delta}{\alpha : A \vee B, \Gamma \Rightarrow \Delta}\ (\vee \text{ left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha : A}{\Gamma \Rightarrow \Delta, \alpha : A \vee B}\ (\vee \text{ right})$$

  (Rules for $\wedge$ are the dual forms of the ones for $\vee$.)

This system is known to be complete in the following sense: $A \in {}^{\cup}\mathbf{R}$ if and only if $\mathbf{LK}^{\cup}\mathbf{R} \vdash \Rightarrow \emptyset : A$ (i.e., the sequent $\Rightarrow \emptyset : A$ is provable in $\mathbf{LK}^{\cup}\mathbf{R}$). On the other hand, there is another labelled sequent calculus, called $\mathbf{LJ}^{\cup}\mathbf{R}$, that is defined from $\mathbf{LK}^{\cup}\mathbf{R}$ by imposing the restriction that *the right-hand side of each sequent is a singleton.* (The names $\mathbf{LK}^{\cup}\mathbf{R}$ and $\mathbf{LJ}^{\cup}\mathbf{R}$ come from the well-known sequent calculi $\mathbf{LK}$ for the classical logic and $\mathbf{LJ}$ for the intuitionistic logic, where $\mathbf{LK}$ and $\mathbf{LJ}$ are equivalent to the label-free parts of $\mathbf{LK}^{\cup}\mathbf{R}$ and $\mathbf{LJ}^{\cup}\mathbf{R}$, respectively.) In this paper, we prove that $\mathbf{LK}^{\cup}\mathbf{R}$ *is a conservative extension of* $\mathbf{LJ}^{\cup}\mathbf{R}$ (although $\mathbf{LK}$ is *not* a conservative extension of $\mathbf{LJ}$); therefore the following three conditions are shown to be equivalent: (1) $A \in {}^{\cup}\mathbf{R}$. (2) $\mathbf{LK}^{\cup}\mathbf{R} \vdash \Rightarrow \emptyset : A$. (3) $\mathbf{LJ}^{\cup}\mathbf{R} \vdash \Rightarrow \emptyset : A$. Moreover, similar results for ${}^{\cup}\mathbf{T}$, ${}^{\cup}\mathbf{RW}$, and ${}^{\cup}\mathbf{TW}$ are proved. These are shown with a delicate transformation of sequent-proofs. Note that these results were already stated in [5], but the proof there was wrong (see the beginning of Section 3).

2. Hilbert-style axiomatization of ${}^{\cup}\mathbf{R}$/${}^{\cup}\mathbf{T}$. We prove that Hilbert-style systems for ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$ are obtained from the orthodox systems $\mathbf{R}$ and $\mathbf{T}$ by adding an extra inference rule. In [3], such a result for ${}^{\cup}\mathbf{R}$ has been proved where the extra rule is slightly different from this paper. Our extra rule corresponds to the "$\vee$ left" rule of $\mathbf{LJ}^{\cup}\mathbf{R}$, and our proof method is "translation of $\mathbf{LJ}^{\cup}\mathbf{R}({}^{\cup}\mathbf{T})$-sequents into formulas", which is clearer than the method of [3]. For ${}^{\cup}\mathbf{T}$, no other completeness proof for Hilbert-style systems has been known. Unfortunately we do not yet get such a result for ${}^{\cup}\mathbf{RW}/{}^{\cup}\mathbf{TW}$.

3. Equivalence between the commutative monoid semantics and the distributive semilattice semantics for ${}^{\cup}\mathbf{RW}/{}^{\cup}\mathbf{TW}$. In [4, 5, 8], two kinds of models are introduced for ${}^{\cup}\mathbf{RW}/{}^{\cup}\mathbf{TW}$: one is "commutative monoid" and the other is "distributive semilattice". In this paper we introduce labelled sequent calculi for both semantics. Then, by the analysis of these systems, we show the equivalence of the two semantics in the sense that *a formula A is valid in any commutative monoid models if and only if A is valid in any distributive semilattice models.* This equivalence for ${}^{\cup}\mathbf{RW}$ and for the $\{\to,\wedge\}$-fragment of ${}^{\cup}\mathbf{TW}$ have been proved in [4, 8] (where the proof methods are different from this paper), but the result for ${}^{\cup}\mathbf{TW}$ with the connective $\vee$ has not been proved so far.

Thus our results are classified into three, where the first one is a fundamental result on the proof theory of the semilattice relevant logics and the second and third ones solve some open problems, making good use of the first result. We hope that some more important property on the semilattice relevant logics will be shown by virtue of the fundamental result; for example, a decision procedure for ${}^{\cup}\mathbf{R}({}^{\cup}\mathbf{T})$ with the cut-free system $\mathbf{LJ}^{\cup}\mathbf{R}({}^{\cup}\mathbf{T})$ is an attractive goal. (It is open whether they are decidable or not.) We also hope to extend these results to other relevant logics, for example, **E** and the neighbors appearing in [6].

The structure of this paper is as follows. In Section 2, we present the definitions of the semilattice relevant logics ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$ with the semilattice models. Then we introduce the labelled sequent calculi $\mathbf{LK}^{\cup}\mathbf{R}$ and $\mathbf{LK}^{\cup}\mathbf{T}$, and we prove the completeness of them. In Section 3, we introduce the labelled sequent calculi $\mathbf{LJ}^{\cup}\mathbf{R}$ and $\mathbf{LJ}^{\cup}\mathbf{T}$, and we prove that $\mathbf{LK}^{\cup}\mathbf{R}$ (or $\mathbf{LK}^{\cup}\mathbf{T}$) is a conservative extension of $\mathbf{LJ}^{\cup}\mathbf{R}$ (or $\mathbf{LJ}^{\cup}\mathbf{T}$, respectively). In Section 4, we show that $\mathbf{LJ}^{\cup}\mathbf{R}$ and $\mathbf{LJ}^{\cup}\mathbf{T}$ can be changed into systems without the "weakening" rule. These systems are called $\mathbf{LI}^{\cup}\mathbf{R}$ and $\mathbf{LI}^{\cup}\mathbf{T}$, which are proved, in Section 5, to be sound with respect to the semilattice models. The results of Sections 2–5 imply the Main Theorem 5.4, which claims the equivalence between the conditions: $A \in X$, $\mathbf{LK}X \vdash \Rightarrow \emptyset : A$, $\mathbf{LJ}X \vdash \Rightarrow \emptyset : A$, and $\mathbf{LI}X \vdash \Rightarrow \emptyset : A$, for $X \in \{{}^{\cup}\mathbf{R}, {}^{\cup}\mathbf{T}\}$. In Section 6, we present Hilbert-style systems for ${}^{\cup}\mathbf{R}$ and ${}^{\cup}\mathbf{T}$, and we show the soundness and completeness of them. In Section 7, we investigate the logics ${}^{\cup}\mathbf{RW}$ and ${}^{\cup}\mathbf{TW}$. We introduce two kinds of labelled sequent calculi which reflect the two kinds of models (commutative monoid and distributive semilattice). All the results of Sections 2–5 are translated for ${}^{\cup}\mathbf{RW}/{}^{\cup}\mathbf{TW}$, and then we obtain the equivalence between the two semantics.

References

[1] A.R. Anderson and N.D. Belnap Jr., **Entailment: The Logic of Relevance and Necessity, Vol. 1**, (Princeton 1975).

[2] A.R. Anderson, N.D. Belnap Jr. and J.M. Dunn. **Entailment: The Logic of Relevance and Necessity, Vol. 2**, (Princeton 1992).

[3] G. Charlwood, *An axiomatic version of positive semilattice relevance logic*, **Journal of Symbolic Logic 46**, 233-239, (1981).

[4] S. Giambrone, R. K. Meyer and A. Urquhart, *A contractionless semilattice semantics*, **Journal of Symbolic Logic 52**, 526-529, (1987).

[5] S. Giambrone and A. Urquhart, *Proof theories for semilattice logics*, **Zeitschrift für Mathematische Logik und Grundlagen der Mathematik 33**, 433-439, (1987).

[6] R. Kashima, *Completeness of Implicational Relevant Logics*, **Logic Journal of the Interest Group in Pure and Applied Logics 8** 761-785, (2000).

[7] R. Kashima and N. Kamide, *Substructural implicational logics including the relevant logic E*, **Studia Logica 63**, 181-212, (1999).

[8] R. K. Meyer, S. Giambrone, A. Urquhart and E. P. Martin, *Further results on proof theories for semilattice logics*, **Zeitschrift für Mathematische Logik und Grundlagen der Mathematik 34**, 301-304, (1988).

[9] A. Urquhart, *Semantics for relevant logics*, **Journal of Symbolic Logic 37**, 159-169, (1972).

Learning Word-to-Meaning Mappings in Logical Semantics

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Siskind (1996, 2000) discusses the problem of learning word-to-meaning mappings from sentences coupled with their meanings. In his setup, sentence meanings are represented by first-order terms built up from constants and function symbols (called *conceptual symbols*), and word meanings are represented by first-order terms with zero or more individual variables. For instance, the meaning of *John lifted Mary* may be represented by $\mathrm{CAUSE}(\mathbf{John}, \mathrm{GO}(\mathbf{Mary}, \mathrm{UP}))$, where CAUSE, **John**, GO, **Mary**, and UP are conceptual symbols. The learner is fed with a stream of word strings coupled with expressions like the one above, and is expected to arrive at a word-to-meaning mapping like:

$$\begin{array}{lcl} \textit{John} & \mapsto & \mathbf{John} \\ \textit{lifted} & \mapsto & \mathrm{CAUSE}(x, \mathrm{GO}(y, \mathrm{UP})) \\ \textit{Mary} & \mapsto & \mathbf{Mary} \end{array} \tag{1}$$

Semantic composition is supposed to be done by substitution. An equivalent way of looking at it is to view word meanings as λ-terms like $\lambda xy.\mathrm{CAUSE}(x, \mathrm{GO}(y, \mathrm{UP}))$, where all subterms are of *first-order* types, and semantic composition as application plus β-reduction.

Siskind shows that one can reduce the complexity of learning by breaking down the learning process into two phases. In the first phase, the learner forms, for each word, the set of conceptual symbols involved in the meaning of that word. This can be done efficiently by an online algorithm that maintains two sets of conceptual symbols: the set of symbols that are possibly in the correct word meaning, and the set of symbols that are necessarily in the correct word meaning. The second phase then builds the correct meaning expression out of the symbol set learned in the first phase. For instance, suppose that the learner has arrived at the following association of words with symbol sets:

$$\begin{array}{lcl} \textit{John} & : & \{\mathbf{John}\} \\ \textit{lifted} & : & \{\mathrm{CAUSE}, \mathrm{GO}, \mathrm{UP}\} \\ \textit{Mary} & : & \{\mathbf{Mary}\} \end{array}$$

Then, given that the sentence *John lifted Mary* has meaning $\mathrm{CAUSE}(\mathbf{John}, \mathrm{GO}(\mathbf{Mary}, \mathrm{UP}))$, the learner can now uniquely determine the word-to-meaning mapping given in (1).

The aim of this paper is to study a generalization of Siskind's problem, where conceptual symbols, as well as variables, can be of arbitrary types, and meaning expressions are typed λ-terms built out of these, using λ-abstraction as well as application. This is the usual setup of logical semantics. For instance, the sentence *every man finds a unicorn* may have a meaning represented by $\forall x[\mathrm{MAN}(x) \to \exists y[\mathrm{UNICORN}(y) \wedge \mathrm{FIND}(x, y)]]$, which is just a shorthand for

$$\forall^{(e\to t)\to t}(\lambda x^{e}.\ \to^{t\to t\to t}(\mathrm{MAN}^{e\to t}(x))\ (\exists^{(e\to t)\to t}(\lambda y^{e}.\ \wedge^{t\to t\to t}(\mathrm{UNICORN}^{e\to t}(y))(\mathrm{FIND}^{e\to e\to t}(y)(x))))). \tag{2}$$

The meanings of the words in this sentence may be (suppressing the type when it is clear):

$$\begin{array}{lcl} \textit{every} & \mapsto & \lambda u^{e\to t} v^{e\to t}.\forall(\lambda x^{e}.\ \to (ux)(vx)) \\ \textit{man} & \mapsto & \mathrm{MAN} \\ \textit{finds} & \mapsto & \mathrm{FIND} \\ \textit{a} & \mapsto & \lambda u^{e\to t} v^{e\to t}.\exists(\lambda y^{e}.\ \wedge\ (uy)(vy)) \\ \textit{unicorn} & \mapsto & \mathrm{UNICORN} \end{array} \tag{3}$$

(I mostly follow the notational convention of Hindley 1997, except that I may write the application of two terms M and N either MN or $M(N)$, depending on which is more readable.)

Just as in the simple first-order case of Siskind, the problem of learning word-to-meaning mappings can be broken down into two subproblems: the problem of finding for each word the set of constants that appear in the meaning of that word, and the problem of building the correct λ-term for each word using the symbol set associated with that word. The first problem can be solved efficiently in exactly the same way as in the case of Siskind. How to solve the second problem was rather obvious in the case of Siskind, but becomes far from so in our generalized setting. I formulate the heart of the problem as a problem in typed λ-calculus.

Mapping Problem. Given m sets of distinct variables

$$\begin{aligned}\vec{x}_1 &= x_{1,1}^{\sigma_{1,1}}, \ldots, x_{1,n_1}^{\sigma_{1,n_1}} \\ &\vdots \\ \vec{x}_m &= x_{m,1}^{\sigma_{m,1}}, \ldots, x_{m,n_m}^{\sigma_{m,n_m}}\end{aligned}$$

and a λI-term $T^\rho[\vec{x}_1, \ldots, \vec{x}_m]$ (with free variables $\vec{x}_1, \ldots, \vec{x}_m$) in $\beta\eta$-normal form, find m λI-terms $S_1^{\tau_1}[\vec{x}_1], \ldots, S_m^{\tau_m}[\vec{x}_m]$ and a $BCI\lambda$-term $D^\rho[y_1^{\tau_1}, \ldots, y_m^{\tau_m}]$, each in $\beta\eta$-normal form, such that

$$D[S_1[\vec{x}_1], \ldots, S_m[\vec{x}_m]] \triangleright_{\beta\eta} T[\vec{x}_1, \ldots, \vec{x}_m].$$

I denote this problem by $T^\rho[\vec{x}_1; \ldots; \vec{x}_m]$, and I call $\langle S_1^{\tau_1}[\vec{x}_1], \ldots, S_m^{\tau_m}[\vec{x}_m]\rangle$ a *solution* to this problem.

In the solution to a mapping problem, $S_i^{\tau_i}[\vec{x}_i]$'s represent the 'meaning recipe' of individual words, where the free variables $\vec{x}_i$ are supposed to be filled by conceptual symbols. $D^\rho[y_1^{\tau_1}, \ldots, y_m^{\tau_m}]$ then gives the meaning recipe for the sentence. Thus I assume that simultaneous binding of multiple occurrences of a variable is possible in word meanings, but not in meaning recipes for sentences.

A 'mapping problem' of a more restricted kind corresponding to Siskind's setup always has at most one solution, and the condition under which a solution exists is easy to state. As for the general version, we have

Theorem 1. *Every mapping problem has infinitely many solutions.*

Theorem 2. *For any sequence of terms $\vec{M} = M_1^{\sigma_1}, \ldots, M_n^{\sigma_n}$, let $U_\rho[\vec{M}] = \lambda y^{\sigma_1 \to \cdots \to \sigma_n \to \rho}.yM_1 \ldots M_n$. Then, for every mapping problem $T^\rho[\vec{x}_1; \ldots; \vec{x}_m]$ such that $T^\rho[\vec{x}_1, \ldots, \vec{x}_m]$ is a $BCI\lambda$-term, $\langle U_\rho[\vec{x_1}], \ldots, U_\rho[\vec{x_m}]\rangle$ solves $T^\rho[\vec{x}_1; \ldots; \vec{x}_m]$.*

'Universal' solutions like $U_\rho[\vec{x}]$ are not very interesting: they not only derive correct sentence meanings but also many unwanted ones. The following example illustrates the case of a λI mapping problem.

Example 3. Consider a mapping problem $T^t[\vec{x}_1; \vec{x}_2]$, where $\vec{x}_1 = x_{1,1}^{(e\to t)\to t}, x_{1,2}^{t\to t\to t}, x_{1,3}^{e\to t}$, $\vec{x}_2 = x_{2,1}^{e\to t}$, and $T^t[\vec{x}_1, \vec{x}_2] = x_{1,1}(\lambda z^e.x_{1,2}(x_{1,3}z)(x_{2,1}z))$. This corresponds to the situation where the learner is faced with a sentence *everyone walks* coupled with the meaning $\forall z(\mathrm{PERSON}(z) \to \mathrm{WALK}(z))$, having arrived at the word-to-symbol-set association:

$$\begin{array}{lcl} \textit{everyone} & : & \{\forall, \to, \mathrm{PERSON}\} \\ \textit{walks} & : & \{\mathrm{WALK}\} \end{array}$$

A solution $\langle U_t[\vec{x}_1, \mathsf{W}^{(e\to e\to t)\to e\to t}],\ x_{2,1}\rangle$, where $\mathsf{W}^{(e\to e\to t)\to e\to t} = \lambda y^{e\to e\to t}x^e.yxx$, corresponds to the word-to-meaning mapping:

$$\begin{array}{lcl} \textit{everyone} & \mapsto & \lambda y^{((e\to t)\to t)\to(t\to t\to t)\to(e\to t)\to((e\to e\to t)\to e\to t)\to t}.y(\forall)(\to)(\mathrm{PERSON})(\mathsf{W}) \\ \textit{walks} & \mapsto & \mathrm{WALK} \end{array}$$

$\langle U_t[\vec{x}_1, \mathsf{W}^{(e\to e\to t)\to e\to t}],\ x_{2,1}\rangle$ solves not only $T^t[\vec{x}_1; \vec{x}_2]$, but also $V^t[\vec{x}_1; \vec{x}_2] = x_{1,1}(\lambda z^e.x_{1,2}(x_{2,1}z)(x_{1,3}z))$, corresponding to $\forall z(\mathrm{WALK}(z) \to \mathrm{PERSON}(z))$. Of course, $\langle \lambda v^{e\to t}.x_{1,1}(\lambda z^e.x_{1,2}(x_{1,3}z)(vz)),\ x_{2,1}\rangle$, corresponding to the word-to-meaning mapping:

$$\begin{array}{lcl} \textit{everyone} & \mapsto & \lambda v^{e\to t}.\forall z(\mathrm{PERSON}(z) \to vz) \\ \textit{walks} & \mapsto & \mathrm{WALK} \end{array}$$

only solves the former.

Certainly, the second solution in the above example is preferable to the first as a hypothesis for the learner. Perhaps what the learner should find is a 'minimal' solution, in the sense of solving as few unwanted problems as possible. To define a suitable sense of minimality, I introduce a 'definability relation' $\preceq$ between terms and a slightly weaker notion $\preceq_\rho$.

Definition 4. Let $M^\sigma[\vec{x}]$ and $N^\tau[\vec{x}]$ be two λ-terms in $\beta\eta$-normal form with the same set of free variables.

(i) We say that $N^\tau[\vec{x}]$ is *BCI-definable by* $M^\sigma[\vec{x}]$ and write $N^\tau[\vec{x}] \preceq M^\sigma[\vec{x}]$ if there is a closed $BCI\lambda$-term $P^{\sigma\to\tau}$ such that

$$P^{\sigma\to\tau}M^\sigma[\vec{x}] \triangleright_{\beta\eta} N^\tau[\vec{x}].$$

We write $N^\tau[\vec{x}] \simeq M^\sigma[\vec{x}]$ if $N^\tau[\vec{x}] \preceq M^\sigma[\vec{x}]$ and $N^\tau[\vec{x}] \succeq M^\sigma[\vec{x}]$; and write $N^\tau[\vec{x}] \prec M^\sigma[\vec{x}]$ if $N^\tau[\vec{x}] \preceq M^\sigma[\vec{x}]$ but $N^\tau[\vec{x}] \not\simeq M^\sigma[\vec{x}]$.

(ii) We write $N^\tau[\vec{x}] \preceq_\rho M^\sigma[\vec{x}]$ if $\lambda y^{\tau\to\rho}.yN^\tau[\vec{x}] \preceq M^\sigma[\vec{x}]$. We write $N^\tau[\vec{x}] \simeq_\rho M^\sigma[\vec{x}]$ if $N^\tau[\vec{x}] \preceq_\rho M^\sigma[\vec{x}]$ and $N^\tau[\vec{x}] \succeq_\rho M^\sigma[\vec{x}]$; and write $N^\tau[\vec{x}] \prec_\rho M^\sigma[\vec{x}]$ if $N^\tau[\vec{x}] \preceq_\rho M^\sigma[\vec{x}]$ but $N^\tau[\vec{x}] \not\simeq_\rho M^\sigma[\vec{x}]$.

