The curse of constructive mathematics
-Can the gadget computer go beyond the NP problem?-

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1 Introduction

Most computer scientists believe that normal computers work in constructive mathematics. In constructive mathematics, "there exists something" means "we can construct something" or "we know the position of something", and usual algorithms of most computers work in this way.

Now, I think about the "$P = NP$" problem. The P problems and the NP problems are decision problems.

When we solve a decision problem, we have only to get either a 'yes' answer or a 'no' answer. However, before we know a 'yes'-'no' answer, we get more information from constructive mathematics.

My main question is if there exists an algorithm which gets a 'yes'-'no' answer before we get more information. I think that most computers have the curse of constructive mathematics, and I doubt whether such a thing is possible using algorithms of normal computers.

In this paper, we try to compute an NP problem by using concrete objects, which is called "gadgets".

In the gadget world, it seems that "there exists something" dose not necessarily mean "we can construct something" or "we know the position of something."

For example, let's think about the existence of a hole in a sheet. In constructive mathematics, "there exists the hole" means "we know the position of the hole correctly." However, in the gadget world, if we want to know the existence of the hole, it is sufficient to illuminate this sheet. We do not have to know the position of the hole beforehand.

In this case, it seems that we can know the existence of the hole before we measure the position. This paper discusses the time complexity between these algorithms.

2 Basic Notation

In this paper, we think about the satisfiability of the conjunctive normal form, which is called SAT.

Definition 1. *Let n be a natural number. Let $U_n = \{x_1, x_2, \ldots, x_n\}$ be a set of n boolean variables.*

Definition 2. *Let k be a natural number, and each $l_i\,(i = 1, 2, \ldots, k)$ be either a boolean variable or a negated boolean variable in U_n. Then,*

$$l_1 \vee l_2 \vee \cdots \vee l_k$$

is said to be the clause.

Definition 3. *Let m be a natural number, and each $C_j\,(j = 1, 2, \ldots, m)$ be a clause. Then,*

$$C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

is said to be the conjunctive normal form.

Definition 4. *Let D be the conjunctive normal form w.r.t. U_n . If there exists the assignment $f : U_n \to \{\mathbf{True}, \mathbf{False}\}$ such that the value of D is* **True** *by substituting the assignment f into variables, then D is satisfied.*

Example 5. *We use* $U_4 = \{x_1, x_2, x_3, x_4\}$.

$$\begin{array}{rccccccccccc} & & (& x_1 & \vee & \neg x_2 & & & & &) \\ & \wedge & (& & & x_2 & \vee & \neg x_3 & & &) \\ D = & \wedge & (& x_1 & \vee & x_2 & \vee & x_3 & & &) \\ & \wedge & & & & \neg x_2 & & & & & \\ & \wedge & (& \neg x_1 & \vee & x_2 & \vee & x_3 & \vee & x_4 &) \end{array}$$

is the conjunctive normal form. And, there exists the assignment

$$f(x) = \begin{cases} \textbf{True} & \text{if } x = x_1, x_4 \\ \textbf{False} & \text{if } x = x_2, x_3 \end{cases}$$

such that the value of D *is* **True** *by the substitution* f. *Therefore,* D *is satisfied.*

We fix the number n of variables. And $\mathbf{N}$ is the set of all natural numbers.

Definition 6. $\overline{U_n}$ *is defined as the set* $\{i \in \mathbf{N} \mid 1 \leq i \leq 2^n\}$.

Definition 7. *Let* x_i *be the element of* U_n. *Then,* $\widetilde{x_i}$ *is defined as*

$$\left\{ j \in \overline{U_n} \;\middle|\; \exists u, v, w \in \mathbf{N} \left(\begin{array}{l} (j = u + v \times w) \wedge \big(1 \leq u \leq 2^{n-i}\big) \\ \wedge \big(1 \leq v \leq 2^{i-1}\big) \wedge \big(w = 2^{n-i+1}\big) \end{array} \right) \right\} .$$

And, $\widetilde{\neg x_i}$ *is defined as the set* $\overline{U_n} - \widetilde{x_i}$.

Example 8. *Let* n *be* 4 . *Then,*

$$\begin{array}{rcccccccccccccccccc} \overline{U_4} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16 & \}, \\ \widetilde{x_1} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8 & & & & & & & & & \}, \\ \widetilde{\neg x_1} & = & \{ & & & & & & & & & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16 & \}, \\ \widetilde{x_2} & = & \{ & 1, & 2, & 3, & 4, & & & & & 9, & 10, & 11, & 12 & & & & & \}, \\ \widetilde{\neg x_2} & = & \{ & & & & & 5, & 6, & 7, & 8, & & & & & 13, & 14, & 15, & 16 & \}, \\ \widetilde{x_3} & = & \{ & 1, & 2, & & & 5, & 6, & & & 9, & 10, & & & 13, & 14 & & & \}, \\ \widetilde{\neg x_3} & = & \{ & & & 3, & 4, & & & 7, & 8, & & & 11, & 12, & & & 15, & 16 & \}. \end{array}$$

Exercise 9. *Let* n *be* 4 . *Then, what are* $\widetilde{x_4}$ *and* $\widetilde{\neg x_4}$*?*

Definition 10. *Let* C *be the clause* $(l_1 \vee l_2 \vee \cdots \vee l_k)$. *Then,* $\overline{C}$ *is defined as*

$$\overline{U_n} - \widetilde{l_1} - \widetilde{l_2} - \cdots - \widetilde{l_k} .$$

Example 11. *Let* n *be 4, and* C *be the clause* $(\neg x_1 \vee x_2 \vee x_3)$. *Then,*

$$\begin{array}{rcl} \overline{C} & = & \overline{U_4} - \widetilde{\neg x_1} - \widetilde{x_2} - \widetilde{x_3} \\ & = & \{7, 8\} . \end{array}$$

Definition 12. *Let* D *be the conjunctive normal form* $(C_1 \wedge C_2 \wedge \ldots \wedge C_m)$. *Then,* $\overline{D}$ *is defined as*

$$\overline{C_1} \cup \overline{C_2} \cup \cdots \cup \overline{C_m} .$$

Theorem 13. *Let* D *be the conjunctive normal form. Then,* $\overline{D} \subsetneqq \overline{U_n}$ *if and only if* D *is satisfied.*

Proof. We use the function

$$g(x) = \begin{cases} 0 & \text{if } x = \textbf{True}, \\ 1 & \text{if } x = \textbf{False}. \end{cases}$$

For each assignment $f : \overline{U_n} \to \{\textbf{True}, \textbf{False}\}$,

$$\sum_{i=1}^{n} g\,(\,f\,(x_i)\,) \times 2^{n-i} \in \overline{U_n} - \overline{D} \quad \leftrightarrow \quad D \text{ is } \textbf{True} \text{ by substituting } f.$$

□

3 The Gadget Computer

First, we fix the conjunctive normal form $D = (C_1 \wedge C_2 \wedge \ldots \wedge C_m)$ w.r.t. U_n .

In this section, we calculate the satisfiability of the conjunctive normal form in the gadget computer. We use m pieces of sheets, whose figures are rectangles 2 height and 2^n wide, as gadgets.