Clearly, $N^\tau[\vec{x}] \preceq M^\sigma[\vec{x}]$ implies $N^\tau[\vec{x}] \preceq_\rho M^\sigma[\vec{x}]$ for all ρ. Also, if τ 'ends in' ρ, $N^\tau[\vec{x}] \preceq_\rho M^\sigma[\vec{x}]$ implies $N^\tau[\vec{x}] \preceq M^\sigma[\vec{x}]$. The two relations do not coincide in general: $x^e \preceq_t \lambda y^{e\to t}.yx$, but $x^e \not\preceq \lambda y^{e\to t}.yx$. Both $\preceq$ and $\preceq_\rho$ are reflexive and transitive. Also, both relations are decidable. The latter follows from the known fact that every type is inhabited by finitely many $BCI\lambda$-terms in $\beta\eta$-normal form (van Benthem 1991).

Theorem 5. *If* $\langle S_1[\vec{x}_1], \ldots, S_m[\vec{x}_m]\rangle$ *solves* $T^\rho[\vec{x}_1; \ldots; \vec{x}_m]$, *and* $S_i[\vec{x}_i] \preceq_\rho P[\vec{x}_i]$, *then* $\langle S_1[\vec{x}_1], \ldots, S_{i-1}[\vec{x}_{i-1}], P[\vec{x}_i], S_{i+1}[\vec{x}_{i+1}], \ldots, S_m[\vec{x}_m]\rangle$ *solves* $T^\rho[\vec{x}_1; \ldots; \vec{x}_m]$.

Even though a mapping problem has infinitely many solutions, many of them turn out to be equivalent with respect to $\simeq$ and $\simeq_\rho$. A trivial example is $T^t[x^e; y^{e\to t}] = yx$, which has just two solutions modulo $\simeq$, namely $\langle x,\ y\rangle$ and $\langle \lambda y^{e\to t}.yx,\ y\rangle$ (which, in turn, are equivalent with respect to $\simeq_t$). I conjecture that every mapping problem has only finitely many solutions modulo $\simeq$.

The relations $\preceq$ and $\preceq_\rho$ can be naturally extended to solutions of mapping problems. In general, a mapping problem $T^\rho[\vec{x}_1; \ldots; \vec{x}_n]$ can have more than one $\preceq_\rho$-incomparable minimal solution.

Example 6. A problem $T^t[x^e; y^{e\to e\to t}; z^e] = yzx$ has the following solutions, among others:

$$\langle \lambda y^{e\to e\to t}z^e.yzx,\ y,\ \lambda y^{e\to e\to t}v^{(e\to e\to t)\to e\to t}.v(\lambda xz.yzx)z\rangle,$$
$$\langle \lambda y^{e\to e\to t}v^{(e\to e\to t)\to e\to t}.vyx,\ y,\ \lambda y^{e\to e\to t}.yz\rangle.$$

Since $\lambda y^{e\to e\to t}z^e.yzx \prec_t \lambda y^{e\to e\to t}v^{(e\to e\to t)\to e\to t}.vyx$ and $\lambda y^{e\to e\to t}v^{(e\to e\to t)\to e\to t}.v(\lambda xz.yzx)z\rangle \succ_t \lambda y^{e\to e\to t}.yz$, the two solutions are incomparable with respect to $\preceq_t$. Their 'meet',

$$\langle \lambda y^{e\to e\to t}z^e.yzx,\ y,\ \lambda y^{e\to e\to t}.yz\rangle,$$

does not solve $T_t[x^e; y^{e\to e\to t}; z^e]$, and indeed the two solutions can be shown to be minimal with respect to $\preceq_t$.

The above example suggests that it may not be desirable for the purpose of learning to collect all minimal solutions to a given mapping problem. A simple algorithm exists that finds a 'reasonably small' solution to a mapping problem, together with a pure application term as a sentence meaning recipe.

References

Benthem, Johan van. 1991. *Language in Action: Categories, Lambdas and Dynamic Logic.* Amsterdam: North-Holland. (Second edition, 1995, Cambridge, Mass.: MIT Press.)

Hindley, J. Roger. 1997. *Basic Simple Type Theory.* Cambridge: Cambridge University Press.

Siskind, J. Mark. 1996. A computational study of cross-situational techniques for learning word-to-meaning mappings. *Cognition* **61**(1–2), 39–91. (Reprinted in Michael Brent, ed., 1996, *Computational Approaches to Language Acquisition*, pp. 39–91, Amsterdam: Elsevier.)

Siskind, J. Mark. 2000. Learning word-to-meaning mappings. In Peter Broeder and Jaap Murre, eds., *Models of Language Acquisition: Inductive and Deductive Approaches*, pp. 121–153. Oxford: Oxford University Press.

Some properties on orthologics

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We have only little knowledge about the class of orthologics now. We intend this study to be a small step to throw light on the problems of these area of mathematical logics. Here we discuss some logical properties, and the structure of the lattice of modular orthologics.

1 Preliminaries

The notion of *orthologics* stems from an algebraic structure called *ortholattice*, which we will define first.

Definition 1 (Ortholattice) An *ortholattice* is an algebraic structure $\mathfrak{A} = \langle \mathrm{A}, \cap, \cup, {}', 0.1 \rangle$, which is a bounded lattice with a unary operation $(\cdot)'$ that satisfies the following: for any $x, y \in A$,

(a) $x'' = x$. (b) $x \cap x' = 0$. (c) $x \leq y$ implies $y' \leq x'$.

∎

Our propositional language consists of the following primitive symbols: a collection of propositional variables $p, q, r, p_0, p_1, \ldots$ etc., a propositional constant $\bot$ (falsity), a unary connective $\neg$ (negation), a binary connective $\wedge$ (conjunction), and a pair of parentheses (,). The set Φ of all formulas of this language is defined in a usual way, and the notion of orthologics is defined in the following way.

Definition 2 (Orthologic) ([1]) An *orthologic* L on the set Φ of formulas is a subset of the product $\Phi \times \Phi$ (write $\alpha \vdash_L \beta$, instead of $\langle \alpha, \beta \rangle \in L$) which includes the following axiom schemes and is closed under the following inference rules:

Axiom schemes:

(Ax1) $\alpha \vdash_L \alpha$

(Ax2) $\neg\neg\alpha \vdash_L \alpha$

(Ax3) $\alpha \vdash_L \neg\neg\alpha$

(Ax4) $\alpha \wedge \beta \vdash_L \alpha$

(Ax5) $\alpha \wedge \beta \vdash_L \beta$

(Ax6) $\alpha \wedge \neg\alpha \vdash_L \beta$

(Ax7) $\bot \vdash_L \alpha$

Inference Rules:

(R1) $$\frac{\alpha \vdash_L \beta \quad \beta \vdash_L \gamma}{\alpha \vdash_L \gamma}$$

(R2) $$\frac{\alpha \vdash_L \beta \quad \alpha \vdash_L \gamma}{\alpha \vdash_L \beta \wedge \gamma}$$

(R3) $$\frac{\alpha \vdash_L \beta}{\neg\beta \vdash_L \neg\alpha}$$

The intersection of all orthologics on Φ, that is called *the* (*smallest*) *orthologic*, is denoted by O. ∎

The class of all ortholattices is defined by a set of identities, which means that the class forms a *variety*. This variety is denoted by $\mathcal{O}$. The orthologics O is complete with respect to the variety $\mathcal{O}$. The most frequently considered subvarieties of $\mathcal{O}$, the variety of orthomodular lattices ($\mathcal{OML}$), the variety

of modular ortholattices ($\mathcal{MOL}$), and the variety of Boolean algebras($\mathcal{BA}$) are obtained by adding the following identities to the defining identities of $\mathcal{O}$.

For $\mathcal{OML}\cdots\cdots x\cap\{(x\cap y)\cup x'\}=x\cap y$ (orthomodular law)

For $\mathcal{MOL}\cdots\cdots x\cap\{(x\cap y)\cup z\}=(x\cap y)\cup(x\cap z)$ (modular law)

For $\mathcal{BA}\cdots\cdots\cdots x\cap(y\cup z)=(x\cap y)\cup(x\cap z)$ (distributive law)

Of course, by adding suitable axiom schemes to O, we consider the (*smallest*) *orthomodular logic* OML, the (*smallest*) *modular orthologic* MOL, and the *classical logic* CL, which correspond to $\mathcal{OML}, \mathcal{MOL}$, and $\mathcal{BA}$ respectively.

2 Variable separation, variable sharing, and interpolation

For a formula α, the set of all propositional variables appeared in α is denoted by $V(\alpha)$. For an orthologic L, the binary relation of logically equivalence in L is denoted by $\equiv_L$. That is, for formulas α, β, $\alpha\equiv_L\beta$ if and only if $\langle\alpha,\beta\rangle\in L$ and $\langle\beta,\alpha\rangle\in L$. Note that $V(\alpha)=\emptyset$ means that α is $\bot$ or α is $\top(=\neg\bot)$.

Definition 3 Let L be an orthologic.

(1) L has the *variable separation property* (VSEP) if and only if for $\alpha,\beta\in\Phi$ such that $V(\alpha)\cap V(\beta)=\emptyset$, $\langle\alpha,\beta\rangle\in L$ implies that $\alpha\equiv_L\top$ or else $\beta\equiv_L\bot$.

(2) L has the *variable sharing property* (VSHP) if and only if for $\alpha_1,\alpha_2,\beta_1,\beta_2\in\Phi$, such that $V(\alpha_1,\alpha_2)\cap V(\beta_1,\beta_2)=\emptyset$, $\langle\alpha_1\wedge\beta_1,\alpha_2\vee\beta_2\rangle\in L$ implies that $\langle\alpha_1,\alpha_2\rangle\in L$ or else $\langle\beta_1,\beta_2\rangle\in L$.

(3) L has the *interpolation property* (IP) if and only if for $\alpha,\beta\in\Phi$, if $\langle\alpha,\beta\rangle\in L$, then there exists a formula γ which satisfies the following:

(a) $\langle\alpha,\gamma\rangle\in L$ and $\langle\gamma,\beta\rangle\in L$.

(b) $V(\gamma)\subseteq V(\alpha)\cap V(\beta)$. ∎

It is easily seen that IP implies VSEP. A sufficient algebraic condition for an orthologic to have IP was obtained in [2], and only O and CL are orthologics that are known to have IP at present. On VSEP and VSHP of orthologics, we consider the following constructions of ortholattices.

1. **reversing**: For an ortholattice $\mathfrak{A}=\langle A,\cap,\cup,(\cdot)',0,1\rangle$, a *reverse* of $\mathfrak{A}$ is an ortholattice $\overline{\mathfrak{A}}=\langle A,\cup,\cap,(\cdot)',1,0\rangle$. That is, only the order relation in $\overline{\mathfrak{A}}$ is redefined as the converse of the order relation in $\mathfrak{A}$.

2. **$\{0,1\}$-pasting**: For ortholattices $\mathfrak{A}=\langle A,\cap,\cup,(\cdot)',0_A,1_A\rangle$ and $\mathfrak{B}=\langle B,\cap,\cup,(\cdot)',0_B,1_B\rangle$, $\{0,1\}$-*pasting* of $\mathfrak{A}$ and $\mathfrak{B}$ is $\mathfrak{A}+\mathfrak{B}=\langle(A\cup B)/\approx,\cap,\cup,(\cdot)',0,1\rangle$, where $\approx$ is an equivalence relation defined $\approx:=\{(x,x)\mid x\in A\cup B\}\cup\{(0_A,0_B),(0_B,0_A),(1_A,1_B),(1_B,1_A)\}$. That is, $\mathfrak{A}+\mathfrak{B}$ is an ortholattice in which 0_A and 0_B are identical to the new smallest element 0, and 1_A and 1_B are identical to the new largest element 1 and others are inherited from the original $\mathfrak{A}$ and $\mathfrak{B}$.

Indeed, the reverse of an ortholattice is proved to be an ortholattice, and moreover, the varieties $\mathcal{OML}$, $\mathcal{MOL}$, and $\mathcal{BA}$ are closed under reversing. On the other hand, $\mathcal{O}$ and $\mathcal{OML}$ are closed under $\{0,1\}$-pasting, whereas $\mathcal{MOL}$ and $\mathcal{BA}$ are not.

Using the construction of $\{0,1\}$-pasting, the next theorem on VSEP can be proved.

Theorem 4 Let $\mathcal{C}$ be a class of ortholattices closed under $\{0,1\}$-pasting, and $L=L(\mathcal{C})$ the orthologic characterized by $\mathcal{C}$. Then L has the variable separation property. □

By the above theorem, we now recognize that the orthologic OML has VSEP. About VSHP, we can show only an algebraic equivalent conditions below at present.

Theorem 5 Let $\mathcal{C}$ be a class of ortholattices closed under reversing and $\{0,1\}$-pasting, and $L=L(\mathcal{C})$ the orthologic characterized by $\mathcal{C}$. Then the following 3 conditions are equivalent.

(1) L has the variable sharing property.

(2) Suppose $\alpha, \beta \in \Phi$, $V(\alpha) \cap V(\beta) = \emptyset$. Then $\alpha \vee \beta \equiv_L \top$ implies $\alpha \equiv_L \top$ or $\beta \equiv_L \top$.

(3) Suppose $\alpha, \beta \in \Phi$, $V(\alpha) \cap V(\beta) = \emptyset$. Then $\alpha \wedge \beta \equiv_L \bot$ implies $\alpha \equiv_L \bot$ or $\beta \equiv_L \bot$. □

It is easily seen that CL has VSHP.

Question: Is CL the only orthologic that has VSHP?

3 A chain of modular orthologics

Finally, we consider extremely simple ortholattices and their corresponding logics. An ortholattice is *lantern-type* if and only if it satisfies that for any x, y, $0 < x \leq y < 1$ implies $x = y$. An element a in a lantern-type ortholattice is a *middle element* if and only if $0 < a < 1$. A lantern-type ortholattice that has k middle elements is denoted by $\mathfrak{A}_k$. Note that k is always even because of the operation $(\cdot)'$. Also, the orthologic characterized by an ortholattice $\mathfrak{A}$ is denoted by $L(\mathfrak{A})$. It is proved easily that for any $k \in \omega$, $L(\mathfrak{A}_k)$ is a *modular orthologic* since $\mathfrak{A}_k$ is a *modular ortholattice*.

On the structure of modular orthologics characterized by lantern-type ortholattices, the next lemma holds.

Lemma 6

(1) For all $n \in \omega$, $L(\mathfrak{A}_{2n}) \supseteq L(\mathfrak{A}_{2n+2})$.

(2) For all $n \geq 1$, $L(\mathfrak{A}_{2n}) \neq L(\mathfrak{A}_{2n+2})$. □

Proof of (1) is straightforward because $\mathfrak{A}_{2n}$ can be embedded into $\mathfrak{A}_{2n+2}$. To show (2) above, employ $2n+1$ propositional variables $p_0, p_1, p_2, \ldots, p_{2n}$, and build up the following formulas:

$$\alpha_{2n} := p_0 \wedge \bigwedge_{1 \leq i < j \leq 2n} (p_i \vee p_j). \qquad \beta_{2n} := \bigvee_{i=1}^{2n} (p_0 \wedge p_i)$$

Then the following proposition can be proved, which leads to Lemma 6 (2).

Proposition 7

(1) For $n \geq 1$, $\langle \alpha_{2n}, \beta_{2n} \rangle \notin L(\mathfrak{A}_{2n+2})$.

(2) For $n \in \omega$, $\langle \alpha_{2n}, \beta_{2n} \rangle \in L(\mathfrak{A}_{2n})$. □

By Lemma 6, we have the following theorem on the lattice of modular orthologics.

Theorem 8

(1) $L(\mathfrak{A}_0) = L(\mathfrak{A}_2) \supset L(\mathfrak{A}_4) \supset \cdots\cdots \supset L(\mathfrak{A}_{2n}) \supset L(\mathfrak{A}_{2n+2}) \supset \cdots$.

(2) Orthologics which are characterized by lantern-type ortholattices form an infinite descending chain.

(3) There exist at least countable modular orthologics. □

These analysis highly depends on their simple form of lantern-type ortholattices. It is unknown at least today whether there exist other classes of ortholattices in which each propositional logic determined by its single member can be distinguished from each other by a special form of pairs of formulas.

Question: Can we axiomatize the orthologic $L := \bigcap \{L(\mathfrak{A}) \mid \mathfrak{A}$ is a lantern-type ortholattice$\}$?

References

[1] Goldblatt R.I., *Semantic analysis of orthologic*, Journal of Philosophical Logic 3, 19–35 (1974).

[2] Miyazaki Y., *The super-amalgamation property of the variety of ortholattices*, Reports on Mathematical Logic 33, 45–63 (1999).

Tableau System and Theorem Prover for Temporal Logic $\mathbf{K_t}$

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1 Introduction

Sequent systems and tableau systems for temporal logics have been discussed in papers [1, 3, 4, 7]. Nishimura [7] and Kashima [3] introduced sequent systems, and Gore [1] studied a tableau system.

In this paper, we introduce sequent systems and tableau ones for the minimal temporal logic $\mathbf{K_t}$. Our aim is to implement a decision algorithm of $\mathbf{K_t}$ based on tableau systems as a theorem prover, since tableau systems are well-suited for proof search by computers. To get an efficient algorithm, we propose a restricted tableau system for $\mathbf{K_t}$. Main difficulty in implementation is that rules of a restricted tableau system for $\mathbf{K_t}$ cause two kinds of loops. Heuerding [2] and Mouri [6] have introduced *histories* into sequent systems for **KT** and **S4** for loop-checking. Matsumoto [5] has applied a special modal operator by Mouri [6] to the tableau system for **KT** to eliminate loops. Therefore, we expect that *histories* will be useful for loop-checking in the tableau system for $\mathbf{K_t}$.

2 Temporal Logic $\mathbf{K_t}$

In temporal logic $\mathbf{K_t}$, $[F]$ and $[P]$ are used as temporal operators. Formulas A, B, $\cdots$ are defined in usual way, and $[F]A$ and $[P]A$ are read as "*at all future times* A" and "*at all past times* A", respectively. Both $\langle F\rangle A$ and $\langle P\rangle A$ are abbreviations of $\neg[F]\neg A$ and $\neg[P]\neg A$, respectively. The temporal logic $\mathbf{K_t}$ has classical tautologies and the following as axioms, and has modus ponens and the rule of necessitation for temporal operators $[F]$ and $[P]$ as rules of inference.

$$\begin{array}{lcllcl} KF & : & [F](A\supset B)\supset([F]A\supset[F]B), & KP & : & [P](A\supset B)\supset([P]A\supset[P]B),\\ 4F & : & [F]A\supset[F][F]A, & 4P & : & [P]A\supset[P][P]A,\\ CF & : & A\supset[F]\langle P\rangle A, & CP & : & A\supset[P]\langle F\rangle A. \end{array}$$

Let W be a non-empty set of possible worlds, and let R_T be a binary relation on W. A Kripke frame for temporal logic is defined to be a pair (W, R_T). Kripke semantics are given in the usual way, and valuations $\models$ for $[F]A$ and $[P]A$ are defined as follows:

$$u\models[F]A \text{ iff for all } v\in W\ (uR_Tv \text{ implies } v\models A),\quad u\models[P]A \text{ iff for all } v\in W\ (vR_Tu \text{ implies } v\models A).$$

Proposition 2.1 *Let (W, R_T) be any Kripke frame. Then, $4F$ and $4P$ are valid on (W, R_T) iff R_T is transitive.*

A Kripke frame (W, R_T) satisfying a condition that R_T is transitive is called $\mathbf{K_t}$-frame. It can be easily to show that the completeness theorem holds for $\mathbf{K_t}$, i.e $\mathbf{K_t}\not\vdash A$ iff there exists $\mathbf{K_t}$-model M such that $M\not\models A$.

Greek capital letters $\Gamma, \Delta\cdots$ denote finite sets of formulas. The set of all subformulas of A and the set of all subformulas of formulas in Γ are denoted by $Sub(A)$ and $Sub(\Gamma)$, respectively. The notation $\neg Sub(\Gamma)$ is abbreviation of $\{\neg A | A\in Sub(\Gamma)\}$. The set of all propositional variables in A is denoted by $Var(A)$.

3 Sequent System for $\mathbf{K_t}$

In this section, we introduce three sequent systems for temporal logic $\mathbf{K_t}$. All of them lack the cut-elimination property. Using Takano's method, however, we can derive some important logical properties for $\mathbf{K_t}$ using these systems.

The sequent system $S(\mathbf{K_t})$ is obtained from the propositional fragment of the sequent system **LK** for the classical logic by adding the following two temporal rules:

$$\frac{[F]\Gamma, \Gamma \to [P]\Delta, [P]\Pi, A}{[F]\Gamma \to [P]\Delta, \Pi, [F]A}\ (T1) \qquad \frac{[P]\Gamma, \Gamma \to [F]\Delta, [F]\Pi, A}{[P]\Gamma \to [F]\Delta, \Pi, [P]A}\ (T2)$$

The cut rule whose cut formula is restricted to a subformula of formulas in the lower sequent is called the *acceptable cut*. The sequent system obtained from $S(\mathbf{K_t})$ by replacing cut rule by acceptable cut rule is denoted by $S(\mathbf{K_t})^*$. Furthermore, a restricted sequent system $S(\mathbf{K_t})^{**}$ is defined as the system $S(\mathbf{K_t})^*$ with conditions $[P]\Pi \subseteq Sub(\Gamma \cup \Delta \cup \{A\})$ in applying the rule $(T1)$ and $[F]\Pi \subseteq Sub(\Gamma \cup \Delta \cup \{A\})$ in applying the rule $(T2)$. Then we can prove that all of sequent systems $S(\mathbf{K_t})$, $S(\mathbf{K_t})^*$ and $S(\mathbf{K_t})^{**}$ are equivalent. The following interpolation theorem for Kt is shown by using Maehara's method in $S(\mathbf{K_t})^*$.

Theorem 3.1 (interpolation theorem) *If $A \supset B$ is provable in $\mathbf{K_t}$, then there exists a formula C such that* (i) *$A \supset C$ and $C \supset B$ are both provable in $\mathbf{K_t}$ and* (ii) *$Var(C) \subseteq Var(A) \cap Var(B)$.*

The sequent system $S(\mathbf{K_t})^{**}$ has the *subformula property*, i.e. every formula A provable in $S(\mathbf{K_t})^{**}$ has such a proof P that every formula appearing in P is a subformula of A. Therefore, we can get a decision procedure for $S(\mathbf{K_t})^{**}$ by using Gentzen's method. However the procedure is not always efficient. In order to obtain a more efficient decision procedure, we propose the procedure with tableau system in the next section.

4 Tableau System for $\mathbf{K_t}$

In this section, we propose a tableau system for $\mathbf{K_t}$. A tableau system consist of a set of tableau rules. A tableau for Γ is a finite tree with the root Γ, each of whose nodes carries a finite set of formulas. To check whether A is provable in a tableau system or not, we give $\neg A$ to the tableau procedure as an input. A tableau is *closed* if all of leaves of the tableau end with $\{f\}$. Otherwise, it is *open*. A tableau system for $\mathbf{K_t}$, denoted by $T(\mathbf{K_t})$, is obtained by adding the following tableau rules to the standard tableau system for classical logic, and the right tableau is an example of a closed tableau for CF:

$$\frac{\Gamma}{\Gamma, A \mid \Gamma, \neg A}\ (cut)$$

$$\frac{\Delta, [F]\Gamma, \neg[F]A}{\Gamma, [F]\Gamma, \neg A}\ (K4F) \qquad \frac{\Delta, [P]\Gamma, \neg[P]A}{\Gamma, [P]\Gamma, \neg A}\ (K4P)$$

$$\frac{\Gamma, \neg[F]A}{\neg[P]\neg\Gamma, \neg A}\ (CF) \qquad \frac{\Gamma, \neg[P]A}{\neg[F]\neg\Gamma, \neg A}\ (CP)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(A \supset [F]\neg[P]\neg A)}{A, \neg[F]\neg[P]\neg A}\ (CF)}{\neg[P]\neg A, \neg\neg[P]\neg A}}{\neg[P]\neg A, [P]\neg A}\ (K4P)}{\neg\neg A, \neg A, [P]\neg A}}{A, \neg A, [P]\neg A}}{f}$$

A closed tableau for CF

Here, we face with a problem: it is hard to search a tableau of $T(\mathbf{K_t})$ for a given formula A, because the cut-formula of (cut) is arbitrary. Therefore, we give conditions for applying some rules of $T(\mathbf{K_t})$ as Takano [8]. A restricted tableau system $T(\mathbf{K_t})^*$ is obtained by replacing (cut), (CF) and (CP) of $T(\mathbf{K_t})$ with the following restricted rules, respectively:

$$\frac{\Gamma}{\Gamma, A \mid \Gamma, \neg A}\ (cut)^* \quad \text{where } A \in Sub(\Gamma)$$

$$\frac{\Gamma, \neg[F]A}{\neg[P]\neg\Gamma, \neg A}\ (CF)^* \quad \text{where } \neg[P]\neg\Gamma \subseteq Sub(A)$$

$$\frac{\Gamma, \neg[P]A}{\neg[F]\neg\Gamma, \neg A}\ (CP)^* \quad \text{where } \neg[F]\neg\Gamma \subseteq Sub(A)$$

¿From these conditions, we can say that any tableau of $T(\mathbf{K_t})^*$ for Γ is constructed by formulas in $Sub(\Gamma) \cup \neg Sub(\Gamma) \cup \{f\}$. This property is very useful for searching tableaus.

Theorem 4.1 *The tableau system $T(\mathbf{K_t})^*$ is sound with respect to $\mathbf{K_t}$-frames.*

In order to show completeness of $T(\mathbf{K_t})^*$ we need some technical machinery (see [1] for the details).

Conjecture 4.2 *The tableau system $T(\mathbf{K_t})^*$ is complete with respect to $\mathbf{K_t}$-frames.*

If we check all nodes of each tableau and find loops, then we stop constructing tableau any more. Hence, we can show the decidability of $T(\mathbf{K_t})^*$. But the loop-checking causes inefficiency.

5 Implementing Theorem Prover for $\mathbf{K_t}$

In implementing theorem prover for $\mathbf{K_t}$, we have to avoid two kinds of loops caused by applications of tableau rules. One is caused by applications of $(K4F)$ or $(K4P)$, the other is caused by alternating applications of (CF) and (CP) as follows:

$$\dfrac{\dfrac{[F]\neg[F]\neg A, \neg[F]A, \neg A}{[F]\neg[F]\neg A, \neg[F]A, \neg A}\ (K4F)}{\vdots}\ (K4F) \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{A, \neg[F]\neg[P]\neg A}{\neg[P]\neg A, \neg\neg[P]\neg A}\ (CF)}{\neg[P]\neg A, [P]\neg A}\ (CP)}{\neg\neg A, \neg[F]\neg[P]\neg A}\ (CP)}{A, \neg[F]\neg[P]\neg A}\ (CF)}{\vdots}$$

To avoid these loops, we will need to introduce "*histories*" as Heuerding [2] and Mouri [6] did. This will be left for further work.

References

[1] R. Goré, *Tableau Methods for Modal and Temporal Logics*, in Handbook of Tableau Methods, eds. by M. D'Agostino, D. M.Gabby, R. Hähnle and J. Posegga, Kluwer Academic Publishers, 1999.

[2] A. Heuerding, M. Seyfried and H. Zimmermann, *Efficient Loop-Check for Backward Proof Search in Some Non-Classical Propositional Logics*, eds. by P. Miglioli, U. Moscato, D. Mundici and M. Ornaghi, Tableaux 96, LNCS 1071, pp. 201-225, 1996.

[3] R. Kashima, *Cut-Free Sequent Calculi for Some Tence Logics*, Studia Logica 53, pp. 199-135, 1994.

[4] A. Maruyama, S. Tojo and H. Ono, *Decidability of Temporal Epistemic Logics for Multi-Agent Models*, Proceedings of the ICLP'01 Workshop on Computational Logic in Multi-Agent Systems (CLIMA-01), pp. 31-40, 2001.

[5] T. Matsumoto and H. Ono, *Theorem Prover for Modal Logics with Tableau Method*, Proceedings of International Workshop RPC'01, pp. 42-59, 2001.

[6] M. Mouri, *An Efficient Construction of Counter-Models for Modal Logic S4*, Proceedings of International Workshop RPC'01, pp. 247-256, 2001.

[7] H. Nishimura, *A Study of Some Tense Logics by Gentzen's Sequential Method*, Publications of the Research Institute for Mathematical Sciences, Kyoto University 16, pp. 343-353, 1980.

[8] M. Takano, *Subformula Property as a Substitute for Cut-Elimination in Modal Propositional Logics*, Mathematica Japonica 37, pp. 1129-1145, 1992.

A Sahlqvist Theorem for Relevant Modal Logics

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1 Introduction

Kripke-completeness of every classical modal logic with Sahlqvist formulas is one of the basic general results on completeness of classical modal logics. This paper shows a Sahlqvist theorem for modal logic over the relevant logic **R** in terms of Routley-Meyer semantics. Our completeness result covers most of completeness results of relevant modal logics so far. It is shown that usual Sahlqvist theorem for classical modal logics can be obtained as a special case of our theorem.

Below, $\&, \Rightarrow, \Leftrightarrow, \forall$ and $\exists$ are used to denote respectively conjunction, implication, equivalence, universal and existential quantifiers in the metalanguage. Further, Prop and Wff denote the set of all propositional variables and formulas, respectively.

2 Preliminaries

A *regular relevant modal logic over* **R** (a regular logic over **R**, for short) is a logic obtained from **R** by adding the axioms

$$\Box A \wedge \Box B \to \Box(A \wedge B) \qquad \Diamond(A \vee B) \to \Diamond A \vee \Diamond B$$

and the rules of inference

$$\frac{A \to B \quad A}{B} \qquad \frac{A \quad B}{A \wedge B} \qquad \frac{A \to B}{\Box A \to \Box B} \qquad \frac{A \to B}{\Diamond A \to \Diamond B}.$$

The least regular logic over **R** is called $\mathbf{R.C}_{\Box\Diamond}$. In the following, we assume that **L** denotes any regular logic over **R**.

Our models are obtained by extending the enlarged **R**-model in the sense of [5]. An $\mathbf{R.C}_{\Box\Diamond}$-*frame* is a 7-tuple $\langle O, W, R, S_\Box, S_\Diamond, ^*, u\rangle$, where W is a non-empty set of all worlds, O is a non-empty subset of W, R is a ternary relation on W, $S_\Box$ and $S_\Diamond$ are binary relations on W, * is an unary operation on W and u, called the universal world, is a element of W. An $\mathbf{R.C}_{\Box\Diamond}$-frame $\langle O, W, R, S_\Box, S_\Diamond, ^*, u\rangle$ satisfies first order sentences in the predicates O, R, $S_\Box$, $S_\Diamond$, * and the constant u which are omitted in this paper.

We call a pair $\langle \mathfrak{F}, V\rangle$ an $\mathbf{R.C}_{\Box\Diamond}$-*model* on an $\mathbf{R.C}_{\Box\Diamond}$-frame $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, u\rangle$ (we simply say an $\mathbf{R.C}_{\Box\Diamond}$-model), where $\mathfrak{F}$ is an $\mathbf{R.C}_{\Box\Diamond}$-frame and V is a mapping from Prop to 2^W, called a *valuation* on $\mathfrak{F}$, which satisfies that for all $a, b \in W$ and all $p \in \mathsf{Prop}$, $u \in V(p)$ and $a \leq b \;\&\; a \in V(p) \Rightarrow b \in V(p)$.

Given an $\mathbf{R.C}_{\Box\Diamond}$-model $\langle O, W, R, S_\Box, S_\Diamond, ^*, u, V\rangle$, for $a \in W$ and $A \in \mathsf{Wff}$, a relation $\models$ between W and Wff is defined inductively as follows:

i. for any $p \in \mathsf{Prop}$, $a \models p$ iff $a \in V(p)$
ii. $a \models A \wedge B$ iff $a \models A \;\&\; a \models B$
iii. $a \models A \vee B$ iff $a \models A$ *or* $a \models B$
iv. $a \models A \to B$ iff $\forall b, c(Rabc \;\&\; b \models A \Rightarrow c \models B)$
v. $a \models\!\sim A$ iff $a^* \not\models A$
vi. $a \models \Box A$ iff $\forall b(S_\Box ab \Rightarrow b \models A)$
vii. $a \models \Diamond A$ iff $\exists b(S_\Diamond ab \;\&\; b \models A)$.

We say that a formula A is *valid* in an $\mathbf{R.C}_{\Box\Diamond}$-frame $\langle O, W, R, S_\Box, S_\Diamond, ^*, u\rangle$ if $a \models A$ for every world $a \in O$ in every $\mathbf{R.C}_{\Box\Diamond}$-model $\langle O, W, R, S_\Box, S_\Diamond, ^*, u, V\rangle$. By the method of using canonical model, we can show that $\mathbf{R.C}_{\Box\Diamond}$ is complete with respect to the class of all $\mathbf{R.C}_{\Box\Diamond}$-frames.

An $\mathbf{R.C}_{\Box\Diamond}$-frame in which all thorems of $\mathbf{L}$ are valid is called an $\mathbf{L}$-*frame*. In general, any regular logic $\mathbf{L}$ over $\mathbf{R}$ is not necessary complete with respect to the class of all $\mathbf{L}$-frames.

On the other hand, the Lindenbaum's method guarantees completeness of any regular logic $\mathbf{L}$ over $\mathbf{R}$ in terms of $\mathbf{L}$-matrices which are modal extensions of Font-Rodríguez's $\mathbf{R}$-matrices. With the idea of frames, we want to make $\mathbf{L}$ complete with respect to frames as well as matrices. Then we consider the restriction of valuation on an $\mathbf{L}$-frame. For realizing this, we introduce the notion of general frames.

A *general* $\mathbf{L}$-*frame* is an 8-tuple $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^*, u, P\rangle$, where $\langle O, W, R, S_\Box, S_\Diamond, ^*, u\rangle$, written by $\kappa\mathfrak{F}$, is an $\mathbf{L}$-frame and P is called a *set of possible values* in $\mathfrak{F}$ (whose definitions are omitted in this paper).

The definition of $\mathbf{L}$-models on general $\mathbf{L}$-frames is the same as that on $\mathbf{L}$-frames, except that V is a mapping from Prop to P. With the help of duality theory, we can see that any regular logic $\mathbf{L}$ over $\mathbf{R}$ is complete with respect to the class of all general $\mathbf{L}$-frames.

We say that a general $\mathbf{L}$-frame is called *descriptive* if it is isomorphic to its bidual. $\mathcal{D}$ denotes the class of descriptive $\mathbf{L}$-frames and let $\kappa\mathcal{D} = \{\kappa\mathfrak{F} \mid \mathfrak{F} \in \mathcal{D}\}$. We put $\mathcal{D}^\star = \mathcal{D} \cup \kappa\mathcal{D}$. Then we say that

- $\mathbf{L}$ is $\mathcal{D}$-*complete* ($\kappa\mathcal{D}$-*complete*) if A is a theorem of $\mathbf{L}$ whenever it is valid in every $\mathbf{L}$-frame in $\mathcal{D}$ (in $\kappa\mathcal{D}$).
- $\mathbf{L}$ is $\mathcal{D}^\star$-*elementary* if there exists a set Φ of first-order sentences in the predicates O, R, $S_\Box$, $S_\Diamond$, * and the constant u such that for every $\mathfrak{F} \in \mathcal{D}^\star$, $\mathfrak{F}$ is an $\mathbf{L}$-frame iff $\mathfrak{F}$ satisfies each sentence in Φ.

Theorem 1 *Let* $\mathbf{L}$ *be any regular logic over* $\mathbf{R}$. *Then* $\mathbf{L}$ *is* $\mathcal{D}$*-complete. Further, if* $\mathbf{L}$ *is* $\mathcal{D}^\star$*-elementary, then it is* $\kappa\mathcal{D}$*-complete.*

3 A Sahlqvist theorem

In this section, we mention a Sahlqvist theorem for relevant modal logics. First of all, we give the definition of Sahlqvist formulas. Below, $\Box^n A$ denotes $\underbrace{\Box \cdots \Box}_{n} A$.

- A modal formula A is *positive* if A contain only $\wedge, \vee, \Box$ and $\Diamond$ as connectives.
- A modal formula of the form $\Box^{m_1} p_1 \wedge \cdots \wedge \Box^{m_k} p_k$ with not necessarily distinct propositional variables $p_1, \cdots, p_k$ is called a *strongly positive formula*.
- A given formula A is *negative* (in a regular logic $\mathbf{L}$ over $\mathbf{R}$) if A is equivalent to $\sim B$ for a positive formula B in $\mathbf{L}$.
- A modal formula A is *untied* (in $\mathbf{L}$) if it can be constructed from negative formulas (in $\mathbf{L}$) and strongly positive formulas using only $\wedge$ and $\Diamond$.
- A modal formula A is *Sahlqvist* (in $\mathbf{L}$) if it is equivalent in $\mathbf{L}$ to a conjunction of formulas of the form $\Box^k(B \to C)$, where $k \geq 0$, B is untied in $\mathbf{L}$ and C is positive.

Theorem 2 *Let* $\mathbf{L}$ *be regular logic over* $\mathbf{R}$. *For any Sahlqvist formula* A*, there exists a first order sentence* $\phi(a)$ *in the predicates* $O, R, S_\Box, S_\Diamond, ^*$ *and the constant* u *having* a *as its only free variable and such that the following holds for every* $\mathbf{L}$*-frame* $\mathfrak{F} \in \mathcal{D}^\star$ *and every* $a \in W$*,*

$$(\mathfrak{F}, a) \models A \quad \textit{iff} \quad \mathfrak{F} \textit{ satisfies } \phi(a),$$

where $(\mathfrak{F}, a) \models A$ *means that* $a \models A$ *under any valuation on* $\mathfrak{F}$*.*

Theorem 3 *Let* $\mathbf{L}_0$ *be a* $\mathcal{D}^\star$*-elementary regular logic over* $\mathbf{R}$*, and* $\mathbf{L}$ *be a logic obtained from* $\mathbf{L}_0$ *by adding a set of Sahlqvist formulas as axioms. Then* $\mathbf{L}$ *is* $\mathcal{D}^\star$*-elementary, and hence* $\kappa\mathcal{D}$*-complete.*

We can see that Theorem 3 covers all completeness results of relevant modal logics in [2], [3] and [4]. Thus, we can apply Theorem 3 to any logic obtained from one of these logics by adding a set of Sahlqvist formulas as axioms.

Moreover, the classical modal logic $\mathbf{K}$ is a $\mathcal{D}^\star$-elementary extension of $\mathbf{R.C}_{\Box\Diamond}$. $\mathbf{K}$ is obtained from $\mathbf{R.C}_{\Box\Diamond}$ by adding the axioms $A \to (B \to A)$, $\Box(A \to B) \to (\Box A \to \Box B)$, $\Diamond A \leftrightarrow \sim \Box \sim A$, and the rule of necessitation. In fact, to obtain frames for $\mathbf{K}$, it suffices to add the frame postulates to those of general $\mathbf{R.C}_{\Box\Diamond}$-frames $\langle O, W, R, S_\Box, S_\Diamond, ^*, u, P\rangle$ (see Section 5.5 of [5]), which can be represented by first order sentences in the predicates $O, R, S_\Box, S_\Diamond, ^*$ and the constant u. As long as we neglect a element u, frames for $\mathbf{K}$ thus obtained are exactly usual Kripke frames for $\mathbf{K}$. Therefore, Theorem 3 covers Sahlqvist theorem for classical modal logics.

References

[1] A.Chagrov and M.Zakharyaschev, *Modal Logic*, Clarendon Press, Oxford, 1997.

[2] A.Fuhrmann, 'Models for relevant modal logics', *Studia Logica 49* (1990), 501-514.

[3] E.D.Mares, 'The semantic completeness of **RK**', *Reports on Mathematical Logic 26* (1992), 3-10.

[4] E.D.Mares, 'Classically complete modal relevant logics', *Mathematical Logic Quarterly 39* (1993), 165-177.

[5] R.Routley, V.Plumwood, R.K.Meyer and R.T.Brady, *Relevant Logics and Their Rivals I*, Ridgeview Publishing Company, Atascadero, 1982.