Each clause C_j correspond to the j-th sheet. And each sheet is in a coordinate plane, such that

$$\{(x, y) \in \mathbf{R} \times \mathbf{R} \mid 0 \leq x \leq 2^n, 0 \leq y \leq 2 \} .$$

Lemma 14. *We can make these sheets, and the number of steps in this process is $O(m \times n)$.*

Proof. We can measure the length 2^n by the n-th folded sheet. □

Definition 15. *Let l be either a boolean variable or a negated boolean variable in U_n. Then, $\widehat{l}$ is defined as*

$$\left\{ (x, y) \in \mathbf{R} \times \mathbf{R} \;\middle|\; \exists\, i \in \widetilde{l}\ (i - 1 < x < i)\,,\ 0 \leq y < 1 \right\} .$$

Definition 16. *Let C_j be the clause $l_1 \vee l_2 \vee \cdots \vee l_k$. Then, $\widehat{C_j}$ is defined as the remainder, such that we cut off parts $l_1, l_2, \ldots, l_k$ from the j-th sheet.*

Lemma 17. *The number of steps in this cutting process is $O(k \times n)$.*

Proof. If we fold this sheet $O(n)$-times and cut twice well, then we can cut off the part $\widehat{l}$ from the sheet. Of course, if n is sufficiently large, then it is physically impossible. However, now we think about the ideal gadget. □

Theorem 18. *Now, we lay m pieces of cut sheets $\widehat{C_1}, \widehat{C_2}, \ldots, \widehat{C_m}$ one on top of another. Then, we can find the hole in the area $\{(x, y) \in \mathbf{R} \times \mathbf{R} \mid 0 \leq x \leq 2^n,\ 0 \leq y \leq 2 \}$, if and only if D is satisfied.*

Proof. It is obvious by Theorem 13. □

Theorem 19. *Using the gadget computer, we can compute the satisfiability of the conjunctive normal form, and the number of steps in this process is the polynomial function of n .*

Proof. It is obvious by above Lemmata and Theorem. □

4 Conclusion

Using the above algorithm of the gadget computer, it seems that we can know a 'yes'-'no' answer before we get more information. However, I can not prove that it is true at present.

Now, we think about the relation of the time complexity between these algorithms.

Theorem 20. *We assume that, for any conjunctive normal form D w.r.t. U_n , we compute the satisfiability problem of D $O(p(n))$-times . If this problem of D has a 'yes' answer, then we can get the satisfied assignment f $O(n \times p(n))$-times.*

Proof. If D is the conjunctive normal form w.r.t. U_n , then the substitute $D[x_i := \mathbf{True}]$ and $D[x_i := \mathbf{False}]$ are conjunctive normal forms w.r.t. U_n also.

We use the next algorithm.

$E := D$
for $i := 1$ **to** n
  if $E[x_i := \mathbf{True}]$ is satisfied **then** $f(x_i) := \mathbf{True}$ and $E := E[x_i := \mathbf{True}]$
    else $f(x_i) := \mathbf{False}$ and $E := E[x_i := \mathbf{False}]$
next i

If the satisfiability problem of D has a 'yes' answer, then we can get the satisfied assignment f using this algorithm. □

This result is not good for my project. Because, even if there exists an algorithm which gets a 'yes'-'no' answer before we get more information, we can get more information almost at the same time by using the algorithm of the same process.

By this thing, I doubts whether it is essential that the NP problem must be a decision problem in the $P = NP$ problem.

Models for Normal Modal Logics over **KR**

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KR is the system of the logic excluding only 'paradox of relevance', which is one of paradoxes of material implication, in classical logic. In **KR** we can identify relevant negation with classical negation, so it is thought that **KR** is too closer system to classical logic. This paper examines this fact from a viewpoint introducing modal operator. In semantical study of relevant logic, Routley-Meyer semantics are often used and there are two kinds of models, the one is an unreduced model and the other is a reduced model. In general, there is a limit to apply a reduced model. So we consider when a reduced model can be used.

1 Unreduced Model

In this section, based on [4], we deal with the system of normal modal logics over **KR** (in below, we call them simply normal logics). And we will prove the completeness for them by using unreduced models.

First of all, we determine the language. The language is as follows: (i) propositional variables $p, q, r, \cdots$; (ii) logical connectives $\rightarrow$ (relevant implication), $\wedge$ (and), $\vee$ (or) and $\neg$ (negation); (iii) modal operator $\Box$; and (iv) auxiliary symbols.

The formulas are defined by the usual way, and they are denoted by the capital letters $A, B, C, \cdots$ (possibly with subscripts). In below Wff, Prop denote the set of all formulas and that of all propositional variables, respectively, and capital Greeks $\Sigma, \Gamma, \Delta, \cdots$ (possibly with prime) denote the non-empty sets of formulas.

First, we define the system **KRN** of the least normal logic. **KRN** is defined as follows:

(a) Abbreviation

$$\Diamond A \stackrel{\text{ab}}{=} \neg\Box\neg A$$

(b) Axioms

(A1) $A \rightarrow A$
(A2) $A \rightarrow ((A \rightarrow B) \rightarrow B)$
(A3) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
(A4) $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
(A5) $A \wedge B \rightarrow A$
(A6) $A \wedge B \rightarrow B$
(A7) $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
(A8) $A \rightarrow A \vee B$
(A9) $B \rightarrow A \vee B$
(A10) $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$
(A11) $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee C$
(A12) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
(A13) $\neg\neg A \rightarrow A$
(A14) $A \wedge \neg A \rightarrow B$
(A15) $\Box A \wedge \Box B \rightarrow \Box(A \wedge B)$
(A16) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(c) Rules of inference

$$\frac{A \rightarrow B \quad A}{B}\text{(Modus Ponens)} \qquad \frac{A \quad B}{A \wedge B}\text{(Adjunction)} \qquad \frac{A}{\Box A}\text{(Necessitation)}$$

Now we define the typical systems of normal logics. Consider the systems obtained by adding the following formulas as axioms to **KRN**:

(A17) $\Box A \to \Diamond A$

(A18) $\Box A \to A$

(A19) $\Box A \to \Box\Box A$

(A20) $A \to \Box\Diamond A$

(A21) $\Diamond A \to \Box\Diamond A$

We call the systems **KRN.D**, **KRN.T**, **KRN.4**, **KRN.B** and **KRN.5** are obtained by adding (A17),(A18),(A19),(A20) and (A21) to **KRN**, respectively. We can combine more than two systems. For example, **KRN.T4** is obtained by adding (A18) and (A19) to **KRN**, or by adding (A19) to **KRN.T**.

In below **L** denote any systems of normal logics. We write $\mathbf{L} \vdash A$ if A is a theorem of **L**.

Next we give the semantics of the systems of normal logics in terms of unreduced models, and prove the soundness for them.

An (*unreduced*) **KRN**-*frame* is a quadruple $\langle O, W, R, S\rangle$ where (a) W is the set of all worlds, (b) O is the non-empty subset of W, (c) R is a ternary relation on W, and (d) S is a binary relarion on W.

A **KRN**-frame $\mathcal{F} = \langle O, W, R, S\rangle$ satisfies the following postulates, for all $a, b, c, d \in W$:

(p1) $\exists x(x \in O \ \&\ Rxab) \Longleftrightarrow a = b$

(p2) $Raaa$

(p3) $Rabc \Longrightarrow Rbac \ \&\ Racb$

(p4) $Rabe \ \&\ Recd \Longrightarrow \exists x(Racx \ \&\ Rxbd)$

(p5) $Rabd \ \&\ Sdc \Longrightarrow \exists x\exists y(Sax \ \&\ Sby \ \&\ Rxyc)$

(p6) $\exists x(x \in O \ \&\ Sxa) \Longrightarrow a \in O$

(*Unreduced*) **L**-*frames* are **KRN**-frame $\langle O, W, R, S\rangle$ which satisfies adding postulates. Adding postulates are as follows, for all $a, b, c \in W$:

(p7) $\exists x(Sax)$

(p8) Saa

(p9) $Sab \ \&\ Sbc \Longrightarrow Sac$

(p10) $Sab \Longrightarrow Sba$

(p11) $Sab \ \&\ Sac \Longrightarrow Sbc$

We call $\mathcal{F}$ a **KRN.D**-frame, **KRN.T**-frame, **KRN.4**-frame, **KRN.B**-frame and **KRN.5**-frame, if $\mathcal{F}$ is a **KRN**-frame satisfying the postulate (p7), (p8), (p9), (p10) and (p11), respectively. As in systems, we can combine some of distinct **L**-frames. For example, if a **L**-frame $\mathcal{F}$ holds postulates (p1) - (p6), (p8) and (p9), then $\mathcal{F}$ is called a **KRN.T4**-frame.