[6] H.Sahlqvist, 'Completeness and correspondence in the first and second order semantics for modal logic', in S.Kanger, editor, *Proceedings of the third Scandinavian logic symposium*, pages 110-143, North-Holland, Amsterdam, 1975.

[7] T.Seki, *A Semantical Study of Relevant Modal Logics*, PhD thesis, submitted, 2002.

An efficient construction of counter–models for modal logic K4

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1 Introduction

In this paper, we will give an efficient construction of counter–models for modal logic K4. Our method is based on the decision procedure with efficient loop-checking in [1]. A major difficulty for K4 lies in the fact that though we can always find a finite counter–model with a tree-form for intuitionistic logic[3], a finite counter–model for K4 will contain some clusters in it. Therefore, constructing a finite counter–model from a refutation tree will need a new idea. This is done in Section 3 by introducing two kinds of blocks.

2 Decision procedure for K4

Our decision procedure for K4 is obtained by modifying the procedure given in [1]. The algorithm is in fact the same, but we choose to implement it as a sequent calculus rather than a tableau system.

The resulting decision procedure is presented below (Figure 1). Two differences with standard sequent systems are that our sequents consist of multisets rather than lists and that we add certain sets of formulas (called 'histories') to sequents, which show when rules for modality are applicable.

$$\frac{}{\Box\Gamma, p_1, \cdots, p_n \to q_1, \cdots, q_m, \Box\Delta \ \langle \Box\Gamma | \Box\Sigma \rangle} \ (init)_s (\Box\Delta \subset \Box\Sigma)$$

$$\frac{}{\Box\Gamma, p_1, \cdots, p_n \to q_1, \cdots, q_m \ \langle \Box\Pi | \Box\Sigma \rangle} \ (init)_t$$

$$\frac{\Gamma, A, B \to \Delta \ \langle \Box\Pi | \Box\Sigma \rangle}{\Gamma, A \wedge B \to \Delta \ \langle \Box\Pi | \Box\Sigma \rangle} \ (\wedge L) \qquad \frac{\Gamma \to \Delta, A \ \langle \Box\Pi | \Box\Sigma \rangle \quad \Gamma \to \Delta, B \ \langle \Box\Pi | \Box\Sigma \rangle}{\Gamma \to \Delta, A \wedge B \ \langle \Box\Pi | \Box\Sigma \rangle} \ (\wedge R)$$

$$\frac{\Gamma \to A, \Delta \ \langle \Box\Pi | \Box\Sigma \rangle}{\Gamma \neg A \to \Delta \ \langle \Box\Pi | \Box\Sigma \rangle} \ (\neg L) \qquad \frac{\Gamma, A \to \Delta \ \langle \Box\Pi | \Box\Sigma \rangle}{\Gamma \to \Delta, \neg A \ \langle \Box\Pi | \Box\Sigma \rangle} \ (\neg R)$$

$$\frac{\Box\Gamma, \Gamma \to A_1 \ \langle \Box\Gamma | \Box\Sigma, \Box\Theta \rangle \quad \cdots \quad \Box\Gamma, \Gamma \to A_m \ \langle \Box\Gamma | \Box\Sigma, \Box\Theta \rangle}{\Box\Gamma, p_1, \cdots, p_n \to \Box A_1, \cdots, \Box A_m, \Box\Delta, q_1, \cdots q_l \ \langle \Box\Gamma | \Box\Sigma \rangle} \ (\Box)_s$$
$$(\Box\Theta \equiv \Box A_1, \cdots \Box A_n, \ \Box\Theta \cap \Box\Sigma = \emptyset, \ \Box\Delta \subset \Box\Sigma)$$

$$\frac{\Box\Gamma, \Gamma \to A_1 \ \langle \Box\Gamma | \Box\Theta \rangle \quad \cdots \quad \Box\Gamma, \Gamma \to A_m \ \langle \Box\Gamma | \Box\Theta \rangle}{\Box\Gamma, p_1, \cdots, p_n \to \Box A_1, \cdots, \Box A_m, q_1, \cdots q_l \ \langle \Box\Pi | \Box\Sigma \rangle} \ (\Box)_t$$
$$(\Box\Theta \equiv \Box A_1, \cdots, \Box A_n, \Box\Gamma \supsetneq \Box\Theta)$$

Figure 1: Decision procedure for K4

We write $A \vee B$ and $A \supset B$ to denote shorted forms of $\neg(\neg A \wedge \neg B)$ and $\neg(A \wedge \neg B)$, respectively.

Note that $(\Box)_s$ and $(\Box)_t$ rules should be understood as 'or'–branch, but not as 'and'–branch. Therfore, if one of the upper sequents is provable, then the lower sequent is provable. In order to distinguish it

from other usual rules, we use a double line (=), instead of a single line.

The original system from [1] is obtained by removing $(\Box)_t$ and $(init)_t$ rules. Though our decision procedure are essentially the same as the one in [1], we need these rules when we construct counter–models. We have the following.

Theorem 2.1 *The system in Figure 1 gives a decision procedure for modal logic K4.*

Proof: See [1].

Finally, we come to 'history'. Take for example the $(init)_t$ rule.

$$\frac{}{\Box\Gamma, p_1, \cdots, p_n \to q_1, \cdots, q_m \ \langle \Box\Pi | \Box\Sigma \rangle} \ (init)_t$$

The pair $\langle \Box\Pi | \Box\Sigma \rangle$ is called a **history**. The pairs occurrencing in other rules are similar. The first element of the pair ($\Box\Pi$ in $(init)_t$ rule) is called **valid history**, because the boxed formulae are valid at the corresponding world in counter–model. The second element of the pair ($\Box\Sigma$ in $(init)_t$ rule) is called **invalid history**. because the boxed formulae are invalid at the corresponding world. Histories provide efficient loop–check in syntax, and give looplessness in semantics.

3 Counter-model construction

In this section, we will show how to construct counter–models.

To construct a Kripke–model, we consider refutation tree which is obtained from the decision procedure for an unprovable sequent. We will define a Kripke–model from the reftation tree.

Definition 3.1 (block) *We call a* **block** *a part of refutation tree beginning with initial sequent,* $(\Box)_s$ *or* $(\Box)_t$ *rule and ending to* $(\Box)_s$ *or* $(\Box)_t$ *rule. A block is a subtree of refutation tree contained between double line(=) and another double line.*
Symmetric block *is a block which begins with* $(\Box)_s$ *or* $(init)_s$*rule.*
Transitive block *is a block which begins with* $(\Box)_t$ *or* $(init)_t$ *rule.*
Initial block *is a block which begins with an initial sequent.*
Symmetric initial block *is a block which begins with* $(init)_s$ *initial sequent.*
transitive initial block *is a block which begins with* $(init)_t$ *initial sequent.*

Definition 3.2 (counter model) *We construct a corresponding Kripke–model and a world for a refutation tree as follows.*
Case 1. symmetric initial block beginning with $\Box\Gamma, p_1, \cdots, p_n \to q_1, \cdots, q_m, \Box\Delta\langle\Box\Gamma|\Box\Sigma\rangle$*: It corresponds to* (M, w) *such that* $M = (W, R, \models)$, $W = \{w\}, R = (w, w)$, $w \models p_1 \wedge \cdots \wedge p_n$.

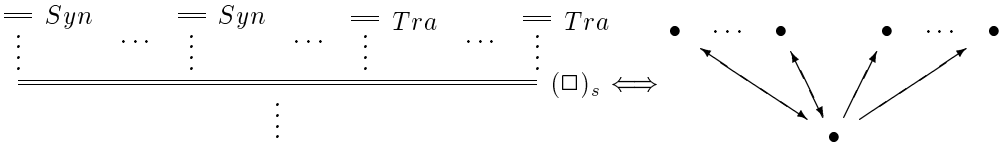
Case 2. transitive initial block: Identical to symmetric case.

Case 3. symmetric block

$$\frac{\cdots \quad \Box\Gamma, \Gamma \to S_i \ \langle \Box\Gamma | \Box\Sigma, \Box\Theta \rangle \quad \cdots \quad \Box\Gamma, \Gamma \to T_j \ \langle \Box\Gamma | \Box\Sigma, \Box\Theta \rangle \quad \cdots}{\Box\Gamma, p_1, \cdots, p_n \to \Box S_1, \cdots, \Box S_m, \Box T_1, \cdots, \Box T_l, \Box\Delta, q_1, \cdots q_k \ \langle \Box\Gamma | \Box\Sigma \rangle} \ (\Box)_s \ (1 \le i \le m, 1 \le j \le l)$$

Let $\Box\Gamma, \Gamma \to S_1\langle\Box\Gamma|\Box\Sigma, \Box\Theta\rangle, \cdots, \Box\Gamma, \Gamma \to S_m\langle\Box\Gamma|\Box\Sigma, \Box\Theta\rangle$ *be end–sequents of symmetric blocks and each of them correspond to* (M_{S_i}, w_{S_i})*, where* $M_{S_i} = (W_{S_i}, R_{S_i}, \models)$*. Let* $\Box\Gamma, \Gamma \to T_1\langle\Box\Gamma|\Box\Sigma, \Box\Theta\rangle, \cdots, \Box\Gamma, \Gamma \to T_l\langle\Box\Gamma|\Box\Sigma, \Box\Theta\rangle$ *be the end sequents of transitive blocks and each of them correspond to* $(M_{T_i} w_{T_i})$*, where* $M_{T_i} = (W_{T_i}, R_{T_i}, \models)$.

The symmetric block above corresponds to (M, w)*, where* $M = (W, R \models, w)$ *such that* $W = \{w\} \cup W_{S_1} \cup \cdots \cup W_{S_m} \cup W_{T_1} \cup \cdots \cup W_{T_l}$, $R' = \{(w, w_{S_1}), \cdots, (w, w_{S_m}), (w, w_{T_1}), \cdots, (w, w_{T_l})\} \cup \{(w_{S_1}, w), \cdots, (w_{S_m}, w)\} \cup R_{S_1} \cup \cdots \cup R_{S_m} \cup R_{T_1} \cup \cdots \cup R_{T_l}$, $R =$ *transitive closer of* R', $w \models p_1 \wedge \cdots \wedge p_n$.

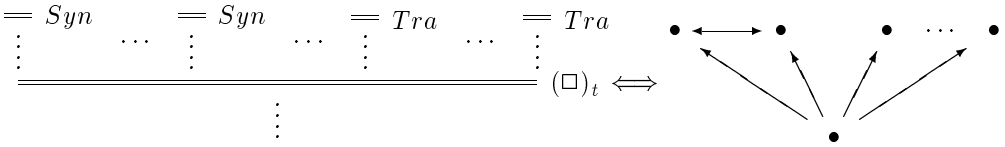


Case 4. transitive block

$$\frac{\cdots \quad \Box\Gamma, \Gamma \to S_i \ \langle \Box\Pi | \Box\Theta \rangle \quad \cdots \quad \Box\Gamma, \Gamma \to T_j \ \langle \Box\Pi | \Box\Theta \rangle \quad \cdots}{\Box\Gamma, p_1, \cdots, p_n \to \Box S_1, \cdots, \Box S_m, \Box T_1, \cdots, \Box T_l, \Box\Delta, q_1, \cdots q_k \ \langle \Box\Gamma | \Box\Sigma \rangle} \ (\Box)_t \ (1 \leq i \leq m, 1 \leq j \leq l)$$

Let $\Box\Gamma, \Gamma \to S_1\langle \Box\Pi | \Box\Theta \rangle, \cdots, \Box\Gamma, \Gamma \to S_m\langle \Box\Pi | \Box\Theta \rangle$ *be end–sequents of symmetric blocks and each of them correspond to* (M_{S_i}, w_{S_i})*, where* $M_{S_i} = (W_{S_i}, R_{S_i}, \models)$*. Let* $\Box \to T_1\langle \Box\Pi | \Box\Theta \rangle, \cdots, \Box \to T_l\langle \Box\Pi | \Box\Theta \rangle$ *be the end sequents of transitive blocks and each of them correspond to* $(M_{T_i} w_{T_i})$*, where* $M_{T_i} = (W_{T_i}, R_{T_i}, \models)$*.*

The symmetric block above corresponds to (M, w)*, where* $M = (W, R \models, w)$ *such that* $W = \{w\} \cup W_{S_1} \cup \cdots \cup W_{S_m} \cup W_{T_1} \cup \cdots \cup W_{T_l}$*,* $R' = \{(w, w_{S_1}), \cdots, (w, w_{S_m}), (w, w_{T_1}), \cdots, (w, w_{T_l})\} \cup \{(w_{S_1}, w_{S_2}), \cdots, (w_{S_1}, w_{S_m}), \cdots, (w_{S_m}, w_{S_1}), \cdots, (w_{S_m}, w_{S_{(m-1)}})\} \cup R_{S_1} \cup \cdots \cup R_{S_m} \cup R_{T_1} \cup \cdots \cup R_{T_l}$*,* $R =$ *transitive closer of* R'*,* $w \models p_1 \wedge \cdots \wedge p_n$*.*



Theorem 3.3 *Models constructed by our method are indeed counter–models.*

Proof: Similar to the proof for S4 in [2].

4 Conclusion

In this paper, we give an effective way of constructing counter–models for modal logic K4 from a decision procedure. By our method, each counter–model is represented as a tree of semi–clusters, where semi–cluster is the structure consisted of a possible world and a cluster. In each semi–cluster, a set of boxed formulae is valid. The higher in the tree structure a semi–cluster is, the more valid boxed formulae it has, as the example in Figure 2 shows.

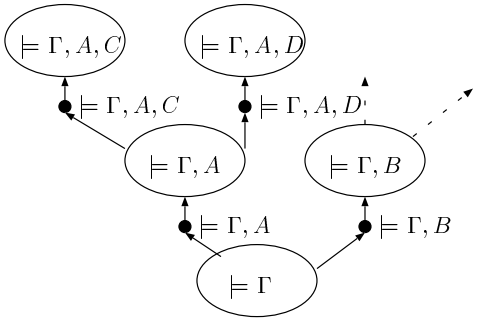


Figure 2: Structure of K4 couner–model

References

[1] Alain Heuerding, Michael Seyfried and Heinrich Zimmermann: ***Efficient loop-check for backward proof search in some non-classical propositional logics***, In P. Miglioli, U. Moscato, D. Mundici, M. Ornaghi, editors, Tableaux 96, LNCS 1071, 210-225, 1996.

[2] Motohiko Mouri ***An efficient contraction of counter–models for modal logic S4***, Bulletin of the Section of Logic, vol 30 No.2, 2001, 79–86.

[3] L. Pinto and R. Dyckhoff, ***Loop–free construction of counter–models for intuitionistic propositional logic***, Symposia Gaussiana, Conf. A (Behara, Fritsch, and Lintz, editors), Walter de Gruyter & Co., Berlin, 1995, 225–232.

On complementary modalities

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1 Problem

The language here is a propositional bimodal language with $\Box$ and [e]. A *complementary frame* is $\langle W, R, \bar{R}\rangle$ where R is a binary relation on W and $\bar{R}$, the complement of R, serves as the accessibility relation to interpret [e].

Humberstone[1] gives an axiomatic system of the logic of valid sentences in the class of all complementary frames with a schema:

1. K for $\Box$ and [e]
2. Let $\mathcal{S}$ be a sequence of $\Box$ and [e], and $\mathcal{W}$ be the sequence of modalities where each $\star$ in $\mathcal{S}$ is substituted by its dual. Then,
$$\mathcal{W}(\Box p \wedge [\mathrm{e}]q) \rightarrow \mathcal{S}(p \vee q) \tag{1}$$

Problem 1. *When R is an equivalence relation, is there any axiomatization of the logic without the axiom schema 1? Is the logic decidable?*

This short paper answers yes to both.

2 An axiomatic system

A system $\mathsf{S}(\Box, [\mathrm{e}])$ consists of:

1. Tautologies of classical propositional logic
2. S5 for $\Box$
3. B for [e]
4. $[\mathrm{e}]p \rightarrow [\mathrm{e}]\Box p \wedge \Box[\mathrm{e}]p$
5. $\Diamond[\mathrm{e}]p \rightarrow [\mathrm{e}]p$
6. $[\mathrm{e}]p \wedge \Box p \rightarrow [\mathrm{e}]^2 p$

Observation 2. $[\mathrm{e}]p \wedge \Box q \rightarrow [\mathrm{e}](\Diamond\neg p \rightarrow q) \wedge \Box(\Diamond\neg p \rightarrow q)$ *is provable in* $\mathsf{S}(\Box, [\mathrm{e}])$.

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3 Completeness and decidability

Here completeness and decidability is to be shown by constructing a finite complementary model via filtration. A "copy-and-paste" of equivalence classes may be required during the construction.

Given a consistent formula A, construct a filtration set as follows:

$$\begin{array}{lcl} X_0(A) & = & Sub(A) \cup \{[\mathrm{e}]\bot\} \\ X_1(A) & = & X_0(A) \cup \{[\mathrm{e}]B, \Box\neg\Box B : \Box B \in X_0(A)\} \\ & & \cup\{\Box B, [\mathrm{e}]\Box B, [\mathrm{e}]^2 B, [\mathrm{e}]\neg[\mathrm{e}]B, [\mathrm{e}]\neg[\mathrm{e}]^2 B, \Box\neg[\mathrm{e}]B : [\mathrm{e}]B \in X_0(A)\} \\ X(A) & = & Sub(X_1(A) \cup \{\neg B : B \in X_1(A)\}) \end{array}$$

Obviously, $X(A)$ is finite. Fix an arbitrary maximal consistent set Γ_A such that $A \in \Gamma_A$.

Let W be the class of maximal consistent sets. Define an equivalence relation $\sim$ as: $\Gamma \sim \Gamma'$ iff $\Gamma \cap \Gamma_A = \Gamma' \cap \Gamma_A$. Let $[W]$ be the quotient set by $\sim$, and $[\Gamma]$ be the equivalence class to which Γ belongs. Define binary relations on $[W]$:

$$\begin{array}{lll} R_0^{\Box}[\Gamma][\Gamma'] & \text{iff} & \{A : \Box A \in \Gamma\} \cap X_A \subseteq \Gamma' \\ R_0^{[\mathrm{e}]}[\Gamma][\Gamma'] & \text{iff} & \{A : [\mathrm{e}]A \in \Gamma\} \cap X_A \subseteq \Gamma'. \end{array}$$

Let $W_0 = \{[\Gamma] \in [W] : (R_0^{\Box} \cup R_0^{[\mathrm{e}]})^*[\Gamma_A][\Gamma]\}$.

Lemma 3. *For each $x, y \in W_0$, either $R_0^{\Box}xy$ or $R_0^{[\mathrm{e}]}xy$.*

Proof. It suffices to show by induction that $(R_0^{\Box} \cup R_0^{[\mathrm{e}]})[\Gamma_A][\Gamma]$ for every $[\Gamma] \in W_0$. The base is clear. Suppose $(R_0^{\Box} \cup R_0^{[\mathrm{e}]})(n+1)[\Gamma_A][\Gamma]$. By induction hypothesis, $(R_0^{\Box} \cup R_0^{[\mathrm{e}]})^2[\Gamma_A][\Gamma]$. Case 1: When $(R_0^{\Box})^2[\Gamma_A][\Gamma]$, a standard routine shows $R_0^{\Box}[\Gamma_A][\Gamma]$. Case 2: The axiom $[\mathrm{e}]p \to [\mathrm{e}]\Box p \wedge \Box[\mathrm{e}]p$ takes care of these two cases: $(R_0^{\Box} \circ R_0^{[\mathrm{e}]})[\Gamma_A][\Gamma]$ and $(R_0^{[\mathrm{e}]} \circ R_0^{\Box})[\Gamma_A][\Gamma]$. Case 3: Suppose $(R_0^{[\mathrm{e}]})^2[\Gamma_A][\Gamma]$. Consider $Y = \{[\mathrm{e}]B \in X(A) : [\mathrm{e}]B \in \Gamma_A\}$. Suppose there is any $[\mathrm{e}]C \in Y$, $C \notin \Gamma$. Then, $[\mathrm{e}]^2 C \notin \Gamma_A$ and $\neg\Box C \in \Gamma$ must hold. Consider arbitrary $\Box D \in \Gamma_A \cap X(A)$. Since $[\mathrm{e}]C \wedge \Box D \in \Gamma_A$, $\neg\Box C \to D \in \Gamma$, which implies $D \in \Gamma$. Therefore, $R_0^{\Box}[\Gamma_A][\Gamma]$. □

W_0 is finite. Enumerate the members of W_0, and for the i-th member $\Delta_i = [\Gamma_i]$, let $x_i = \{B \in X(A) : B \in \Gamma\}$. Let W_1 be the set of x_i's. Where $x_j = \{B \in X(A) : B \in \Gamma_j\}$ $(j = 1, 2)$, define: $R_1^{\star}x_1x_2$ iff $R_0^{\star}[\Gamma_1][\Gamma_2]$ $(\star = \Box, [\mathrm{e}])$.

The case which requires extra caution is when there is any $x \in W_1$ with $R_1^{[\mathrm{e}]}xx$.

If there is no such x, the construction ends. Define $v_c : Prop \to \wp(W_1)$ to be $v_c(p) = \{x : p \in x\}$. Take $M_c = \langle W_1, R_1^{\Box}, R_1^{[\mathrm{e}]}, v_c\rangle$ as a filteration model.

Otherwise, such $R_1^{[\mathrm{e}]}$-reflexive points are to split. Let $Z \subseteq W_1$ be the set of $R_1^{[\mathrm{e}]}$-reflexive points. $W_c = (W_1 \setminus Z) \cup (Z \uplus Z)$. For each $z \in Z$, let $(z, 0), (z, 1)$ denote the results of splitting.

$$\begin{array}{lll} R_c^{\Box}xy & \text{iff} \quad R_1^{\Box}xy & \text{if } x, y \in (W_1 \setminus Z) \\ & & \quad \text{or } x = (x, i), y = (y, j) \text{ and } i = j \\ R_c^{[\mathrm{e}]}xy & \text{iff} \quad \text{not } R_c^{\Box}xy & \end{array} \tag{2}$$

Identify $(z, 0), (z, 1)$ with z. Let $v_c : Prop \to \wp(W_c)$ be $v_c(p) = \{x \in W_c : p \in x\}$. Let $M_c = \langle W_c, R_c^{\Box}, R_c^{[\mathrm{e}]}, v_c\rangle$.

In each case, M_c is made to be a complementary model. The finite construction leads via a standard induction to:

Lemma 4. *(Truth lemma) For each formula $B \in Sub(A)$, $M^c, x \models B$ iff $B \in x$.*

Corollary 5. *$S(\Box, [\mathrm{e}])$ is complete.*

Corollary 6. *$S(\Box, [\mathrm{e}])$ is decidable.*

References

[1] I. L. Humberstone(1983) Inaccessible Worlds. *Notre Dame Journal of Formal Logic*, **24**, 346–352.

A Map of Common Knowledge Logics*

(with M. Kaneko, T. Nagashima and Y. Tanaka)

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Abstract

In order to capture the concept of common knowledge in the framework of multi-modal (epistemic) logics, various extensions of multi-modal epistemic logics, such as finitary ones and infinitary ones, have been proposed. The case of propositional epistemic logics, these extensions turn out to be all Kripke complete. F. Wolter showed that any finitary predicate common knowledge logic is Kripke incomplete. However, if we go further to an infinitary extension, the Kripke completeness would be recovered. Thus there is some gap in predicate case. The aim of this article is to give an overview of the finitary and infinitary approaches to common knowledge logics.

Introduction

Multi-agent epistemic logics have been developed for various investigations such as AI (see e.g. Meyer-van der Hoek [8]) and game theoretical problems (see e.g. Kaneko [3]). In such situations, common knowledge is important in discussing knowledge (or beliefs) shared among agents. Various extensions of multi-agent logics have been proposed in order to capture the concept of common knowledge in the framework of multi-modal (epistemic) logics. These extensions are divided into two types: the *finitary* approach such as Halpern-Moses' fixed-point logics (Halpern-Moses [1]), and the *infinitary* approach such as Kaneko-Nagashima's game logics (Kaneko-Nagashima [4, 5]). Some of them are given as propositional logics and some others as predicate logics. Also, some are considered from the model-theoretic viewpoint and some others from the proof-theoretic viewpoint. The aim of this article is to give an overview of finitary and infinitary approaches to common knowledge logics mainly from the viewpoint of Kripke completeness.

In the propositional case, these extensions turn out to be all Kripke complete, and are directly comparable in the sense that either two logics are deductively equivalent or if one is an extension of another, it is also a conservative extension. In contrast, F. Wolter [12] proved in the predicate case that the set of valid formulas in the Kripke semantics (with constant domain) is *not recursively enumerable* in the presence of common knowledge. This implies the *Kripke-incompleteness theorem* that any finitary predicate extension of the Halpern-Moses type common knowledge logic cannot capture the Kripke semantics with common knowledge. In other words, the latter has no *finitary* proof theory. Nevertheless, it is also known from Tanaka-Ono [10] and Tanaka [11] that this difficulty does not occur in the infinitary approach. Thus, the predicate case has some great difference between the finitary and infinitary approaches. In presenting our overview of common knowledge logics, we will focus especially on what this difference is.

Diagram 1 gives some extant extensions of multi-agent epistemic logics in proof theoretical setting. Since we discuss a variety of common knowledge extensions, we make a particular choice of basic epistemic axioms on knowledge operators. We adopt KD4-type axioms, starting with propositional multi-agent epistemic logic $\mathrm{KD4}^n$, where n is the number of agents. The prefix Q means the predicate extension of a propositional logic together with the Barcan axiom. Logic HM is the fixed-point type extension of $\mathrm{KD4}^n$ due to Halpern-Moses [1], where the common knowledge of a formula is determined by adding one axiom

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and one inference rule to $KD4^n$. The *game logic* GL_ω is an infinitary extension of $KD4^n$, and QGL_ω is its predicate extension, due to Kaneko-Nagashima [4] and [5], where the common knowledge of a formula is expressed as an infinitary conjunctive formula.

	Finitary Base Logic	Finitary	Infinitary
Propositional	$KD4^n$	HM	GL_ω
Predicate	$QKD4^n$	QHM	QGL_ω

Diagram 1

The Kripke completeness of $KD4^n$ and that of $QKD4^n$ are known as variants of the completeness results given in the modal logic literature (cf., Hughes-Cresswell [2]). It is also known from Halpern-Moses [1] that HM is Kripke complete. It follows from Tanaka-Ono [10] that GL_ω is also Kripke complete. These completeness results in the propositional case imply that if one logic is an extension of another, then it is a conservative extension.

On the other hand, it is known in the predicate case that there is a gap between QHM and QGL_ω. As already mentioned, the Kripke completeness of $QKD4^n$ is known, and that of QGL_ω is proved in Tanaka [11]. However, we have Wolter's Kripke-incompleteness theorem for QHM.

From the syntactical point of view, the distance between QHM and QGL_ω is large in that QGL_ω allows infinitary conjunctions and disjunctions as well as infinitary proofs. To consider the question of where the gap occurs, we provide two other logics, propositional CX and CY, and their predicate extensions QCX and QCY. These new logics are intermediate between HM, QHM and GL_ω, QGL_ω in that they have the set of finitary formulas in the same way as HM and QHM but admit infinitary proofs similar to GL_ω and QGL_ω.

In the propositional case, both CX and CY are shown to be deductively equivalent to HM. In the predicate case, QCY is Kripke complete, which is shown in Tanaka [11]. This result together with Wolter's incompleteness theorem implies that QCY is deductively different from QHM. The completeness of QCX remains open, though QCX looks more natural as a common knowledge logic than QCY.

The deductive equivalence of HM, CX and CY to GL_ω, as far as relevant formulas are concerned, is shown by proving that HM, CX and CY are faithfully embedded into GL_ω. In the predicate case, only QCY is faithfully embedded into to QGL_ω. The faithful embedding of QCX into QGL_ω remains open.

Concluding Remarks

1. From our considerations, we draw Diagram 2. The map has four rows: The above two give the logics considered in the propositional and predicate cases. The two additional rows explain the characteristics of each logic, e.g., QHM has finitary formulae and finitary proofs, while QCX has finitary formulae but infinitary proofs. The arrow $\to$ denotes the relationship of deductive strength and $\rightsquigarrow$ that of being faithfully embedded.

	$KD4^n$	$\to$	HM	$\longleftrightarrow$	CX	$\longleftrightarrow$	CY	$\rightsquigarrow$	GL_ω
	$\downarrow$		$\downarrow$		$\downarrow$		$\downarrow$		$\downarrow$
	$QKD4^n$	$\to$	QHM	$\to$	QCX	$\to$	QCY	$\rightsquigarrow$	QGL_ω
formulae	finitary		finitary		finitary				infinitary
proofs	finitary		finitary		infinitary				infinitary

Diagram 2

No serious problems remain in the propositional case. All the logics from $KD4^n$ to GL_ω are Kripke complete, and HM, CX, CY are deductively equivalent. They can be faithfully embedded into GL_ω. Also, GL_ω is a conservative extension of $KD4^n$. On the other hand, we have met various problems in the predicate case. The logics $QKD4^n$, QCY, and QGL_ω are Kripke complete, but QHM is Kripke incomplete. The Kripke completeness of QCX remains open. The logic QCY is faithfully embedded into QGL_ω, and QGL_ω is a conservative extension of $QKD4^n$. Though faithfulness of embedding of QHM into QGL_ω fails to hold. The faithful embedding problem of QCX into QGL_ω remains still open.

In Diagram 2, the gap exists around QHM and QCX. There is a formula that is valid in the Kripke semantics but is not provable in QHM. However it is also an open problem to find a concrete one.

2. Recall that MH is Kripke complete but QHM is not. We may see the difference between the above facts from two points of view.

First, let us see the above theorems from the finitary point of view. It is well-known that both $KD4^n$ and $QKD4^n$ are Kripke complete and have recursively enumerable sets of provable formulas. In the propositional case, HM is complete and $\{A \mid A$ is propositional and $\models A\}$ is also recursively enumerable, while in the predicate case, F. Wolter [12] proved that QHM is incomplete and $\{A \mid A$ is predicate and $\models A\}$ is not recursively enumerable. Although HM and QHM are obtained from $KD4^n$ and $QKD4^n$, respectively, by adding both finitary axiom schema and inference rule, only the logic QHM turns out to be incomplete. In this sense, this incompleteness of QHM seems to be an unexpected jump.

Second, let us see the above two theorems from the infinitary point of view. In the infinitary approach, completeness results are known. Indeed, Kripke completeness is recovered in the sense that the strengthened provability in an infinitary approach captures $\{A \mid A$ is predicate and $\models A\}$. In the propositional case, Kaneko [6] proved that HM can be regarded as a fragment of infinitary propositional epistemic logics. Hence we can regard HM as well as QHM as having already an infinitary aspect in part. The non-recursive-enumerability of $\{A \mid A$ is predicate and $\models A\}$ is better understood from this point of view. In this sense, from the infinitary point of view, we may regard the completeness of HM as an unexpected result, rather than the incompleteness of QHM.

3. We have been focussed on extensions of QHM rather than on its fragments. On the other hand, Sturm-Wolter-Zakharyaschev [9] considered the *monodic* fragment of QHM. They proved the Kripke completeness of the monodic fragment of QHM. This fragment is located between HM and QHM. The gap between QHM and QCY occurs after the monodic fragment of QHM.

References

[1] Halpern, J. Y., and Y. Moses, *A guide to completeness and complexity for modal logics of knowledge and beliefs,* **Artificial Intelligence** 54 (1992), 319–379.

[2] Hughes, G. E., and M. J. Cresswell, **A Companion to Modal Logic**, Methuen &Co. London.

[3] Kaneko, M., *Epistemic logics and their applictions: Introduction,* to appear in the special issue "Logic and Economics" of **Economic Theory**.

[4] Kaneko, M., and T. Nagashima, *Game logic and its applications I,* **Studia Logica** 57 (1996), 325–354.

[5] Kaneko, M., and T. Nagashima, *Game logic and its applications II,* **Studia Logica** 58 (1997), 273–303.

[6] Kaneko, M., *Common knowledge logic and game logic*, **Journal of Symbolic Logic** 64 (1999), 685–700.

[7] Kaneko, M., T. Nagashima, N.-Y. Suzuki and Y. Tanaka, *A map of common knowledge logics,* to appear in **Studia Logica**.

[8] Meyer, J.-J. Ch. and W. van der Hoek, **Epistemic Logic for AI and Computer Science**, Cambridge University Press, 1995.

[9] Sturm, H., F. Wolter and M. Zakharyaschev, *Monodic epistemic predicate logic,* to appear in the special issue "Logic and Economics" of **Economic Theory**.

[10] Tanaka, Y., and H. Ono, *The Rasiowa-Sikorski lemma and Kripke completeness of predicate and infinitary modal logics*, **Advances in Modal Logics Vol.2** (2001), CSLI Publications, pp.401–419.

[11] Tanaka, Y., *Some proof systems for common knowledge predicate logic,* Mimeo.

[12] Wolter, F., *First order common knowledge logics,* **Studia Logica** 65 (2000), 249–271.

On Bimodal Provability Logics

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Provability logic **GL** is one of the normal modal logics, which is obtained from the smallest normal modal logic **K** by adding Löb's axiom. The name "*provability logic*" derives from Solovay's theorem in Solovay [Sol76]. He proved that **GL** is complete for the formal provability interpretation in Peano arithmetic **PA**. By his theorem, we might as well say a formula $\Box A$ asserts that A^* is provable in **PA**, where A^* is an arithmetic sentence corresponding to A.

Smoryński [Sym85] considered bimodal provability logics by extending **GL** with another modal operator ∇. An arithmetic interpretation of ∇A is a provability of another arithmetic theory. After that, bimodal provability logics for other arithmetic theories have been considered in several papers, especially Beklemishev [Bek96] gives a detailed survey of this topic.

Here we treat two bimodal provability logics **MOS** and $\mathbf{PRL}_1$ among the logics in [Sym85] and give cut-free sequent systems for these two logics.

1 Bimodal provability logics

In this section, we introduce bimodal provability logics **MOS** and $\mathbf{PRL}_1$ and their Kripke semantics(cf. [Sym85]). The language of the bimodal provability logics consists of propositional variables, $\bot$, $\wedge$, $\vee$, $\supset$, $\Box$ and ∇. Formulas are defined as usual.

Definition. The modal logic **MOS** is the smallest set of formulas containing
all tautologies,
K: $\Box(A \supset B) \supset (\Box A \supset \Box B)$,
L: $\Box(\Box A \supset A) \supset \Box A$,
$A1$: $\nabla A \supset \Box \nabla A$,
$A2$: $\Box A \supset \nabla A$ and
$A3$: $\Box(A \supset B) \supset (\nabla A \supset \nabla B)$
and closed under modus ponens and necessitation.

Definition. The modal logic $\mathbf{PRL}_1$ is the smallest set of formulas containing all tautologies, K, L, $A1$, $A2$ and
$A4$: $\nabla(A \supset B) \supset (\nabla A \supset \nabla B)$
and closed under modus ponens and necessitation.

Definition. A Kripke model for **MOS** is a tuple $\mathcal{K} = \langle K, <, F, \models \rangle$, where
(1) K is a non-empty set,
(2) $<$ is an irreflexive and transitive binary relation on K,
(3) F is a mapping from K to $\mathcal{P}(\mathcal{P}(K))$ satisfying
  (3.1) $X \in F(\alpha)$ implies $X \subseteq \{\gamma \mid \alpha < \gamma\}$,
  (3.2) $\{\gamma \mid \alpha < \gamma\} \in F(\alpha)$ and
  (3.3) $\alpha < \beta$ and $X \in F(\alpha)$ imply $X \cap \{\gamma \mid \beta < \gamma\} \in F(\beta)$,
(4) $\models$ is a valuation satisfying, in addition to the usual boolean laws,
  (4.1) $\alpha \models \Box A$ iff for any $\beta \in \{\gamma \mid \alpha < \gamma\}$, $\beta \models A$ and

(4.2) $\alpha \models \nabla A$ iff there exists $X \in F(\alpha)$ such that for any $\beta \in X$, $\beta \models A$.

Definition. A Kripke model for $\mathbf{PRL}_1$ is a tuple $\mathcal{K} = \langle K, <, R, \models \rangle$, where
(1) $\langle K, < \rangle$ is as in the above definition,
(2) R is a binary relation on K satisfying
  (2.1) $\alpha R \beta$ implies $\alpha < \beta$ and
  (2.2) $\alpha < \beta R \gamma$ implies $\alpha R \gamma$
(3) $\models$ is a valuation satisfying, in addition to the usual boolean laws,
  (3.1) $\alpha \models \Box A$ iff for any $\beta \in \{\gamma \mid \alpha < \gamma\}$, $\beta \models A$ and
  (3.2) $\alpha \models \nabla A$ iff for any $\beta \in \{\gamma \mid \alpha R \gamma\}$, $\beta \models A$.

Let $\mathcal{K}$ be a Kripke model for **MOS** or $\mathbf{PRL}_1$. We say that a formula A is valid in a Kripke model if $\alpha \models A$ for any $\alpha \in K$.

Lemma 1.1([sym85]).
(1) $A \in \mathbf{MOS}$ *iff* A *is valid for any Kripke model for* **MOS**.
(2) $A \in \mathbf{PRL}_1$ *iff* A *is valid for any Kripke model for* $\mathbf{PRL}_1$.

2 Sequent systems

In this section, we introduce sequent systems for **MOS** and $\mathbf{PRL}_1$ and prove cut-elimination theorems. For a finite set of formulas, we use Greek letters $\Gamma, \Delta, \Sigma, \cdots$. The expressions $\Box\Gamma$ and $\nabla\Gamma$ denote the set $\{\Box A \mid A \in \Gamma\}$ and $\{\nabla A \mid A \in \Gamma\}$. By a sequent, we mean an expression $\Gamma \to \Delta$. For brevity's sake, we write

$$A_1, \cdots, A_n, \Gamma_1, \cdots, \Gamma_m \to \Delta_1, \cdots, \Delta_k, B_1, \cdots, B_l$$

instead of

$$\{A_1, \cdots, A_n\} \cup \Gamma_1 \cup \cdots \cup \Gamma_m \to \Delta_1 \cup \cdots \cup \Delta_k \cup \{B_1, \cdots, B_l\}.$$

The sequent system **GMOS** is the system obtained from **LK** by adding the following inference rules.

$$\frac{\Box A, \Gamma, \Box\Gamma, \nabla\Sigma \to A}{\Box\Gamma, \nabla\Sigma \to \Box A}(\Box)$$

$$\frac{\nabla A, B, \nabla B, \Gamma, \Box\Gamma, \nabla\Sigma \to A}{\nabla B, \Box\Gamma, \nabla\Sigma \to \nabla A}(\nabla_1)$$

$$\frac{\nabla A, \Gamma, \Box\Gamma, \nabla\Sigma \to A}{\Box\Gamma, \nabla\Sigma \to \nabla A}(\nabla_2)$$

The sequent system $\mathbf{PRL}_1$ is the system obtained from **LK** by adding the above inference rule $(\Box)$ and the following inference rule.

$$\frac{\nabla A, \Gamma, \Box\Gamma, \Sigma, \nabla\Sigma \to A}{\Box\Gamma, \nabla\Sigma \to \nabla A}(\nabla_3)$$

Lemma 2.1.
(1) *If a sequent* $A_1, \cdots, A_m \to B_1, \cdots, B_n$ *is provable in* **GMOS**, *then* $(A_1 \wedge \cdots \wedge A_m) \supset (B_1 \vee \cdots \vee B_n)$ *is valid for any Kripke model for* **MOS**.
(2) *If a sequent* $A_1, \cdots, A_m \to B_1, \cdots, B_n$ *is provable in* $\mathbf{GPRL}_1$, *then* $(A_1 \wedge \cdots \wedge A_m) \supset (B_1 \vee \cdots \vee B_n)$ *is valid for any Kripke model for* $\mathbf{PRL}_1$.

For a proof of the above lemma, we use Lemma 1.1 and the sequent system for **GL** described in Valentini [Val83] and Avron [Avr84].

Theorem 2.2.

(1) *If there exists no cut-free proof figure for* $\Gamma \to \Delta$ *in* **GMOS**, *then there exist Kripke model* $\mathcal{K}$ *for* **MOS** *and* $\alpha \in K$ *such that* $\alpha \models A$ *for* $A \in \Gamma$ *and* $\alpha \not\models B$ *for* $B \in \Delta$.

(2) *If there exists no cut-free proof figure for* $\Gamma \to \Delta$ *in* $\mathbf{GPRL}_1$, *then there exist Kripke model* $\mathcal{K}$ *for* $\mathbf{PRL}_1$ *and* $\alpha \in K$ *such that* $\alpha \models A$ *for* $A \in \Gamma$ *and* $\alpha \not\models B$ *for* $B \in \Delta$.