We call a quintuple $\langle O, W, R, S, v\rangle$ **L**-*model* on a frame $\langle O, W, R, S\rangle$ (in below, simply **L**-model), where $\langle O, W, R, S\rangle$ is a **L**-frame and $v : \mathsf{Prop}\times W \to \{t, f\}$ is called a *valuation*. For a given model $\langle O, W, R, S, v\rangle$, the relation $a \models A$ for $a \in W$ and $A \in \mathsf{Wff}$ is defined inductively as follows:

i. for any $p \in \mathsf{Prop}$, $a \models p$ iff $v(p, a) = t$

ii. $a \models A \wedge B$ iff $a \models A$ and $a \models B$

iii. $a \models A \vee B$ iff $a \models A$ or $a \models B$

iv. $a \models A \to B$ iff $\forall x\forall y(Raxy \ \&\ x \models A \Rightarrow y \models B)$

v. $a \models \neg A$ iff $a \not\models A$

vi. $a \models \Box A$ iff $\forall x(Sax \Rightarrow x \models A)$.

By the abbreviation and an interpretation defined above, $a \models \Diamond A$ iff $\exists x(Sax \ \&\ x \models A)$.

We say that A is **L**-*valid*, if $x \models A$ for every $x \in O$ in any **L**-model $\langle O, W, R, S, v\rangle$.

Theorem 1 (Soundness of L) *For any* **L**, *if* A *is a theorem of* **L**, *then* A *is* **L**-*valid.*

Finally, we prove the completeness for the systems of normal logics. Basically we follows the method in section 4.6 of [8]. To prove the completeness, the following notions play a key role:

- Let $\Sigma \neq \mathsf{Wff}$ and $\Delta \neq \mathsf{Wff}$. Then (i) $\mathbf{L} \vdash \Sigma \to \Delta$ iff there exist $A_1, \cdots, A_m \in \Sigma (m > 0)$ and $B_1, \cdots, B_n \in \Delta (n > 0)$ such that $\mathbf{L} \vdash A_1 \wedge \cdots \wedge A_m \to B_1 \vee \cdots \vee B_n$, and (ii) (Σ, Δ) is a **L**-*pair* iff (a) $\mathbf{L} \not\vdash \Sigma \to \Delta$ and (b) $\Sigma \cup \Delta = \mathsf{Wff}$.

- Σ is a **L**-*theory* iff (a) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$ and (b) if $\mathbf{L} \vdash A \to B$ and $A \in \Sigma$, then $B \in \Sigma$.

- Let Σ be a **L**-theory. Then (i) Σ is *regular* iff Σ contains all theorems of **L**, and (ii) Σ is *maximal* iff either $A \in \Sigma$ or $\neg A \in \Sigma$ and both do not hold for all $A \in \mathsf{Wff}$

- Let Σ, Γ, Δ be **L**-theories. Then (i) $R\Sigma\Gamma\Delta$ iff if $A \to B \in \Sigma$ and $A \in \Gamma$ then $B \in \Delta$, and (ii) $S\Sigma\Gamma$ iff if $\Box A \in \Sigma$ then $A \in \Gamma$.

Note that $S\Sigma\Gamma$ iff if $A \in \Gamma$ then $\Diamond A \in \Sigma$.

To show the completeness, we see the unreduced case in [8], though there are slight differences in some terminologies and we must replace the part of 'prime' by 'maximal'. Here we pick up the following lemma which does not appear in [8].

Lemma 2 *If Σ is a maximal* **L***-theory such that $\Box C \notin \Sigma$, then there exist maximal* **L***-theory Γ' such that $S\Sigma\Gamma'$ and $C \notin \Gamma'$.*

Then it is established that there exists a regular maximal **L**-theory Π such that $A \notin \Pi$ whenever $\mathbf{L} \nvdash A$. Then we construct a canonical **L**-model $\langle O', W', R', S', v' \rangle$ as follows: (1) W' is the set of all maximal **L**-theories; (2) O' is the set of all regular maximal **L**-theories; (3) R' is a ternary relation R restricted on W'; (4) S' is a binary relation S restricted on W'; (5) v' is defined by $v'(p, \Sigma) = t$ iff $p \in \Sigma$, for all $p \in \mathsf{Prop}$ and $\Sigma \in W'$.

Theorem 3 (Completeness of L) *If A is* **L***-valid, then A is a theorem of* **L***.*

2 Reduced Model

In this section, we consider a reduced model. First we define the system **KR4** in order to use a reduced model. The system **KR4** is the corresponding system of modal logic over **KR** to **NR** in [7] or **R4** in [5], which are the systems of modal logics over **R**. Though **NR** and **R4** are different systems, adding $A \wedge \neg A \to B$ as an axiom to them, the result systems are equivalent.

KR4 is obtained from **KRN.T4** by adding an axiom

$$(\mathrm{A}\Box) \qquad \text{if } A \text{ is an axiom, then so is } \Box A$$

instead of eleminating (Necessitation).

If A is a theorem of **KR4**, we write $\mathbf{KR4} \vdash A$.

Next we consider a reduced model and give the semantics of **KR4** and prove the soundness for it.

A **KR4**-*frame* is a quadruple $\langle 0, W_r, R_r, S_r \rangle$ where (a_r) W_r is the set of all worlds, (b_r) $0 \in W_r$ is called a base world, (c_r) R_r is a ternary relation on W_r, and (d_r) S_r is a binary relation on W_r, and satisfies the following postulates, for all $a, b, c, d \in W_r$:

(pr1) $\exists x(S_r 0x \;\&\; R_r xab) \Longleftrightarrow a = b$

(pr2) $R_r aaa$

(pr3) $R_r abc \Longrightarrow R_r acb$

(pr4) $R_r abe \;\&\; R_r ecd \Longrightarrow \exists x(R_r acx \;\&\; R_r xbd)$

(pr5) $R_r abd \;\&\; S_r dc \Longrightarrow \exists x \exists y(S_r ax \;\&\; S_r by \;\&\; R_r xyc)$

(pr6) $S_r aa$

(pr7) $S_r ab \;\&\; S_r bc \Longrightarrow S_r ac$

We call a quintuple $\langle 0, W_r, R_r, S_r, v_r \rangle$ **KR4**-*model* on a frame $\langle 0, W_r, R_r, S_r \rangle$ (in below, simply **KR4**-model), where $\langle 0, W_r, R_r, S_r \rangle$ is a **KR4**-frame and $v_r : \mathsf{Prop} \times W_r \to \{t, f\}$ is called a *valuation*. For a given **KR4**-model $\langle 0, W_r, R_r, S_r, v_r \rangle$, the relation $a \models A$ for $a \in W_r$ and $A \in \mathsf{Wff}$ is defined by the same way in section 1.

We say that A is **KR4**-*valid*, if $0 \models A$ for all **KR4**-model $\langle 0, W_r, R_r, S_r, v_r \rangle$.

Theorem 4 (Soundness) *If A is a theorem of* **KR4***, then A is* **KR4***-valid.*

Finally we prove the completeness theorem for **KR4**. We prove it by using the style of [8] based on the argument in [7] or [5].

First we give several notions in order to prove the completeness theorem. Note that there are slight differences between in section 1 and in this section. The key notions are as follows.

- Let $\Sigma \neq \mathsf{Wff}$ and $\Delta \neq \mathsf{Wff}$. Then (i) $\mathbf{KR4} \vdash \Sigma \to \Delta$ iff there exist $A_1, \cdots, A_m \in \Sigma (m > 0)$ and $B_1, \cdots, B_n \in \Delta (n > 0)$ such that $\mathbf{KR4} \vdash \Box(A_1 \wedge \cdots \wedge A_m \to B_1 \vee \cdots \vee B_n)$, and (ii) (Σ, Δ) is a **KR4**-*pair* iff (a) $\mathbf{KR4} \nvdash \Sigma \to \Delta$ and (b) $\Sigma \cup \Delta = \mathsf{Wff}$.
- Σ is a **KR4**-*theory* iff (a) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$ and (b) if $\mathbf{KR4} \vdash \Box(A \to B)$ and $A \in \Sigma$, then $B \in \Sigma$.

- Let Σ be a **KR4**-theory. Then (i) Σ is *regular*, iff Σ contains all theorems of **KR4**, and (ii) Σ is *maximal*, iff either $A \in \Sigma$ or $\neg A \in \Sigma$ and both do not hold for all $A \in \mathsf{Wff}$.

Let Π be a regular maximal **KR4**-theory.

- Let $\Sigma \neq \mathsf{Wff}$ and $\Delta \neq \mathsf{Wff}$. Then (i) $\Pi \vdash \Sigma \rightarrow \Delta$ iff there exist $A_1, \cdots, A_m \in \Sigma (m > 0)$ and $B_1, \cdots, B_n \in \Delta (n > 0)$ such that $\Box(A_1 \wedge \cdots \wedge A_m \rightarrow B_1 \vee \cdots \vee B_n) \in \Pi$, and (ii) (Σ, Δ) is a Π-*pair* iff (a) $\Pi \nvdash \Sigma \rightarrow \Delta$ and (b) $\Sigma \cup \Delta = \mathsf{Wff}$.
- Σ is a Π-*theory* iff (a) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$ and (b) if $\Box(A \rightarrow B) \in \Pi$ and $A \in \Sigma$, then $B \in \Sigma$.
- A Π-theory Σ is *maximal*, iff either $A \in \Sigma$ or $\neg A \in \Sigma$ and both do not hold for all $A \in \mathsf{Wff}$.
- Let Σ, Γ, Δ be Π-theories. Then (i) $R\Sigma\Gamma\Delta$ iff if $A \rightarrow B \in \Sigma$ and $A \in \Gamma$ then $B \in \Delta$, and (ii) $S\Sigma\Gamma$ iff if $\Box A \in \Sigma$ then $A \in \Gamma$.

To show the completeness, we see the reduced case in [8], though there are slight differences in some terminologies and we must replace the part of 'prime' by 'maximal'. Here we pick up the following lemma which does not appear in [8].

Lemma 5 *If Σ is a maximal Π-theory such that $\Box C \notin \Sigma$, then there exist maximal Π-theory Γ' such that $S\Sigma\Gamma'$ and $C \notin \Gamma'$.*

Then it is established that there exists a regular maximal **KR4**-theory Π such that $A \notin \Pi$ whenever **KR4** $\nvdash A$. Using this Π, we construct a canonical model $\langle 0', W'_r, R'_r, S'_r, v'_r \rangle$ as follows: (1') $0' = \Pi$; (2') W'_r is the set of all maximal Π-theories; (3') R'_r is a ternary relation R restricted on W'_r; (4') S'_r is a binary relation S restricted on W'_r; (5') v'_r is defined by $v'_r(p, \Sigma) = t$ iff $p \in \Sigma$, for all $p \in \mathsf{Prop}$ and $\Sigma \in W'_r$,

Theorem 6 (Completeness) *If A is* **KR4***-valid, then A is a theorem of* **KR4***.*

3 Unreduced Models and Reduced Models

In this section we consider the relationship between unreduced models, dealt with sections 1, and reduced models, dealt with section 2. Generally, unreduced models can be applied wider than reduced models. We shall see that unreduced models can be translated into reduced models in special case.

The following fact holds:

Lemma 7 *For any $A \in \mathsf{Wff}$,* **KRN.T4** $\vdash A$ *iff* **KR4** $\vdash A$.

Hence **KR4** can be regarded the same system as **KRN.T4**, so **KR4** is a normal logic. So a **KR4**-frame can be said a (reduced) **KRN.T4**-frame.

As in [5], there is a way to construct an unreduced frame from a reduced frame in the argument of the soundness of **R4**. However, this method cannot be applied to the argument of the completeness. So we consider the generalized method to obtain an unreduced model from a reduced model.

In this paper, there is a reduced model only for **KR4**. We shall consider constructing an unreduced **KR4**-model from a reduced **KR4**-model. Note that an unreduced **KR4**-model can be identified an unreduced **KRN.T4**-model by Lemma 7.

There is most difference between the postulates (p1) and (p5) in an unreduced **KRN.T4**-frame and the postulates (p'1) in a reduced **KR4**-frame. Though (p3) and (p'3) are different, they can be identified because the postulate $Rabc \Longrightarrow Rbac$ can be derived in a **KR4**-frame. To construct an unreduced **KRN.T4**-model $\langle O, W, R, S, v \rangle$ from a reduced **KR4**-model $\langle 0, W_r, R_r, S_r, v_r \rangle$, we define a translation σ from a reduced **KR4**-model $\langle 0, W_r, R_r, S_r, v_r \rangle$ into an unreduced **KRN.T4**-model $\langle O, W, R, S, v \rangle$ as follows, for $a, b, c \in W_r$:

- $W = \{\sigma(x) \mid x \in W_r\}$
- $O = \{\sigma(x) \mid S_r 0x\}$
- $R\sigma(a)\sigma(b)\sigma(c)$ iff $R_r abc$
- $S\sigma(a)\sigma(b)$ iff $S_r ab$
- $v(p, \sigma(a)) = v_r(p, a)$, for all $p \in \mathsf{Prop}$.

Lemma 8 *$\langle O, W, R, S, v\rangle$ defined above is an unreduced* **KRN.T4***-frame.*

For the converse, there are some restrictions to make a reduced model. We can use a reduced model only if the system of the normal logic can be derived the following formula:

$$\text{(A18)} \quad \Box A \to A \qquad \text{and} \qquad \text{(A19)} \quad \Box A \to \Box\Box A.$$

$\mathbf{L_{T4}}$ denotes the system of normal logics in which (A18) and (A19) can be derived. For example, we can take **KRN.T4** and **KRN.T5** as $\mathbf{L_{T4}}$.

Using $\mathbf{L_{T4}}$, we can say as follows: for any $\mathbf{L_{T4}}$, there is a reduced model for $\mathbf{L_{T4}}$.

In order to construct a reduced $\mathbf{L_{T4}}$-model from an unreduced $\mathbf{L_{T4}}$-model, we introduce the translation τ from an unreduced $\mathbf{L_{T4}}$-model $\langle O, W, R, S, v\rangle$ into a reduced $\mathbf{L_{T4}}$-model $\langle 0, W_r, R_r, S_r, v_r\rangle$ as follows, for $a, b, c \in W$:

- $W_r = \{\tau(x) \mid x \in W\}$
- for all $x \in O$, $\tau(x) = 0$
- $R_r\tau(a)\tau(b)\tau(c)$ iff $Rabc$
- $S_r\tau(a)\tau(b)$ iff Sab
- $v_r(p, \tau(a)) = v(p, a)$, for all $p \in \mathsf{Prop}$.