For a proof of the above theorem, we use the method in [Avr84]. We construct a Kripke model $\mathcal{K}$ with a set K of saturated sequents, and for the least saturated sequent α, we show that $\alpha \models A$ for $A \in \Gamma$ and that $\alpha \not\models B$ for $B \in \Delta$.

Corollary 2.3.

(1) $A_1, \cdots, A_m \to B_1, \cdots, B_n$ *is provable in* **GMOS** *iff* $(A_1 \wedge \cdots \wedge A_m) \supset (B_1 \vee \cdots \vee B_n) \in \mathbf{MOS}$.

(2) $A_1, \cdots, A_m \to B_1, \cdots, B_n$ *is provable in* $\mathbf{GPRL}_1$ *iff* $(A_1 \wedge \cdots \wedge A_m) \supset (B_1 \vee \cdots \vee B_n) \in \mathbf{PRL}_1$.

Corollary 2.4.

(1) *If a sequent* $\Gamma \to \Delta$ *is provable in* **GMOS**, *then there exists a cut-free proof figure for* $\Gamma \to \Delta$.

(2) *If a sequent* $\Gamma \to \Delta$ *is provable in* $\mathbf{GPRL}_1$, *then there exists a cut-free proof figure for* $\Gamma \to \Delta$.

References

[Avr84] A. Avron, *On modal systems having arithmetical interpretations*, The Journal of Symbolic Logic, 49, 1984, pp. 935–942.

[Bek96] L. D. Beklemishev, *Bimodal logics for extensions of arithmetical theories*, The Journal of Symbolic Logic, 61, 1996, pp. 91–124.

[Smo85] C. Smoryński, *Self-reference and Modal logic*, Springer-Verlag, 1985.

[Sol76] R. M. Solovay, *Provability interpretations of modal logic*, Israel Journal of Mathematics, 25, 1976, pp. 287–304.

[Val83] S. Valentini, *The modal logic of provability: cut-elimination*, Journal of Philosophical logic, 12, 1983, pp. 471–476.

Elementary ordinal representation systems for KP theories

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Elementary ordinal representation systems (which we abbreviate to EORSs) have been developed for establishing proof theoretic ordinals of set theoretical or arithmetical systems. This article is devoted to mention some relationships between several EORSs, which are (essentially) define by W.Buchholz and M.Rathjen (cf. [1], [2], [5] and [3]).

In order to obtain proof theoretic ordinals of set theoretic systems based on KP (Kripke-Platek set theory) such as KPi and KPM by using the method called "collapsing method", one can deal with several large cardinals and admissible ordinals. For example, he can first consider a certain cardinal κ and an admissible ordinal α corresponding to κ and a system KP^* which is based on KP and characterized by α, and next he can define certain sets of ordinals and functions by employing κ (which are called "Skolem hulls and collapsing functions") and, by using such sets and functions, establish an EORS which bears $|\mathrm{KP}^*|$ (the proof theoretic ordinal of KP^*).

Observing these inventions such as Skolem hulls, collapsing functions, and EORSs, we can expect a possibility of existence of primitive recursive ordinals corresponding to large cardinals (and admissible ordinals) which are employed for establishing the proof theoretic ordinals. In fact, in [6] and [7], we define "proof-theoretically regular ordinals" which are primitive recursive analogues of several regular cardinals. Since these ordinals are defined by using some EORSs, they deeply depends on the EORSs with which we define them. Therefore, it is necessary to check relationships between EORSs. The results of this article imply that EORSs defined in this article share a certain initial segment of proof-theoretically regular ordinals (see Co.10 and Co.17).

Let $+$ denote ordinary (noncommutative) ordinal addition. An ordinal $\alpha > 0$ is called an *additive principal number* iff $\forall\, \beta, \gamma\, (\beta, \gamma < \alpha \Rightarrow \alpha + \beta < \alpha)$. Let AP denote the class of additive principal numbers. The Veblen function φ is defined by: $\varphi\alpha\beta$ is the β^{th} element γ of AP such that $\forall\, \xi < \alpha\ (\varphi\xi\gamma = \gamma)$. For example, $\varphi 0\alpha$ is the α^{th} additive principal number $(= \omega^{1+\alpha})$, and $\varphi 1\alpha$ is the α^{th} ε-number $(= \varepsilon_\alpha)$. An ordinal $\gamma > 0$ is said to be *strongly critical* iff $\forall\, \alpha, \beta\ (\beta, \alpha < \gamma\ \Rightarrow\ \varphi\alpha\beta < \gamma)$. For each ordinal α, let Γ_α denote the α^{th} strongly critical ordinal.

Let Reg denote the class of (uncountable) regular cardinals, Ω_α denote the α^{th} cardinal, and let I denote the least weakly inaccessible cardinal.

Definition 1 (Skolem hulls and Collapsing functions based on I: Buchholz [2]) For each $\alpha < \Gamma_{I+1}$ and $\beta < I$, we define a set $C^I(\alpha, \beta)$ and a function $\psi_I^\alpha : Reg \cap I \to I$ by recursion on α, as follows.
(I1) $\beta \cup \{0, I\} \subset C^I(\alpha, \beta)$.
(I2,3) If $\gamma,\ \delta \in C^I(\alpha, \beta)$, then $\gamma + \delta,\ \varphi\gamma\delta \in C^I(\alpha, \beta)$.
(I4) If $\gamma \in C^I(\alpha, \beta)$, then $\Omega_\gamma \in C^I(\alpha, \beta)$.
(I5) If $\gamma \in C^I(\alpha, \beta) \cap \alpha$ and $\kappa \in C^I(\alpha, \beta) \cap Reg \cap I$, then $\psi_I^\gamma(\kappa) \in C^I(\alpha, \beta)$.
$\psi_I^\alpha(\kappa) := \min\{\rho < \kappa :\ C^I(\alpha, \rho) \cap \kappa = \rho \,\&\, \kappa \in C^I(\alpha, \rho)\}$.

Definition 2 We define a set $\mathcal{T}(I)$, as follows. [1]
($\mathcal{T}_I$1) $0, I \in \mathcal{T}(I)$.
($\mathcal{T}_I$2) If $\alpha, \beta \in \mathcal{T}(I) \cap (\alpha + \beta)$, then $\alpha + \beta \in \mathcal{T}(I)$.

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[1] One can consider each EORS to be formalized as a set of finite strings of suitable symbols (for example, see Sec.2 in [4]). Indeed, we can regard each EORS as a primitive recursive order structure.

($\mathcal{T}_I3$) If $\alpha, \beta \in \mathcal{T}(I) \cap \varphi\alpha\beta$ and ($\varphi\alpha\beta < I$ or ($\alpha = 0$ & $\beta > I$)), then $\varphi\alpha\beta \in \mathcal{T}(I)$.
($\mathcal{T}_I4$) If $\alpha \in \mathcal{T}(I) \cap \Omega_\alpha$ and $0 < \alpha < I$, then $\Omega_\alpha \in \mathcal{T}(I)$.
($\mathcal{T}_I5$) If $\gamma = \psi_I^\alpha(\kappa)$, $\alpha \in C^I(\alpha, \gamma)$ and $\alpha, \kappa \in \mathcal{T}(I)$, then $\gamma \in \mathcal{T}(I)$.

Lemma 3 Every element of $\mathcal{T}(I)$ has a unique expression by symbols $0, I, +, \varphi, \Omega$ and ψ_I.

Theorem 4 (Buchholz [2] (See also [3].)) $\psi_I^{\varepsilon_{I+1}}(\Omega_1) = \mathcal{T}(I) \cap \Omega_1 = |\mathrm{KPi}|$.

We next define an EORS for KPM, which is based on [5]. Let M denote the least weakly Mahlo cardinal.

Definition 5 (Skolem hulls and Collapsing functions based on M) For each $\alpha < \Gamma_{M+1}$ and $\beta < M$, we define a set $C^M(\alpha, \beta)$ and functions χ^α, ψ_M^α by recursion on α, as follows.
(M1) $\beta \cup \{0, M\} \subset C^M(\alpha, \beta)$.
(M2,3) If $\gamma,\ \delta \in C^M(\alpha, \beta)$, then $\gamma + \delta,\ \varphi\gamma\delta \in C^M(\alpha, \beta)$.
(M4) If $\gamma \in C^M(\alpha, \beta)$, then $\Omega_\gamma \in C^M(\alpha, \beta)$.
(M5) If $\gamma, \delta \in C^M(\alpha, \beta)$ and $\gamma < \alpha$, then $\chi^\gamma(\delta) \in C^M(\alpha, \beta)$.
(M6) If $\gamma \in C^M(\alpha, \beta) \cap \alpha$ and $\kappa \in C^M(\alpha, \beta) \cap Reg \cap M$, then $\psi_M^\gamma(\kappa) \in C^M(\alpha, \beta)$.
$\chi^\alpha(\delta) :=$ the δ^{th} regular cardinal $\pi < M$ s.t. $C^M(\alpha, \pi) \cap M = \pi$.
$\psi_M^\alpha(\kappa) \simeq \min\{\rho < \kappa :\ C^M(\alpha, \rho) \cap \kappa = \rho\ \&\ \kappa \in C^M(\alpha, \rho)\}$.

Definition 6 We define a set $\mathcal{T}(M)$, as follows.
($\mathcal{T}_M1$) $0,\ M\ \in\ \mathcal{T}(M)$.
($\mathcal{T}_M2$) If $\alpha, \beta \in \mathcal{T}(M)\ \cap\ (\alpha + \beta)$, then $\alpha + \beta \in \mathcal{T}(M)$.
($\mathcal{T}_M3$) If $\alpha, \beta \in \mathcal{T}(M) \cap \varphi\alpha\beta$ and ($\varphi\alpha\beta < M$ or ($\alpha = 0$ & $\beta > M$)), then $\varphi\alpha\beta \in \mathcal{T}(M)$.
($\mathcal{T}_M4$) If $\alpha \in \mathcal{T}(M) \cap \Omega_\alpha$ and $0 < \alpha < M$, then $\Omega_\alpha \in \mathcal{T}(M)$.
($\mathcal{T}_M5$) If $\gamma = \chi^\xi(\alpha)$, $\alpha < \gamma$, $\xi \in C^M(\xi, \gamma)$ and $\xi, \alpha \in \mathcal{T}(M)$, then $\gamma \in \mathcal{T}(M)$.
($\mathcal{T}_M6$) If $\gamma = \psi_M^\alpha(\kappa)$, $\alpha \in C^M(\alpha, \gamma)$ and $\alpha, \kappa \in \mathcal{T}(M)$, then $\gamma \in \mathcal{T}(M)$.

Lemma 7 (1) Every element of $\mathcal{T}(M)$ has a unique expression by symbols $0, M, +, \varphi, \Omega, \chi$ and ψ_M.
(2) $\psi_M^{\varepsilon_{M+1}}(\Omega_1) = \mathcal{T}(M) \cap \Omega_1$.

Definition 8 We define a mapping $*$ from $\mathcal{T}(I)$ to $\mathcal{T}(M)$ by recursion on the structure of each element of $\mathcal{T}(I)$, as follows.
(i) $0^* := 0$ and $I^* := \chi^0(0)$.
(ii) $(\alpha + \beta)^* := \alpha^* + \beta^*$.
(iii) $(\varphi\alpha\beta)^* := \varphi\alpha^*\beta^*$.
(iv) $(\Omega_\alpha)^* := \Omega_{\alpha^*}$.
(v) $(\psi_I^\alpha(\kappa))^* := \psi_M^{\alpha^*}(\kappa^*)$.

Theorem 9 Let $*\lceil_{\mathcal{T}(I)\cap\Omega_1}$ denote the restriction of $*$ to $\mathcal{T}(I) \cap \Omega_1$. Then, $*\lceil_{\mathcal{T}(I)\cap\Omega_1}$ is the identity function from $\mathcal{T}(I) \cap \Omega_1$ to $\psi_M^{\varepsilon_{I+1}}(\Omega_1)$, that is, for each $\gamma \in \mathcal{T}(I) \cap \Omega_1$, $\gamma^* = \gamma$.

The most important point of the proof of Th.9 is to check that $*$ is well-defined to be a strictly order-preserving function from $\mathcal{T}(I)$ to $\mathcal{T}(M)$ and that every γ less than $\psi_M^{\varepsilon_{I+1}}(\Omega_1)$ contains no symbol in $\mathcal{T}(M)$ which can not be interpreted with symbols in $\mathcal{T}(I)$.

Corollary 10 (1) $\psi_I^{\varepsilon_{I+1}}(\Omega_1) = \psi_M^{\varepsilon_{I+1}}(\Omega_1) = |\mathrm{KPi}|$.
(2) $\forall\tau \in \mathcal{T}(I) \cap Reg\ (\ \psi_I^\tau(\Omega_1) \in \mathcal{T}(I)\ \Rightarrow\ \tau^* \in \mathcal{T}(M) \cap Reg\ \&\ \psi_M^{\tau^*}(\Omega_1) \in \mathcal{T}(M)\ \&\ \psi_I^\tau(\Omega_1) = \psi_M^{\tau^*}(\Omega_1)\ )$.

In [6], we define a "proof-theoretically regular ordinal based on an EORS $\mathcal{T}$" to be an element of $\mathcal{T}$ which is of the form $\psi^\kappa(\Omega_1)$ for some $\kappa \in Reg$, where ψ^κ is a collapsing function with which $\mathcal{T}$ is constructed (see [6] or [7] for details). Therefore, Co.10.(2) means that every proof-theoretically regular ordinal based on $\mathcal{T}(I)$ is that based on $\mathcal{T}(M)$.

We next compare $\mathcal{T}(M)$ with another EORS denoted by $\mathcal{T}_B(M)$ which is (essentially) defined in [1].

Definition 11 For ordinals $\alpha < \Gamma_{M+1}$ and $\beta < M$, we define a set $B^M(\alpha,\beta)$ and a function $\psi^\alpha_B : Reg \cap (M+1) \to On$ by recursion on α, as follows.
(B1) $\beta \cup \{0, M\} \subset B^M(\alpha,\beta)$.
(B2,3) If $\gamma,\ \delta \in B^M(\alpha,\beta)$, then $\gamma+\delta,\ \varphi\gamma\delta \in B^M(\alpha,\beta)$.
(B4) If $\gamma \in B^M(\alpha,\beta)$, then $\Omega_\gamma \in B^M(\alpha,\beta)$.
(B5) If $\gamma \in B^M(\alpha,\beta) \cap \alpha$, $\kappa \in B^M(\alpha,\beta) \cap Reg \cap (M+1)$ and $\gamma,\kappa \in B^M(\gamma, \psi^\gamma_B(\kappa))$, then $\psi^\gamma_B(\kappa) \in B^M(\alpha,\beta)$.
$\psi^\alpha_B(\kappa) := \min\{\rho \in \mathcal{D}_\kappa(\alpha) :\ B^M(\alpha,\rho) \cap \kappa \subseteq \rho\}$, where

$$\mathcal{D}_\kappa(\alpha) := \begin{cases} \{\zeta \in Reg \cap (M+1) :\ \alpha \in B^M(\alpha,M) \Rightarrow \alpha \in B^M(\alpha,\zeta)\} & \text{if } \kappa = M; \\ \{\zeta :\ \kappa \in B^M(\alpha,\kappa) \Rightarrow \kappa \in B^M(\alpha,\zeta)\} & \text{if } \kappa < M. \end{cases}$$

Definition 12 We define a set $\mathcal{T}_B(M)$, as follows.
($\mathcal{T}_B$1) $0,\ M\ \in\ \mathcal{T}_B(M)$.
($\mathcal{T}_B$2) $\alpha,\beta \in \mathcal{T}_B(M) \cap (\alpha+\beta)\ \Rightarrow\ \alpha+\beta \in \mathcal{T}_B(M)$.
($\mathcal{T}_B$3) If $\alpha,\beta \in \mathcal{T}_B(M) \cap \varphi\alpha\beta$ and ($\varphi\alpha\beta < M$ or ($\alpha = 0$ & $\beta > M$)), then $\varphi\alpha\beta \in \mathcal{T}_B(M)$.
($\mathcal{T}_B$4) If $\alpha \in \mathcal{T}_B(M) \cap \Omega_\alpha$ and $0 < \alpha < M$ then $\Omega_\alpha \in \mathcal{T}_B(M)$.
($\mathcal{T}_B$5) If $\gamma = \psi^\alpha_B(\kappa)$, $\alpha,\kappa \in C^I(\alpha,\gamma)$ and $\alpha,\kappa \in \mathcal{T}_B(M)$, then $\gamma \in \mathcal{T}_B(M)$.

Lemma 13 Every element of $\mathcal{T}_B(M)$ has a unique expression by symbols $0, M, +, \varphi, \Omega$ and ψ_B.

Theorem 14 (Buchholz [1]) $\psi^{\varepsilon_{M+1}}_B(\Omega_1) = \mathcal{T}_B(M) \cap \Omega_1 \geq |\mathrm{KPM}|$.

Definition 15 We define a mapping $\sharp$ from $\mathcal{T}_B(M)$ to $\mathcal{T}(M)$ by recursion on the structure of each element of $\mathcal{T}_B(M)$, as follows.
(i) $0^\sharp := 0$ and $M^\sharp := M$.
(ii) $(\alpha+\beta)^\sharp := \alpha^\sharp + \beta^\sharp$.
(iii) $(\varphi\alpha\beta)^\sharp := \varphi\alpha^\sharp\beta^\sharp$.
(iv) $(\Omega_\alpha)^\sharp := \Omega_{\alpha^\sharp}$.
(v) If $\gamma =_{\mathrm{nf}} \psi^\alpha_B(\kappa)$, then $(\psi^\alpha_B(\kappa))^\sharp := \begin{cases} \chi^{\alpha^\sharp}(0) & \text{if } \kappa = M; \\ \psi^{\alpha^\sharp}_M(\kappa^\sharp) & \text{if } \kappa < M. \end{cases}$

Theorem 16 (1) $\sharp$ is well-defined to be a strictly order-preserving function from $\mathcal{T}_B(M)$ to $\mathcal{T}(M)$.
(2) $\sharp\lceil_{\psi_B^{\varepsilon_{\psi^0_B(M)+1}}(\Omega_1)}$ is the identity function from $\psi_B^{\varepsilon_{\psi^0_B(M)+1}}(\Omega_1)$ to $\psi^{\varepsilon_{I+1}}_M(\Omega_1)$.

Corollary 17 (1) $\psi^{\varepsilon_{M+1}}_M(\Omega_1) \geq \psi^{\varepsilon_{M+1}}_B(\Omega_1)$.
(2) $\forall\tau \in \mathcal{T}(I) \cap Reg\ (\ \psi^\tau_I(\Omega_1) \in \mathcal{T}(I)\ \Rightarrow\ \exists\kappa \in \mathcal{T}_B(M) \cap Reg\ (\ \psi^\kappa_B(\Omega_1) \in \mathcal{T}_B(M)\ \&\ \psi^\tau_I(\Omega_1) = \psi^\kappa_B(\Omega_1)\ )\ )$.

References

[1] W. Buchholz, A note on the ordinal analysis of KPM, *Proceedings Logic Colloquium '90 (Edited by J. Väänänen)*, (1992) p1-9.

[2] W. Buchholz, A simplified version of local predicativity, *Proof Theory (Edited by P. Aczel, H. Simmons and S. Wainer)*, (1993) p117-147.

[3] W. Pohlers. Subsystems of set theory and second order number theory, *Handbook of Proof Theory (Edited by S. R. Buss)*, (1998) p209-335.

[4] M. Rathjen, Proof-theoretic analysis of KPM, *Archive for Mathematical Logic*, 30 (1991) p377-403.

[5] M. Rathjen, The higher infinite in proof theory, *Logic Colloquium '95, Lecture Notes in Logic*, 11 (1998) p275-304.

[6] O. Takaki, Primitive recursive analogues of the least regular cardinal and the least weakly inaccessible cardinal, *Submitted for publication.*

[7] O. Takaki, Proof-theoretically inaccessible ordinals and Proof-theoretically Mahlo ordinals, *Preprint.*

Groups in Allegories

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Abstract

Groups are one of the most fundamental notions in mathematics. This paper provides a foundation of group theory in allegories. Almost all results in the paper can be applied to theory of fuzzy groups.

1 Introduction

The motivation of the paper arose from the following three fundamental exercises in group theory:

(a) Let $G = (G, \cdot)$ be a semigroup, that is, a binary operation $\cdot : G \times G \to G$ is associative. Show that if $\forall x \in G : xe = x$ and $\forall x \in G \exists y \in G : xy = e$ for some element $e \in G$, then $\forall x \in G : ex = x$ and $\forall x \in G \exists y \in G : yx = e$.