Lemma 9 *$\langle 0, W_r, R_r, S_r, v_r\rangle$ defined above is* $\mathbf{L_{T4}}$*-model.*

Note that $\mathbf{L_{T4}}$ has (Necessitation) as the rule of inference. From Lemma 7 and Lemma 9, we show the method constructing a reduced $\mathbf{L_{T4}}$-model from an unreduced $\mathbf{L_{T4}}$-model as follows:

1. Eliminate (Necessitation) from $\mathbf{L_{T4}}$ and add the following axiom to axioms of $\mathbf{L_{T4}}$:

   if A is an axiom, then so is $\Box A$.

2. Construct a reduced $\mathbf{L_{T4}}$-model $\langle 0, W_r, R_r, S_r, v_r\rangle$ from an unreduced $\mathbf{L_{T4}}$-model $\langle O, W, R, S, v\rangle$ by using the translation τ defined above.

When we enjoy the completeness by using a reduced $\mathbf{L_{T4}}$-model, we should argue the same way as in section 2.

The argument in this section can be applied to normal logics over **R**. Therefore this fact give us the information about the relationship between an unreduced model, investigated in [4], and a reduced model, investigated in [7].

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Subframe formulas for **S4**-type intuitionistic modal logic

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The subframe formulas Y for **S4**-type intuitionistic modal logic $\mathbf{L}_0$ is introduced. It is shown that the logic $\mathbf{L}_0 + Y + bw_n$ has the finite model property.

1 Intuitionistic modal logics

In this section, we repeat some definitions and notations in [3].

Formulas are constructed using by propositional variables, $p_0, p_1, \cdots$, logical symbols, $\wedge$, $\vee$, $\supset$, $\neg$, $\Box$, $\top$ and $\bot$ (not included $\Diamond$).

The logic $\mathbf{L}_0$ is the logics obtained from **H** by adding the following axioms,

- $\Box p \supset p$,
- $\Box p \supset \Box\Box p$,
- $\Box(p \supset q) \supset (\Box p \supset \Box q)$,

and closed under the rules of substitution, necessitation and modus ponens.

The intuitionistic modal logics are extension of $\mathbf{L}_0$ and closed the above rules.

A triple $(N, \leq, R)$ is called an **I**-frame if the following conditions are satisfied.

- N is a non-empty set.
- $\leq$ is a partial order on N.
- R is a reflexive and transitive relation on N such that $x \leq y$ implies xRy for each $x, y \in N$.

Let $\mathbf{M}$ be an **I**-frame $(M, \leq, R)$. $\mathbf{M}$ is said to be rooted if it has a least element with respect to R, that is, there is an element $0 \in M$ such that $0Ra$ and $a \not\leq 0$ for any other element $a \in M$. This element 0 is said to be a root. If $\mathbf{M}$ is an **I**-frame, then we let

$$\begin{aligned} IS_{\leq}(a) &= \{b : a \leq b\} \\ IS_R(a) &= \{b : aRb\} \end{aligned}$$

$aR^{\sim}b$ denotes aRb and bRa.

w is said to be maximal with respect to A if

$$A \equiv B \supset C, w \not\models A \Rightarrow \forall v(w \leq v \text{ and } v \models A), \tag{1}$$

$$A \equiv \neg B, w \not\models A \Rightarrow \forall v(w \leq v \text{ and } v \models A), \tag{2}$$

$$A \equiv \Box B, w \not\models A \Rightarrow \forall v(w \overrightarrow{R} v \text{ and } v \models A). \tag{3}$$

A pair $(\mathbf{M}, \models)$ is said to be an **I**-model if, for all $a \in M$, any propositional variable p and any formulas A, B,

1. $a \models p$ and $a \leq b \Rightarrow b \models p$,
2. $a \models A \wedge B \Leftrightarrow a \models A$ and $a \models B$,

3. $a \models A \vee B \Leftrightarrow a \models A$ or $a \models B$,
4. $a \models A \supset B \Leftrightarrow \forall b(a \leq b) b \not\models A$ or $b \models B$,
5. $a \models \neg A \Leftrightarrow \forall b(a \leq b$ implies $b \not\models A)$,
6. $a \models \Box A \Leftrightarrow \forall b(aRb$ implies $b \models A)$.

2 Preliminaries

In this section, we show some property of **I**-models. We can easily verify the next propositions.

Proposition 2.1 $\mathbf{L}_0 \vdash \Box\neg(\Box p \vee \Box\neg\Box p) \supset \bot$
$\mathbf{L}_0 \vdash \neg(\neg p \vee p) \supset \bot$

Let $\mathbf{L} \supseteq \mathbf{L}_0$, $\mathbf{M_L}$ be a canonical model for $\mathbf{L}$ and $u \in \mathbf{M_L}$.

Proposition 2.2 *u is an element in maximal cluster iff $\Box A \vee \Box\neg\Box A \in u$ for every wff A.*

Proposition 2.3 *u is maximal with respect to $\leq$ iff $\neg A \vee A$ for every wff A.*

Proposition 2.4 *For all $u \in \mathbf{M_L}$, there exists a maximal element with respect to $\leq$.*

Proposition 2.5 *For all $u \in \mathbf{M_L}$, there exists a maximal element with respect to R.*

Let bw_n be the following formula:

$$bw_n = \bigvee_{i=0}^{n} \Box(\Box p_i \rightarrow \bigvee_{i \neq j} \Box p_j)(n \geq 1).$$

Lemma 2.6 **M** *validates bw_n iff each point in* **M** *has at most n R-incomparable successors.*

For a formula A, we define the relation $\sim_{F(A)}$ to get results on model equivalence with respect to A between two models. Let **M** and **N** are **I**-frames. We define $\sim_{F(p)}$, $\leq_{F(p)}$, $\equiv_{F(A)}$ and $\sim_{F(A)}$ as follows:

$$\begin{aligned}
(\mathbf{M}, a) \sim_{F(p)} (\mathbf{N}, v) &\Leftrightarrow \forall p(a \models p \Leftrightarrow w \models' p) \\
(\mathbf{M}, a) \leq_{F(A \supset B)} (\mathbf{N}, v) &\Leftrightarrow \forall b \in M, \exists v \in N(a \leq b \rightarrow u \leq' v) \text{ and} \\
&\quad (M, b) \sim_{F(A)} (N, v) \text{ and} \\
&\quad (M, b) \sim_{F(B)} (N, v)) \\
(\mathbf{M}, a) \leq_{F(\neg A)} (\mathbf{N}, v) &\Leftrightarrow \forall b \in M, \exists v \in N(a \leq b \rightarrow u \leq' v) \text{ and} \\
&\quad (M, b) \sim_{F(A)} (N, v) \\
(\mathbf{M}, a) \leq_{F(\Box A)} (\mathbf{N}, v) &\Leftrightarrow \forall b \in M, \exists v \in N(aRb \rightarrow uR'v), \text{ and} \\
&\quad (M, b) \sim_{F(A)} (N, v) \\
(\mathbf{M}, a) \equiv_{F(A)} (\mathbf{N}, v) &\Leftrightarrow (\mathbf{M}, a) \leq_{F(A)} (\mathbf{N}, v) \text{ and} \\
&\quad (\mathbf{N}, v) \leq_{F(A)} (\mathbf{M}, a) \\
(\mathbf{M}, a) \sim_{F(A)} (\mathbf{N}, v) &\Leftrightarrow (\mathbf{M}, a) \sim_{F(p)} (\mathbf{N}, v) \text{ and} \\
&\quad (\mathbf{M}, a) \equiv_{F(A)} (\mathbf{N}, v)
\end{aligned}$$

For $(\mathbf{M}, w) \sim_{F(A)} (\mathbf{M}, v)$, we write $w \sim_{F(A)} v$ if **M** is understood.
$|w|_{F(A)}$ denotes the set $\{v \in M : w \sim_{F(A)} v\}$.

Lemma 2.7 *Let* **M** *and* **N** *be* **I***-frames. For any formula A and $a \in M, v \in N$,*

$$(\mathbf{M}, a) \sim_{F(A)} (\mathbf{N}, v) \Rightarrow (v \models A \rightarrow a \models A)$$

Proof. By the induction on the construction of A ■

Lemma 2.8 *Suppose $A \supset B \notin w$ (respectively, $\neg A \notin w$) for $w \in \mathbf{M}_L$, w is not maximal element with respect to $A \supset B$ (respectively, $\neg A$). Then there exists $v (v \in \mathbf{M}_L, w \leq v)$ which v is maximal with respect to $A \supset B$ (respectively, $\neg A$).*

Lemma 2.9 *Suppose $\Box A \in w$, for $w \in \mathbf{M}_L$, w is not maximal element with respect to $\Box A$. Then there exists $v (v \in \mathbf{M}_L, wRv)$ which v is maximal with respect to $\Box A$.*

Definition 2.10 (p-morphism,subreduction) *Let $\mathbf{M}$ and $\mathbf{N}$ be $\mathbf{I}$-frames. A map f from $\mathbf{N} = (N, \leq_N, R_N)$ to $\mathbf{M} = (M, \leq_M, R_M)$ is called a p-morphism if the following conditions are satisfied, for any $u, v \in N$ and $a \in M$,*

(p-1) *f is onto,*

(p-2) *If $u \leq_N v$ then $f(u) \leq_M f(v)$,*

(p-3) *If $uR_N v$ then $f(u)R_M f(v)$,*

(p-4) *If $f(u) \leq_M a$ then there exists v such that $u \leq_N v$ and $f(v) = a$,*

(p-5) *If $f(u)R_M a$ then there exists v such that $uR_N v$ and $f(v) = a$.*

A map f from $\mathbf{N}$ to $\mathbf{M}$ is called a subreduction if f is a p-morphism from a subframe of $\mathbf{N}$ to $\mathbf{M}$. And then, $\mathbf{N}$ is subreducible to $\mathbf{M}$.

3 subframe formulas

In this section, we introduce subframe formulas for $\mathbf{L}_0$ by the method as Shimura defined for the intuitionistic logic in [4]. Fine showed the completeness of subframe logics $Y(\mathbf{M}, 0)$ for **K4** [2]. It was shown by selecting finite points from canonical models. We show the completeness some logics containing $\mathbf{L}_0$ by the same method as Fine.

For a rooted and finite $\mathbf{I}$-frame, we define the subframe formulas by the induction on the cardinality of $a\uparrow_{\leq}$, $a\uparrow_R$ as follows:

$$
\begin{aligned}
X_{\leq}(\mathbf{M}, a) &= \bigvee_{b \not\leq a} p_b \vee \bigvee_{\substack{d \in IS_{\leq}(a) \\ a \not\tilde{R} d}} Y(\mathbf{M}, d) \vee \bigvee_{\substack{e \in IS_{\leq}(e) \\ a R^{\sim} e}} X(\mathbf{M}, e), \\
X_R(\mathbf{M}, a) &= \bigvee_{c \not R a} \Box q_c \vee \bigvee_{e \in IS_R(a)} \Box Y(M, e), \\
X(\mathbf{M}, a) &= p_a \wedge \Box q_a \supset p \vee X_{\leq}(\mathbf{M}, a) \vee X_R(\mathbf{M}, a), \\
Y_R(\mathbf{M}, a) &= \bigwedge_{a' \in Cl(a)} (\bigwedge_{a'' R^{\sim} a'} \Box(\Box X(\mathbf{M}, a'') \supset X(\mathbf{M}, a'))), \\
Y(\mathbf{M}, a) &= Y_R(\mathbf{M}, a) \supset X(\mathbf{M}, a).
\end{aligned}
$$

Proposition 3.1 *Let $\mathbf{M}$ be a finite and rooted $\mathbf{I}$-frame and a, $b \in M$. Then,*

1. *if $b \leq a$ does not hold, $\mathbf{L}_0 \vdash p_b \supset Y(\mathbf{M}, a)$,*
2. *if bRa does not hold, $\mathbf{L}_0 \vdash \Box q_b \supset Y(\mathbf{M}, a)$,*
3. *if $a \leq b$ hold, $Y(\mathbf{M}, b) \supset Y(\mathbf{M}, a)$,*
4. *if $a \leq b$ hold, $Y(\mathbf{M}, b) \supset Y(\mathbf{M}, a)$,*
5. *if aRb hold, $\Box Y(\mathbf{M}, b) \supset Y(\mathbf{M}, a)$.*

Lemma 3.2 *Let* $(\mathbf{N}, \models)$ *be an* **I***-frame,* $u \in N$ *and* $\mathbf{M}$ *be a finite and rooted* **I***-frame. If* $u \not\models Y(\mathbf{M}, a)$ *then there exists* $v \in N$ *such that*

$$u \leq v, v \models p_a \wedge \Box q_a \text{ and } v \not\models Y(\mathbf{M}, a).$$

Lemma 3.3 *Suppose there is a subreduction* f *from* $\mathbf{N}$ *to* $\mathbf{M}$*, formula* E *and formulas sets* $\{F_a; a \in M\}$ *and* $\{\Box G_a; a \in M\}$ *such that*

$$\exists v(a \not\leq f(v), u \leq v) \to u \not\models F_a$$
$$\exists v(a \not R f(v), uRv) \to u \not\models \Box G_a$$
$$\exists v(uRv, v \in dom(f)) \to u \not\models E$$

Then the formula obtained from $Y(\mathbf{M})$ *by substituting* F_a *(respectively,* G_a*) for* p_a *(respectively,* q_a*) for every* $a \in M$ *and* E *for* p *is refuted on* $\mathbf{N}$*.*

Theorem 3.4 *Let* $\mathbf{M}$ *be a finite and rooted* **I***-frame. Then the following are equivalent, for any* **I***-frame* $\mathbf{N}$*,*

- $\mathbf{N} \not\models Y(\mathbf{M})$*,*
- *there is a subreduction* f *from* $\mathbf{N}$ *to* $\mathbf{M}$*.*

Proof. ($\Rightarrow$) Assume $\mathbf{N} \not\models Y(\mathbf{M})$. Define a partial map f from N to M

$$f(u) = \begin{cases} a & (u \models p_a \wedge \Box q_a \text{ and } u \not\models Y(\mathbf{M}, a)), \\ \text{undefined} & (\text{otherwise}). \end{cases}$$

Then, f is a subreduction from $\mathbf{N}$ to $\mathbf{M}$.
($\Leftarrow$) For $u \in N$, define a valuation $\models$ on $\mathbf{N}$ as follows:

$$\exists v(a \not\leq f(v), u \leq v) \to u \not\models p_a$$
$$\exists v(a \not R f(v), uRv) \to u \not\models \Box q_a$$
$$\exists v(uRv, v \in dom(f)) \to u \not\models p$$