(b) Let H be a subgroup of a group $G = (G, \cdot, e, \cdot^{-1})$. Prove that the (binary) relation $x \equiv y$ on G, defined by $x^{-1}y \in H$, is an equivalence relation.

(c) Show that the set of all normal subgroups of a group G forms a modular lattice. In other words, the following modular law holds for three normal subgroups S, T and U of G:

$$S \subseteq U \;\Rightarrow\; ST \cap U \subseteq S(T \cap U).$$

To define group objects in categories [6] is not new, but it is difficult to directly treat algebraic relations, such as residual relations (b) induced by subgroups, in the ordinary category theory. Allegories [2], as a kind of relation categories, give a natural and suitable setting for manipulating algebraic binary relations in group theory, lattice theory [5] and so on. This paper provides a foundation of group theory in allegories, and solves the above three questions (a), (b) and (c).

2 Allegories

In this section we recall the fundamentals on relation categories, called allegories [2].

Throughout this paper, a morphism α from an object X into an object Y in an allegory (which will be defined below) will be denoted by a half arrow $\alpha : X \rightharpoonup Y$, and the composite of a morphism $\alpha : X \rightharpoonup Y$ followed by a morphism $\beta : Y \rightharpoonup Z$ will be written as $\alpha\beta : X \rightharpoonup Z$. Also we will denote the identity morphism on X as id_X.

Definition 2.1 An allegory $\mathcal{A}$ is a category satisfying the following:
D1. [Meet Semi-Lattice] For all pairs of objects X and Y the hom-set $\mathcal{A}(X, Y)$ consisting of all morphisms of X into Y is a meet semi-lattice with the greatest morphism ∇_{XY}. Its semi-lattice structure will be denoted by

$$\mathcal{A}(X, Y) = (\mathcal{A}(X, Y), \sqsubseteq, \sqcap, \nabla_{XY}).$$

D2. [Converse] There is given a converse operation $^\sharp : \mathcal{A}(X, Y) \to \mathcal{A}(Y, X)$. That is, for all morphisms $\alpha, \alpha' : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$, the following converse laws hold:

(a) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, (b) $(\alpha^\sharp)^\sharp = \alpha$, (c) If $\alpha \sqsubseteq \alpha'$, then $\alpha^\sharp \sqsubseteq \alpha'^\sharp$
for all morphisms $\alpha, \alpha' : X \rightharpoonup Y$ and $\beta : Y \rightharpoonup Z$.
D3. [Dedekind Formula] For all morphisms $\alpha : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$ and $\gamma : X \rightharpoonup Z$ the Dedekind formula $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$ holds.
D4. [Sub-Distributivity] The composition preserves order: If $\alpha \sqsubseteq \alpha'$ and $\beta \sqsubseteq \beta'$, then $\alpha\beta \sqsubseteq \alpha'\beta'$.

The fundamental properties of relational categories is referred to [1, 2, 7, 4]. A morphism $\alpha : X \rightharpoonup Y$ is *total* if $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp$ (or equivalently, $\alpha\nabla_{YX} = \nabla_{XX}$). A morphism $f : X \rightharpoonup Y$ such that $f^\sharp f \sqsubseteq \mathrm{id}_Y$ (*univalent*) is called a *function* and may be introduced as $f : X \to Y$. In what follows the word *relation* is used as synonym for morphisms in allegories.

Definition 2.2 A pair (A, B) of objects in an allegory $\mathcal{A}$ *has a relational product* if there exists a pair $(p : A \times B \to A, q : A \times B \to B)$ of total functions such that $p^\sharp q = \nabla_{AB}$ and $pp^\sharp \sqcap qq^\sharp = \mathrm{id}_{A\times B}$. The pair $(p : A \times B \to A, q : A \times B \to B)$ is called a pair of projections for (A, B). An allegory $\mathcal{A}$ *has a relational product* if every pair of objects in $\mathcal{A}$ has a relational product.

An object I in an allegory $\mathcal{A}$ is called a *strict unit* if $\nabla_{II} = \mathrm{id}_I$ and $\nabla_{XI}\nabla_{IX} = \nabla_{XX}$ for all objects X in $\mathcal{A}$.

Throughout the rest of the paper we assume that $\mathcal{A}$ is a fixed allegory with relational products and a strict unit I.

Let $\xi : A \rightharpoonup X$ and $\eta : B \rightharpoonup Y$ be a pair of relations. We define a relation $\xi \times \eta : A \times B \rightharpoonup X \times Y$ by $\xi \times \eta = p\xi\top q\eta = p\xi p_0^\sharp \sqcap q\eta q_0^\sharp$. If $f : A \to X$ and $g : B \to Y$ are total functions, then $f \times g$ is a unique function such that $(f \times g)p_0 = pf$ and $(f \times g)q_0 = qg$.

$$\begin{array}{ccccc} A & \xleftarrow{p} & A \times B & \xrightarrow{q} & B \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\xi\times\eta} & & \downarrow{\scriptstyle\eta} \\ X & \xleftarrow[p_0]{} & X \times Y & \xrightarrow[q_0]{} & Y \end{array}$$

3 Binary Operations

Throughout of the rest of the paper we assume that G is an object in an allegory $\mathcal{A}$ and $(p : G \times G \to G, q : G \times G \to G)$ is a pair of projections for (G, G).

Definition 3.1 A binary operation on G is a relation $\mu : G \times G \rightharpoonup G$ in $\mathcal{A}$.

Let $\mu : G \times G \rightharpoonup G$ be a binary operation. Then we define $\alpha \odot \beta = (\alpha p^\sharp \sqcap \beta q^\sharp)\mu$ for a pair of relations $\alpha, \beta : X \rightharpoonup G$. Remark that $p \odot q = \mu$.

Theorem 3.2 *A binary operation $\mu : G \times G \rightharpoonup G$ satisfies the associative law $(\mu \times \mathrm{id}_G)\mu = a(\mathrm{id}_G \times \mu)\mu$ if and only if $(\alpha \odot \beta) \odot \gamma = \alpha \odot (\beta \odot \gamma)$ for all relations $\alpha, \beta, \gamma : X \rightharpoonup G$. (Where a stands for the associative isomorphism.)*

$$\begin{array}{ccccc} (G \times G) \times G & & \xrightarrow{\mu\times id_G} & & G \times G \\ {\scriptstyle a}\downarrow & & & & \downarrow{\scriptstyle\mu} \\ G \times (G \times G) & \xrightarrow[id_G\times\mu]{} & G \times G & \xrightarrow[\mu]{} & G \end{array}$$

The first qeustion (a) reflects the following theorem on our framework:

Theorem 3.3 *Let $\varepsilon : I \rightharpoonup G$ and $\iota : G \rightharpoonup G$ be relations satisfying $\mathrm{id}_G \odot \nabla_{GI}\varepsilon = \mathrm{id}_G$ and $\mathrm{id}_G \odot \iota = \nabla_{GI}\varepsilon$. Then the following four conditions are equivalent:*

(a) ι *is a total function,*

(b) $\iota \odot \iota^2 = \nabla_{GI}\varepsilon$,

(c) $\nabla_{GI}\varepsilon \odot \iota^2 = \mathrm{id}_G$ *and* $\nabla_{GI}\varepsilon \odot \mathrm{id}_G = \mathrm{id}_G$,

(d) $\iota^2 = \mathrm{id}_G$.

4 Groups

In this section we define the notion of groups, subgroups and normal subgroups in allegories, and state the main theorems. However we limit all the operations of group structures to be total functions for the sake of simplicity.

Definition 4.1 A group $G = (G, m, e, i)$ in an allegory $\mathcal{A}$ is a quartet of an object G and three total functions $m : G \times G \to G$, $e : I \to G$ and $i : G \to G$ satisfying the following conditions:

(Associative Law) $(m \times \mathrm{id}_G)m = a(\mathrm{id}_G \times m)m$,
(Right Unitary) $\mathrm{id}_G \odot \nabla_{GI}e = \mathrm{id}_G$,
(Right Inverse) $\mathrm{id}_G \odot i = \nabla_{GI}e$.

Remark in the above definition that $\alpha \odot \beta = (\alpha p^\sharp \sqcap \beta q^\sharp)m$ for a pair of relations $\alpha, \beta : X \rightharpoonup G$.

Definition 4.2 A nonempty relation $\rho : I \rightharpoonup G$ is a *subgroup* of G if $\rho i \sqsubseteq \rho$ and $\rho \odot \rho \sqsubseteq \rho$.

The following theorem is a generalization of the second question (b).

Theorem 4.3 *If $\rho : I \rightharpoonup G$ is a subgroup, then $\theta = \mathrm{id}_G \odot \nabla_{GI}\rho$ is an equivalence relation on G.*

Normal subgroups in allegories will be simply defined as follows:

Definition 4.4 A subgroup $\rho : I \rightharpoonup G$ of G is called *normal* if $\mathrm{id}_G \odot \nabla_{GI}\rho = \nabla_{GI}\rho \odot \mathrm{id}_G$.

It is trivial that a subgroup $\rho : I \rightharpoonup G$ is normal iff $\alpha \odot \nabla_{XI}\rho = \nabla_{XI}\rho \odot \alpha$ for all relations $\alpha : X \rightharpoonup G$. Finally we can answer to the third question (c) that the set of all normal subgroups in allegories forms a modular lattice, as in the classical case:

Theorem 4.5 *Let $\rho, \sigma, \tau : I \rightharpoonup G$ be normal subgroups of a group G. If $\rho \sqsubseteq \tau$, then $(\rho \odot \sigma) \sqcap \tau \sqsubseteq \rho \odot (\sigma \sqcap \tau)$.*

References

[1] C. Brink, W. Kahl and G. Schmidt (eds.), Relational methods in computer science. Advances in Computing Science, (Springer, Wien, New York, 1997).

[2] P. Freyd and A. Scedrov, Categories, allegories (North-Holland, Amsterdam, 1990).

[3] J. Desharnais, *Monomorphic characterization of n-ary direct products*, Information Sciences **119**(1999), 275–288.

[4] Y. Kawahara, *Relational set theory*, Lecture Notes in Computer Science, **953**(1995), 44–58.

[5] Y. Kawahara, *Lattices in Dedekind categories*, In: Orlowska, E. and Szalas, A. (Eds), Relational Methods for Computer Science Applications, Physica-Verlag, 2001.

[6] S. Mac Lane, *Categories for the working mathematician*, (Springer-Verlag, 1972).

[7] G. Schmidt and T. Ströhlein, Relations and graphs – Discrete Mathematics for Computer Science – (Springer-Verlag, 1993).

[8] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35**(1971), 512–517.

[9] A. Tarski, *On the calculus of relations*, J. Symbolic Logic **6** (1941) 73–89.

A remark about elementarity

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1 Introduction

Two notions related to modal logics—and thus to varieties of modal algebras—that have achieved a certain notoriety are *elementarity* and *canonicity*. Let us recall that a modal logic Λ is called elementary if Λ is determined by an elementary class of Kripke-frames, and Λ is called canonical if the corresponding variety $\mathcal{V}(\Lambda)$ of modal algebras is closed under canonical extensions (cf. e.g., [1]). Fine proved in [3] that every elementary logic is canonical. It is still not known whether the converse implication holds. Goldblatt (cf. e.g., [4]) has proposed an algebraic condition of *singleton-persistence* under which elementarity is equivalent to canonicity (and further to completeness). However, canonicity is known not to imply singleton-persistence (cf. again [4]).

On the other hand, Goldblatt [5] proves that a logic Λ is elementary if and only if Λ is determined by its class of Kripke-frames, and that class is closed under ultrapowers. In what follows, we define an algebraic counterpart of closure under ultraproducts of Kripke-frames and show that it is indeed equivalent to elementarity, providing thereby a "purely algebraic" characterisation of elementarity.

2 Definitions

We assume familiarity with basic notions from the area, such as *complex algebras* of Kripke-frames, *canonical extensions* of modal algebras, etc. For a Kripke-frame $\mathfrak{F} = \langle W; R\rangle$, the full complex algebra of $\mathfrak{F}$ will be denoted by $\mathrm{Cm}\mathfrak{F}$; For a modal algebra $\mathbf{A}$ its canonical extension will be denoted by $\mathbf{A}^\sigma$.

For an atomic algebra $\mathbf{A}$ let $\mathfrak{A}$ stand for the *atom-structure* of $\mathbf{A}$, i.e., the structure $\langle \mathrm{At}\mathbf{A}; R\rangle$, where $R \subseteq \mathrm{At}\mathbf{A} \times \mathrm{At}\mathbf{A}$ is defined by xRy iff $fy \geq x$. If $\mathbf{A}$ is a full complex algebra, $\mathrm{Cm}\mathfrak{A}$ is isomorphic to $\mathbf{A}$ itself. A modal algebra $\mathbf{A}$ is *completely additive* if for every $X \subseteq A$, whenever $\bigvee X$ exists, we have: $f(\bigvee X) = \bigvee\{fx : x \in X\}$. This property does not need to hold even if the boolean reduct of $\mathbf{A}$ is *complete*, i.e., $\bigvee X$ exists for all $X \subseteq A$. However, all full complex algebras are completely additive. We will follow the custom of calling a modal algebra complete if it is complete as a boolean algebra and completely additive. Thus, all full complex algebras are complete and *vice versa*.

We will also make use of *minimal extensions* of modal algebras, as introduced by Monk [2]. There are several equivalent ways of defining these, we will however omit the technicalities altogether. All we need is contained in the next lemma.

Lemma 1 [Monk] *If a modal algebra $\mathbf{A}$ is completely additive, then its minimal extension $\mathbf{A}^c$ exists. Moreover, $\mathbf{A}^c$ is complete, preserves all existing joins and meets from $\mathbf{A}$, and is unique up to isomorphism.*

For a Kripke-frame $\mathfrak{F}$, the *singleton-algebra* of $\mathfrak{F}$, denoted $\mathrm{Sn}\mathfrak{F}$ is the subalgebra of $\mathrm{Cm}\mathfrak{F}$ generated by the singletons of $\mathfrak{F}$. A variety $\mathcal{V}$ of modal algebras is *singleton-persistent* if for any Kripke-frame $\mathfrak{F}$, we have $\mathrm{Sn}\mathfrak{F} \in \mathcal{V}$ implies $\mathrm{Cm}\mathfrak{F} \in \mathcal{V}$.

Lemma 2 [Goldblatt] *For any Kripke-frame $\mathfrak{F}$, the algebra $\mathrm{Cm}\mathfrak{F}$ is the minimal completion of $\mathrm{Sn}\mathfrak{F}$.*

Let $(\mathbf{A}_i)_{i\in I}$ be a family of complete modal algebras, and U an ultrafilter on I. Consider the ultraproduct $\prod_{i\in I}\mathbf{A}_i/U$.

Lemma 3 *The algebra $\prod_{i\in I}\mathbf{A}_i/U$ is atomic and completely additive.*

Let $\left(\mathbf{A}_i\right)_{i\in I}$ and U be as above. The minimal completion of $\prod_{i\in I}\mathbf{A}_i/U$, i.e., the algebra $(\prod_{i\in I}\mathbf{A}_i/U)^c$ will be called *complete ultraproduct* of $\left(\mathbf{A}_i\right)_{i\in I}$ over U.

Lemma 4 *Complete ultraproduct of $\left(\mathbf{A}_i\right)_{i\in I}$ over U is isomorphic to the full complex algebra of the ultraproduct $\prod_{i\in I}\mathfrak{A}_i/U$, where $\mathfrak{A}_i$ is the atom-structure of $\mathbf{A}_i$.*

A variety $\mathcal{V}$ of modal algebras is *complete* if $\mathcal{V}$ is generated by a class of complete algebras. $\mathcal{V}$ is *elementarily generated* if $\mathcal{V}$ is generated by the class $\{\mathrm{Cm}\mathfrak{F} : \mathfrak{F}\in\mathcal{K}\}$ for some elementary class $\mathcal{K}$ of Kripke-frames. We have the following characterisation of elementarity:

Theorem 1 [Goldblatt] *A variety $\mathcal{V}$ of modal algebras is elementarily generated if and only if $\mathcal{V}$ is generated by the class $\mathcal{C} = \{\mathbf{A}\in\mathcal{V} : \mathbf{A}$ is complete$\}$ and the class of atom-structures of algebras from $\mathcal{C}$ is closed under ultrapowers.*

3 Corollaries and a question

We finish off by stating a number of facts that are immediate corollaries of the definitions (and a reasonable amount of folklore). The first of these provides the algebraic characterisation of elementarity, announced at the beginning of the paper.

Corollary 1 *A variety $\mathcal{V}$ of modal algebras is elementarily generated if and only if $\mathcal{V}$ is complete and closed under complete ultrapowers.*

The second corollary records the fact that singleton-persistence implies closure under complete ultraproducts. The converse does not hold. A counterexample is provided by the variety $\mathcal{V}(K4M)$ of *transitive McKinsey* logic, which is elementary and thus closed under complete ultraproducts, yet not singleton-persistent (cf. [4]).

Corollary 2 *If a variety $\mathcal{V}$ of modal algebras is singleton-persistent, then $\mathcal{V}$ is closed under complete ultraproducts.*

The third simply closes the circle with the help of Corollary 1.

Corollary 3 *If a variety $\mathcal{V}$ of modal algebras is closed under complete ultrapowers, then $\mathcal{V}$ is elementarily generated iff $\mathcal{V}$ is canonical iff $\mathcal{V}$ is complete.*

The fourth restates another theorem of Goldblatt's in a purely algebraic language:

Corollary 4 *Let $\mathbf{A}$ be complete. Then, $\mathbf{A}^\sigma\subseteq(\mathbf{A}^I/U)^c$, for some index set I and a certain ω-incomplete ultrafilter U on I.*

Every elementarily generated variety is canonical. Since canonical extensions commute with subalgebras and homomorphic images, it can also be seen as a corollary of Corrollaries 1 and 4 above. Thus, the question "is every canonical logic elementary" is equivalent to the following:

Question 1 *Is every canonical variety closed under complete ultrapowers?*

References

[1] B. Jónsson and A. Tarski, Boolean Algebras with Operators, Part I, *American Journal of Mathematics* 73, 891–939, 1951.

[2] J.D. Monk, Completions of Boolean Algebras with Operators, *Mathematische Nachrichten* 46, 47–55, 1970.

[3] K. Fine, Some Connections between Elementary and Modal Logic, in: S. Kanger (ed.) *Proceedings of the Third Scandinavian Logic Symposium*, North-Holland, 1975.

[4] R. Goldblatt, Persistence and Atomic Generation for Varieties of Boolean Algebras with Operators, *Studia Logica* 68, No. 2, 155–171, 2001.

[5] R. Goldblatt, Varieties of Complex Algebras, *Annals of Pure and Applied Logic* 44, 173–242, 1989.

Games Played on Finite Graphs and Temporal Logic

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Our aim is to study reactive finite state systems (e.g. communication networks, banking systems, airtraffic control systems) by means of game-theoretic methods. A reactive system acts upon the inputs from environment by changing its states. The goal of the system is to satisfy given specifications no matter how environment behaves. We model this situation using games played on finite graphs first introduced by McNaughton [6].

Informally, an interaction between the system and environment (which is assumed to be a finite state system) is a sequence, called **run**, $\pi = s_0, a_0, s_1, a_1, \ldots$ of states of the system and the environment. The success of the run depends upon whether or not the system satisfies a given specification. Since both systems are finite, some states in the run appear infinitely often. We denote the set of these states by $Inf(\pi)$. An idea suggested by this observation is that specifications can be expressed in terms of infinity sets $Inf(\pi)$ of the runs. The run $\pi = s_0, a_0, s_1, a_1, \ldots$ can be thought as a play between two players, Survivor and Adversary. Survivor moves to a_i from s_i, and Adversary responds by moving to s_{i+1} from a_i for $i \in \omega$. The goal of Survivor is to satisfy the specification given while the goal of Adversary is not to allow the specification to be satisfied. We are now ready to give the following definition.

Definition 1 *A* **game** Γ *is a tuple* $(S \bigcup A, E, W, \Omega)$*, where*

1. *S and A are disjoint finite sets called the sets of* **positions** *for Survivor and Adversary, respectively;*
2. *E is the subset of $A \times S \bigcup S \times A$ so that for all $s \in S$ there is an $a \in A$ such that $(s, a) \in E$ and for all $a \in A$ there is an $s \in S$ such that $(a, s) \in E$,;*
3. *W is a subset of $S \bigcup A$; and*
4. *Ω is a subset of 2^W.*

The graph (V, E), where $V = S \cup A$, is called the **system** *(or the graph of the game), and the pair (W, Ω) is a* **specification**.