■

Lemma 3.5 *Let* $\mathbf{L}_0 + \{Y(\mathbf{M}_i)\}_{i \in I} + bw_n$*. There no finite subframe of* $\mathbf{N}_\mathbf{L}$ *is reducible to* $\mathbf{M}_i$*.*

Lemma 3.6 *Suppose that* $w \in \mathbf{M}_\mathbf{L}$*,* $w \not\models A$*. Then there is a finite submodel* $\mathbf{N} = (N, \leq', R', \models')$ *of* $w \uparrow_R$ *and* $v \not\models A$ *for* $v \in N$*.*

Proof. By the induction on the construction A.
(i) If A is a propositional variable, let $N = \{w\}$ and $w = v$.
(ii) Otherwise. By Proposition 2.4 Proposition 2.5, there exists a point w' such that $w' \not\models \Box A$ and w' is maximal with respect to R.
We define N by the induction on the cardinality of $\{|v|_{F(A)} : w' \leq v\}$ and $\{|v|_{F(A)} : w' \overrightarrow{R} v\}$.
Let $N' = \{w'\}$. Suppose $U = \{u : w' \sim u\}$, $X = \{|u|_{F(A)} : u \in U\}$. Pick representatives x_1, x_2, $\cdots$, x_n from X such that each $x_i (i = 1 \cdots n)$ is maximal with respect to $\leq$. Let $rank_{x_i} = |\{|u|_{F(A)} : x_i \leq u \text{ and } u \in U\}|$.
If $rank_{x_i} = 1$ then $U_{x_i} = \{x_i\}$. Otherwise, let $Y = \{|x_{ij}|_{F(A)} : x_i \leq x_{ij} \text{ and } x_{ij} \in U\}$. Pick representatives y_1, y_2, $\cdots$, y_l from Y such that each $y_i (i = 1 \cdots l)$ is maximal with respect to $\leq$. Let $U_{x_i} = U_{y_1} \cup U_{y_2} \cup \cdots \cup U_{y_l}$ and $U = \bigcup_{i=1}^{n} U_{x_i}$.
Next for each $w_i \in U$, let $S = \{|s|_{F(A)} : w_i \leq s \text{ and } w_i \overrightarrow{R} s\}$, and $S' = \{|s'|_{F(A)} \backslash S : w_i \overrightarrow{R} s'\}$.
If $S = \{|w_i|_{F(A)}\}$ or $S = \emptyset$ then let $T_{w_i}^{\leq} = \emptyset$. Otherwise, pick representatives s_1, s_2, $\cdots$, s_m from S such that each $s_i (i = 1 \cdots m)$ is maximal with respect to $\leq$. And Let $T_{w_i}^{\leq} = N_{s_1} \cup N_{s_2} \cup \cdots \cup N_{s_m}$.
If $S' = \{|w_i|_{F(A)}\}$ or $S' = \emptyset$ then let $T_{w_i}^{R} = \emptyset$. Otherwise, pick representatives s'_1, s'_2, $\cdots$, s'_n from S' such that each $s_i (i = 1 \cdots n)$ is maximal with respect to R. And Let $T_{w_i}^{R} = N_{s'_1} \cup N_{s'_2} \cup \cdots \cup N_{s'_n}$.
Set $T_{w_i} = T_{w_i}^{\leq} \cup T_{w_i}^{R}$ and $T = T_{w_1} \cup T_{w_2} \cdots T_{w_p}$.
Let $N = N' \cup U \cup T$. ■

Theorem 3.7 *$L_0 + \{Y(\mathbf{M}_i)\}_{i\in I} + bw_n$ has finite model property.*

Corollary 3.8 *$L_0 + \{Y(\mathbf{M}_i)\}_{i\in I} + bw_n$ is complete.*

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Residuation Theory and Substructural Logics

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1 Introduction

For these years, the author has been working at an algebraic study of substructural logics, in particular of logics over $\mathbf{FL_{ew}}$ (see e.g. [6]). Here, by $\mathbf{FL_{ew}}$ we mean the logic obtained from the intuitionistic logic by deleting the contraction rule, if it is formulated as a sequent system. The corresponding algebraic structures are *residuated lattices*. An important aspect of algebraic study of these logics will come from the fact that algebraic methods enable us to discuss various logics and related algebraic structures within a uniform framework. For instance, by discussing algebraic structures for Łukasiewicz's many-valued logics and also for fuzzy logics together with hoops and pocrims from universal algebra, it becomes possible to see strong interrelations behind them.

In this paper, we will show the importance of the notion of *residuation* in the study of substructural logics, and thus *residuated structures* based on residuation theory offer us algebraic structures which are suitable for substructural logics.

A part of the content of the present paper was presented also at University of Torino, September, 2000. After this, T. Kowalski informed me of the Ph.D thesis [3] by K. Blount and of a book on residuation theory [4] by T.S. Blyth and M.F. Janowitz. Then the author has realized that roles played by the *residuation* in substructural logics can be stated more explicitly by the help of residuation theory.

2 Residuated maps

In this section, we will give a brief survey of basic facts in residuation theory. For more information, see e.g. [4]. See also [3].

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. A map f: $\mathbf{P} \to \mathbf{Q}$ is *residuated* if there exists a map f^*: $\mathbf{Q} \to \mathbf{P}$ such that the following holds for any $p \in P$ and any $q \in Q$:

$$f(p) \leq q \Longleftrightarrow p \leq f^*(q).$$

In the above case, we say that f and f^* form a *residuated pair*, (or, an *adjunction* in category theory) and that f^* is a *residual* of f. We can show the following results.

Lemma 2.1 *If f and f^* form a residuated pair, both f and f^* are monotone increasing.*

Lemma 2.2 *Maps f: $\mathbf{P} \to \mathbf{Q}$ and f^*: $\mathbf{Q} \to \mathbf{P}$ form a residuated pair if and only if both $id_P \leq f^* \circ f$ and $f \circ f^* \leq id_Q$ hold.*

Lemma 2.3 *If a map f is residuated then its residual is determined uniquely by f.*

Hereafter, f^* denotes the residual of a map f when f is residuated. Similarly to Lemma 2.3, we can show that f is determined uniquely by f^* when f and f^* form a residuated pair. In fact, the following holds;

1. $f^*(q) = \max\{p \in P; f(p) \leq q\}$,
2. $f(p) = \min\{q \in Q; p \leq f^*(q)\}$.

Proposition 2.4 *Suppose that f and f^* form a residuated pair between $\mathbf{P}$ and $\mathbf{Q}$, and that $X \subseteq P$ and $Y \subseteq Q$. If $\sup X$ exists then $\sup f(X)$ ($= \sup\{f(p); p \in X\}$) exists and $f(\sup X) = \sup f(X)$. Also if $\inf Y$ exists then $\inf f^*(Y)$ exists and $f^*(\inf Y) = \inf f^*(Y)$.*

When both $\mathbf{P}$ and $\mathbf{Q}$ are lattices in the above proposition, we have the following.

Corollary 2.5 *Suppose that f and f^* form a residuated pair between lattices $\mathbf{L}$ and $\mathbf{M}$. Then $f(a \cup b) = f(a) \cup f(b)$ for all $a, b \in \mathbf{L}$, and $f^*(c \cap d) = f^*(c) \cap f^*(d)$ for all $c, d \in \mathbf{M}$.*

If both $\mathbf{L}$ and $\mathbf{M}$ are moreover complete, we have the converse of Proposition 2.4.