Given a game Γ, a play (from p_0) is an infinite sequence $\pi = p_0, p_1, \ldots$ such that $(p_i, p_{i+1}) \in E$ for each $i \in \omega$. Survivor wins the play if $Inf(\pi) \in \Omega$; otherwise Adversary wins. We say that a player wins the game if the player has a winning strategy in the game. **To decide game** Γ means to find all positions q in the game from which Survivor wins. We denote this set of position by $Win(S)$. Similarly, $Win(A)$ is the set of all winning positions for Adversary. It is not hard to see that the games defined are Borel games, and hence each game is determined by the known result of Martin [5]. Therefore $Win(S) \bigcup Win(A) = S \bigcup A$. In [6] McNaughton proved that there is an algorithm that given a game finds the sets $Win(S)$ and $Win(A)$. In fact, the proof implicitly tells us that the winner of any of these games has a finite state winnng strategy. McNaughton's algorithm is quite inefficient however. In [7] Nerode, Remmel and Yakhnis improved MacNaughton algorithm by showing that it takes $O(|W|!2^{|W|}|W||E|)$ time to decide a given game Γ. Thus, a natural question arises as under which conditions (which can be put either on the specifications or systems) the games can be decided more efficiently. Here is a list of some results related to this question.

One of the natural specifications consists of requiring Survivor to update every node of the system infinitely often. This is formalized as follows.

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Definition 2 *A game* Γ *is an* **update game** *if* $W = S \bigcup A$ *and* $\Omega = \{S \bigcup A\}$. *If Survivor wins this game, we then call the game an* **update network**.

It turns out that update games can be decided in a polynomial time as shown in [1]:

Theorem 1 *There exists an algorithm that given a game* Γ *decides in* $O(|V||E|)$ *time whether or not the game is an update network.*

The proof of this theorem is based on finding certain natural structural properties of update networks. See [1] for details.

In [2] Theorem 1 has been generalized as follows. We need a definition.

Definition 3 *A game* Γ *is a* **relaxed update game** *if* Ω *consists of pairwise disjoint subsets of* W, *that is for distinct* $V, U \in \Omega$ *we have* $V \cap U = \emptyset$. *If Survivor wins this game, we then call the game a* **relaxed update network**.

Using certain forcing techniques it is possible to prove the following theorem [2]:

Theorem 2 *There exists an algorithm that given a game decides in* $O(|V|^3)$ *time whether or not the game is a relaxed update network.*

The proof of this theorem does not use the techniques provided in the proof of Theorem 1. Instead, based on the notion of rank defined in Gurevich-Harrington [3], certain natural concepts (such as forcing) are introduced for the investigation of relaxed update games.

Recently we have been able to prove the following theorem whose proof consists of generalizing the ideas exploited in Theorem 1 and Theorem 2. We first give the following definition.

Definition 4 *We say that the game* Γ *is* **fully separated** *if* $W = S \bigcup A$ *and for each* $V \in \Omega$ *there is a* w *such that* $w \in V$ *but* w *belongs to no other* V' *from* Ω.

Here is our theorem.

Theorem 3 *(Ishihara, Khoussainov: 2001) There exists an algorithm that given a fully separated game* Γ *decides in* $O(|V|^3)$ *time whether or not Survivor wins* Γ.

The proof uses techniques developed in both [1] and [2]. A technical notion that is used in the proof is the notion of S-closedness which is defined in [2]:

Definition 5 *A winning set* $V \in \Omega$ *is* S**-closed** *if it satisfies the following conditions:*

1. *For each* $s \in V \cap S$ *there is an* $a \in V \cap A$ *such that* $(s, a) \in E$.
2. *For each* $a \in V \cap A$ *and all* s *such that* $(a, s) \in E$ *we have* $s \in V$.

Thus, informally this definition tells us that Survivor is able to always keep plays inside V no matter what the oppenent does. The proof of the theorem uses the idea of reducing the game Γ to subgames of the type $(V, \{V\})$ for S-closed $V \in \Omega$. These are update games, and hence Theorem 1 can be applied in the line of the proof.

Now we would like to say a few words in relation to connections with temporal logic. There are several ways to think about these games using the language of temporal logic. For example, one way to think about the specifications (W, Ω) is to identify them with a classes of formulas of temporal logic. More formally, this can be established as follows as it is done in [4]. Given a system $(S \bigcup A, E)$ we can form propositions of temporal logic by identifying each $p \in S \cup A$ as an atomic proposition. In the inductive step, if ϕ and ψ are propositions then their Boolean combinations and the expressions $G\phi$ and $F\phi$ also propositions. Now semantics for these propsitions are runs of the system $(S \bigcup A, E)$. Let π be a run and ϕ be formula. One now can define what it means ϕ to be true on π. Thinking of G as "globally" or "always", and of F as "future", we can actually represent specifications (W, Ω) in the language of

temporal logic. Thus, given a system $(S \bigcup A, E)$ and a specification ϕ we can now ask whether or not the system satisfies ϕ. The satisfaction can be expressed in terms of winning. Namely, the system **satisfies** ϕ if Survivor has a strategy so that in every play π consistent with the strategy the formula ϕ is true. For example, for the system $(S \bigcup A, E)$ with nodes $\{p_0, \ldots, p_{n-1}\}$, the specification

$$(p_0 \vee \ldots \vee p_n) \& \&_{0 \leq i \leq n-1} G(p_i \rightarrow F p_{i+1(mod\ n)})$$

tells that Survivor must visit every node infinitely often. This is in fact a specification of an update network game in terms of temporal logic. Thus, one can study the following natural questions:

1. (Model checking complexity) Given a (fixed) system, what is the time complexity of finding whether or not a given temporal formula is staisfied in the system?

2. (Implementation complexity) Given a (fixed) formula ϕ, what is the complexity of finding whether or not a given system satisfies ϕ?

3. (Combined complexity) What is the complexity of finding, given a formula ϕ and the system $\mathcal{A}$, that $\mathcal{A}$ satisfies ϕ?

Fore more details on topics of research in the line of verification and specifications of systems by means of temporal logic see Vardi [8].

References

[1] M. J. Dinneen and B. Khoussainov. Update networks and their routing strategies. In *Proceedings of the 26th International Workshop on Graph-Theoretic Concepts in Computer Science, WG2000*, volume 1928 of *Lecture Notes on Computer Science*, pages 127–136. Springer-Verlag, June 2000.

[2] H.L. Bodlaender, M.J. Dinneen and B. Khoussainov. On Game-Theoretic Models of Networks, in Algorithms and Computation (ISAAC 2001 proceedings), LNCS 2223, P. Eades and T. Takaoka (Eds.), p. 550-561, Springer-Verlag Berlin Heidelberg 2001.

[3] Y. Gurevich and L. Harrington. Trees, Automata, and Games, STOCS, 1982, pages 60–65.

[4] J. F. Knight and B. Luense. Control Theory, Modal Logic, and Games, In *Hybrid Systems IV*. Panos J. Antsaklis, Wolf Kohn, Anil Nerode, Shankar Sastry (Eds.), volume 1273 of *Lecture Notes in Computer Science*, pages 160–173. Springer, 1997.

[5] D. Martin. Borel Determinacy. Ann. Math. Vol 102, 363-375, 1975.

[6] R. McNaughton. Infinite games played on finite graphs. *Annals of Pure and Applied Logic*, 65:149–184, 1993.

[7] A. Nerode, J. Remmel, and A. Yakhnis. McNaughton games and extracting strategies for concurrent programs. *Annals of Pure and Applied Logic*, 78:203–242, 1996.

[8] M. Vardi. An automata-theoretic approach to linear temporal logic. Proceedings of the VIII Banff Higher Order Workshop. Springer Workshops in Computing Series, Banff, 1994.

Standard completeness of predicate logic MTL∀ for left-continuous t-norms

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1 Fuzzy logics of left-continuous t-norms and residuated lattices

The present paper is based on my joint paper with F. Montanga [8] in which we discussed Kripke semantics, undecidability and standard completeness of predicate logic MTL∀ for left-continuous t-norm, introduced by Esteva and Godo [2]. Here, we will focus our attention on the standard completeness.

In fuzzy set theory, the unit interval [0,1] is taken for the set of truth values. It is linearly ordered, and hence forms a bounded complete lattice. A binary function T from $[0,1] \times [0,1]$ to $[0,1]$ is called a *t-norm*, if the following holds for x, y, z;

1. $T(x, T(y,z)) = T(T(x,y), z)$,
2. $T(x,y) = T(y,x)$,
3. $T(x,1) = x$,
4. $x \leq y$ implies $T(x,z) \leq T(y,z)$.

Thus, $\langle [0,1], *, 1; \leq \rangle$ forms a partially ordered commutative monoid with a product $*$, which is defined by $x * y = T(x,y)$. We say that T is *continuous* (and *left-continuous*), when it is continuous (and left-continuous, respectively) as a function over the interval [0,1]. (Hereafter, we use $*$ instead of T.) Typical examples of continuous t-norms are the following: min, $\times$ and $\max\{x + y - 1, 0\}$.

Algebraic structures corresponding to the logic without contraction rule $\mathbf{FL_{ew}}$ are called *residuated lattices*. Here, a *residuated lattice* is an algebra $\mathbf{A} = \langle A, \cap, \cup, \cdot, \rightarrow, 0, 1 \rangle$ satisfying the following:

(i) $\langle A, \cap, \cup, 0, 1 \rangle$ is a bounded lattice with the greatest element 1 and the least 0,

(ii) $\langle A, \cdot, 1 \rangle$ is a commutative monoid,

(iii) for $x, y \in A$, $\quad x \cdot y \leq z$ if and only if $x \leq y \rightarrow z \quad$ (*the law of residuation*).

For more information on residuated lattices, see e.g. [6, 9]. A residuated lattice $\mathbf{A}$ is *complete* if it is complete as a lattice. We can show that in any complete residuated lattice, the following holds:

$$(\sup x_i) \cdot y = y \cdot (\sup x_i) = \sup(x_i \cdot y).$$

Note that when $\cdot$ is a t-norm, the above condition means the left-continuity. We have the following.

Proposition 1 *Suppose that an algebra $\mathbf{A} = \langle A, \cap, \cup, \cdot, 0, 1 \rangle$ satisfies the above conditions (i),(ii) and $\langle A, \cap, \cup \rangle$ is moreover complete. Then $\mathbf{A}$ is residuated (i.e. $\rightarrow$ is defined in $\mathbf{A}$ and satisfies the law of residuation) if and only if $\cdot$ is left-continuous.*

To show that the if-part, it suffices to define $\rightarrow$ by $y \rightarrow z = \max\{x : x \cdot y \leq z\}$.

Corollary 2 *Let $\mathbf{A}$ be any structure of the form $\langle [0,1], \min, \max, *, 0, 1 \rangle$ with a t-norm $*$. Then, $\mathbf{A}$ forms a residuated lattice with the residuum $\rightarrow$ defined as above if and only if $*$ is left-continuous.*

Thus, we can say that any structure of the above form with a left-continuous t-norm determines a logic over $\mathbf{FL_{ew}}$. For instance, t-norms min, $\times$ and $\max\{x + y - 1, 0\}$ give us Dummett-Gödel logic, product logic and Łukasiewicz logic, respectively.

2 Basic logic BL of continuous t-norms

To capture the logic of continuous t-norms, Hájek [4] introduced the basic propositional logic **BL**, which is obtained from $\mathbf{FL_{ew}}$ by adding the prelinearlity axiom $(\varphi \supset \psi) \vee (\psi \supset \varphi)$ and the axiom for hoops $(\varphi \wedge \psi) \supset (\varphi * (\varphi \supset \psi))$. Then, the following is shown in [1], which is called the *standard completeness* of **BL**.

Proposition 3 *Basic propositional logic* **BL** *is complete with respect to continuous t-norms over the unit interval* $[0,1]$. *More precisely, a formula* φ *is provable in* **BL** *if and only if it is valid in any complete residuated lattice on* $[0,1]$ *with a continuous t-norm.*

The continuity is essential in validating the axiom for hoops. In fact, the following holds.

Lemma 4 *Let* $\mathbf{A}$ *be any complete residuated lattice on* $[0,1]$ *with a left-continuous t-norm* $*$. *Then, the axiom for hoops is valid in* $\mathbf{A}$ *if and only if* $*$ *is continuous.*

Here, we will give a proof of the if-part (see [4]). For this purpose, it is enough to show that for any x, y in $[0,1]$, $x > y$ implies that $y \leq x * (x \to y)$. Define a mapping f on $[0,1]$ by $f(z) = x * z$. By our assumption, f is continuous and weakly increasing satisfying $f(0) = 0$ and $f(1) = x$. By the intermediate value theorem, $f(w) = y$ for some w. Let $V = \{w : f(w) = y\}$ and $v = \sup V$. Then, using the left-continuity of f, we have that $f(v) = y$. Therefore, $v = \max\{w : x * w \leq y\} = x \to y$. Hence, $x * (x \to y) = f(x \to y) = y$.

The basic predicate logic **BL**$\forall$ is the logic obtained from the predicate logic $\mathbf{FL_{ew}}$ by adding the above two axioms with the axiom of constant domains: $\forall x(\phi \vee \psi) \supset (\phi \vee \forall x\psi)$, where x is not free in ϕ. We have the following.

Proposition 5 *The standard completeness does not hold for* **BL**$\forall$.

For, while the set of all formulas provable in **BL**$\forall$ is obviously recursively enumerable, the set of all formulas valid in every residuated lattice on [0,1] with a continuous t-norm is not recursively enumerable; even is not arithmetical (see [7]).

3 Logic MTL of left-continuous t-norms and the standard completeness

In [2], Esteva and Godo introduced a propositional logic **MTL**, called the *monoidal t-norm based logic*, and showed that **MTL** is complete with respect to linearly ordered **MTL**-*algebras*. Here, **MTL** is obtained from **BL** just by dropping the axiom for hoops. The predicate logic **MTL**$\forall$ is defined similarly.

They conjectured that it is complete with respect to the left-continous t-norms over [0,1]. Jenei and Montagna [5] succeeded to show the following standard completeness for **MTL**.

Proposition 6 *The propositional logic* **MTL** *is complete with respect to left-continuous t-norms over* $[0,1]$.

The technique used there is applied also to some extensions of **MTL**, in order to get the standard completeness for them (see [3]). In the rest of the paper, we will give an outline of the proof of the standard completeness of **MTL**$\forall$. First, by applying the method developed in [4], we can show the following (see also [2]).

Proposition 7 *For any predicate formula* φ, φ *is provable in* **MTL**$\forall$ *if and only if* $g(\varphi) = 1$ *in every countable linearly ordered residuated lattice and every* safe *evaluation* g *on it.*

Suppose now that a formula ψ is not provable in **MTL**$\forall$. Then, by the above proposition, there is a countable linearly ordered residuated lattice $\mathbf{A}$ and a safe evaluation g such that $g(\psi) < 1$. To get the standard **MTL**$\forall$, it is necessary to show that $\mathbf{A}$ is embedded into a residuated lattice $\mathbf{B}$ with the underlying set [0,1] in which the left-continuity holds, and that this embedding is moreover a *complete embedding*, i.e. an embedding which preserves all exisiting meets and joins in A.

Lemma 8 *Any countable linearly ordered residuated lattice* $\mathbf{A}$ *can be embedded by a complete embedding* f *into a countable dense linearly ordered commutative monoid* $\mathcal{S}$ *with a monoidal operation* $\circ$ *which is left-continuous in the order toplogy. Moreover,* f *can be taken so as to preserve all the residua in* $\mathbf{A}$.

Since $\mathcal{S}$ is order isomorphic to $\mathbf{Q} \cap [0,1]$, we can suppose that the underlying set of S is $\mathbf{Q} \cap [0,1]$ and that the order of $\mathcal{S}$ is the natural order of $\mathbf{Q} \cap [0,1]$. Then, the identity mapping is a complete embedding of $\mathcal{S}$ into a linearly ordered commutative monoid $\mathcal{R}$ on $[0,1]$, with the monoid operation $\cdot$ by letting $\alpha \cdot \beta = 0$ if either $\alpha = 0$ or $\beta = 0$, and letting, for $\alpha, \beta \in (0,1]$,

$$\alpha \cdot \beta = \sup\{a \circ b : a, b \in \mathbf{Q} \cap [0,1],\ a < \alpha \text{ and } b < \beta\}$$

As in [5], we can see that $\cdot$ is commutative, associative, weakly increasing, and that $1 \cdot \alpha = \alpha$ for every α, and that $\cdot$ coincides with $\circ$ on the rationals (for the last claim we need the left continuity of $\circ$). Moreover, $\cdot$ is left-continuous: let $\langle \alpha_n : n \in \omega \rangle$ and $\langle \beta_n : n \in \omega \rangle$ be weakly increasing sequences of reals in $[0,1]$ such that $\sup\{\alpha_n : n \in \omega\} = \alpha$, and $\sup\{\beta_n : n \in \omega\} = \beta$. Then clearly $\alpha \cdot \beta$ is an upper bound of $\{\alpha_n \cdot \beta_n : n \in \omega\}$. Moreover, if $q \in \mathbf{Q} \cap [0,1]$ and $q < \alpha \cdot \beta$, then by the definition of $\cdot$ there are $r, s \in \mathbf{Q} \cap [0,1]$ such that $r < \alpha$, $s < \beta$, and $q \leq r \circ s$. But then for some $n \in \omega$, $r < \alpha_n$ and $s < \beta_n$ (since $\alpha = \sup\{\alpha_n : n \in \omega\}$ and $\beta = \sup\{\beta_n : n \in \omega\}$). Thus $q \leq r \circ s \leq \alpha_n \cdot \beta_n$ for some n. It follows that for all $q \in \mathbf{Q} \cap [0,1]$, $q < \alpha \cdot \beta$ iff for some $n \in \omega$, $q < \alpha_n \cdot \beta_n$ iff $q < \sup\{\alpha_n \cdot \beta_n : n \in \omega\}$. Hence $\sup\{\alpha_n \cdot \beta_n : n \in \omega\} = \alpha \cdot \beta$.

Thus, $\mathcal{R}$ equipped with the residuum $\Rightarrow$ of $\cdot$ forms a complete residuated lattice, which we call $\mathbf{B}$. Moreover, the mapping f given in the above lemma can be regarded as an complete embedding of $\mathbf{A}$ into $\mathbf{B}$ which preserves the residua. Now, define an evaluation h on $\mathbf{B}$ by letting for every atomic formula γ, $h(\gamma) = f(g(\gamma))$. Using the fact the f is a complete embedding, by induction on φ we obtain that for every closed formula φ, $h(\varphi) = f(g(\varphi))$. It follows that $h(\psi) < 1$. Thus, we have the standard completeness of **MTL**$\forall$.

Theorem 9 *The predicate logic* **MTL**$\forall$ *is complete with respect to left-continuous t-norms on* $[0,1]$.

References

[1] R. Cignoli, F. Esteva, L. Godo and A. Torrens, *Basic fuzzy logic is the logic of continuous t-norms and their residua*, Soft Computing 4, (2000) 106-112.

[2] F. Esteva and L. Godo, *Monoidal t-norm based logic: towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems 124, (2001) 271-288.

[3] F. Esteva, L. Godo, J. Gispert and F. Montagna, *On the standard completeness of some axiomatic extensions of the monoidal t-norm logic*, to appear in Studia Logica.

[4] P. Hájek, *Metamathematics of Fuzzy Logic*, Trends in Logic 4, Kluwer Academic Publishers, 1998.

[5] S. Jenei and F. Montagna, *A proof of standard completeness for Esteva's and Godo's logic MTL*, to appear in Studia Logica.

[6] T. Kowalski and H. Ono, *Residuated Lattices: An algebraic glimpse at logics without contraction*, monograph, March, 2001.

[7] F. Montanga, *Three complexity problems in predicate fuzzy logic*, Studia Logica 68, (2001) 143-152.

[8] F. Montagna and H. Ono, *Kripke semantics, undecidability and standard completeness for Esteva and Godo's logic MTL*$\forall$, draft, 2001.

[9] H. Ono, *Logics without the contraction rule and residuated lattices I*, Festschrift on the occasion of R.K. Meyer's 65^{th} birthday, to appear.