Proposition 2.6 *Suppose that both $\mathbf{L}$ and $\mathbf{M}$ are complete lattices and that f is a map from $\mathbf{L}$ to $\mathbf{M}$. Then, f is residuated if and only if f preserves all (possibly infinite) joins. Dually, a map $f^* : \mathbf{M} \to \mathbf{L}$ is the residual of a map $f : \mathbf{L} \to \mathbf{M}$ if and only if f^* preserves all (possibly infinite) meets.*

3 Residuated structures

Now we will apply results in the previous section to the case where posets are ordered groupoids. Then, we will discuss particular classes of ordered groupiods which originate from algebras for substructural logics.

A structure $\mathbf{G} = \langle G, \cdot, \leq \rangle$ is an *ordered groupoid* if $\cdot$ is a binary operation on a poset $\langle G, \leq \rangle$ which satisfies the following monotonicity (see [4]) :

$$x \leq x' \text{ and } y \leq y' \text{ imply } x \cdot y \leq x' \cdot y'$$

Only for the simplicity's sake, *we assume moreover the commutativity of $\cdot$ in the rest of this section.* For each $x \in G$, define a map $f_x : \mathbf{G} \to \mathbf{G}$ by $f_x(y) = x \cdot y$. When f_x is residuated for any $x \in G$, we say that the ordered commutative groupoid $\mathbf{G}$ is *residuated*. The residual ${f^*}_x$ of f_x is expressed as ${f^*}_x(y) = x \to y$. (One may call such an operation $\cdot$, a *fusion*, and $\to$, an *implication*.) Then, the following must hold:

$$f_x(y) \leq z \Longleftrightarrow y \leq {f^*}_x(z),$$
$$\text{i.e. } x \cdot y \leq z \Longleftrightarrow y \leq x \to z$$

which is called *the law of residuation*.

When $\mathbf{G}$ has also a lattice structure, we have the following equations by Corollary 2.5: For all $a, b, c \in \mathbf{G}$

1. $(a \cup b) \cdot c = (a \cdot c) \cup (b \cdot c)$,
2. $a \to (b \cap c) = (a \to b) \cap (a \to c)$.

In particular, suppose that $\mathbf{G}$ is a Heyting algebra. In this case, $\cap$ and $\to$ form a residuation pair. Thus, the above 1 becomes the distributive law. This gives the well-known fact that any Heyting algebra is distributive. Similarly, when $\mathbf{G}$ is a complete lattice, by using Proposition 2.6 we have the following.

1. $(\bigcup_i a_i) \cdot c = \bigcup_i (a_i \cdot c)$,
2. $a \to (\bigcap_i c_i) = \bigcap_i (a \to c_i)$.

There are many residuated structures related to logic and universal algebra. We have already mentioned Heyting algebras in the above. A *pocrim* is a (partially) ordered commutative residuated groupoid which is also an *integral* monoid, i.e. a monoid whose unit is the greatest element. If a pocrim has moreover a lattice structure, it is called a *residuated lattice*. Pocrims and residuated lattices are algebraic structures for (the implication-fusion fragment) of logics without contraction rule (more precisely, logics over $\mathbf{FL_{ew}}$). See e.g. [2, 6] for the details. Note here that in the definition of "residuated lattices" in

[3] neither the commutativity nor the integrality are assumed and therefore they are related to extensions of Lambek calculus.

In the folllowing, we will take two more examples of residuated structures from logic.

1. Commutative quantales A structure $\mathbf{A} = \langle A, \cap, \cup, \cdot \rangle$ is a *commutative quantale* if

1. $\langle A, \cap, \cup \rangle$ is a complete lattice (and hence is bounded),
2. $\langle A, \cdot \rangle$ is a commutative semigroup,
3. $(\bigcup_i x_i) \cdot y = \bigcup_i (x_i \cdot y)$ for $x_i, y \in A$.

By Proposition 2.6, $\mathbf{A}$ is residuated. If $\langle A, \cdot \rangle$ has moreover the unit element, $\mathbf{A}$ becomes an *intuitionistic phase structure*. See [5].

2. Triangular norms in fuzzy logic A *triangular norm* (or, simply, a t-norm) T is a map from $[0,1]^2$ to $[0,1]$ which satisfies the following conditions:

1. $T(x, T(y,z)) = T(T(x,y), z)$,
2. $T(x,y) = T(y,x)$,
3. $T(x,1) = x$,
4. $x \leq y$ implies $T(x,z) \leq T(y,z)$.

Define $*$ by $x * y = T(x,y)$. The above conditions mean that $\langle [0,1], *, 1 \rangle$ is a partially-ordered commutative integral monoid. It is obvious that $\langle [0,1], \min, \max, 0, 1 \rangle$ forms a complete lattice. Typical examples of t-norms are $\min\{x,y\}$ (*Heyting*), $x \times y$ (*product*) and $\max\{x+y-1, 0\}$ (*Łukasiewicz*).

Now, a question is when the above structure becomes residuated. By Proposition 2.6, it is residuated if and only if

$$T(\bigcup_i x_i, y) = \bigcup_i T(x_i, y).$$

This condition is equivalent to the following. For x, y

$$T(x,y) = \lim_{z \to x-0} T(z,y),$$

i.e. T is *left-continuous*. Note that any of t-norms given above is left-continuous.

Thus, we can say that every left-continuous t-norm determines a residuated lattice and therefore determines a substructural logic over $\mathbf{FL_{ew}}$.

4 Concluding remark: why and how far the framework of substructural logics works well

Ten years have passed after a study of nonclassical logics under the name "substructural logics" started. The study seems to be quite promising, as so many different kinds of nonclassical logics introduced so far can be discussed within this framework. On the other hand, it seems to be not clear yet why it works so well, or more precisely why restrictions on "structural rules" play key roles in it. Here we will give an explanation of it by using the notion of residuation.

One will agree that the "implication" is the most important logical connective. At the same time, it is not so easy to analyze its functions and meanings from both mathematical and philosophical points of view. Suppose here that our implication satisfies the following relation in a sequent system of a given logic $\mathbf{L}$ under consideration:

$$\vdash \Gamma, A \Longrightarrow B \text{ if and only if } \vdash \Gamma \Longrightarrow A \to B,$$

where $\vdash$ means the provability in the logic **L**. The above relation holds when we have both *introduction rule* and *elimination rule* for $\rightarrow$. Now, we introduce a new logical connective $*$, called *fusion*, to represent commas (in the left-hand side) of sequents. Then, the above relation means that implication is the residual of fusion. From the mathematical point of view, fusion may be much more manageable than implication. In such a case, we can shift our attention from implication to fusion, since residuation theory tells us that each fusion determines uniquely an implication and vice versa.

It is easy to see that some of basic properties of fusions (as groupoid operations) can be expressed by structural rules. For example, the commutativity of fusions is expressed by exchange rules. This might be the main but hidden reason why the study of substructural logics has begun with restrictions on structural rules.

So our conclusion is that the essense of substructural logics comes from residuation between fusion and implication, rather than from their structural rules.

Of course, the above explanation may oversimplifies the problem. For example, it is not clear for a given implicational logic whether or not it is possible to introduce fusion as a new connective, whose residual is exactly a given implication. As an example, consider the logic **BCK**. In this case, an appropriate fusion can be introduced. In fact, this is proved either by using the cut elimination theorem of a sequent system of **BCK** with the fusion, or by showing the embedding theorem of **BCK**-algebras into **BCK**-algebras with the fusion (see e.g. [2, 7] for the latter). In order to apply residuation theory, we need to make sure of the existence of such a fusion. But, we cannot assume this always. For instance, in [1] authors introduce the "basic logic" **B** and consider its extensions without the *deductive face*, i.e. logics in which the law of residuation does not hold. In algebraic terms, each algebraic structure for **B** is a (partially) ordered commutative monoid but without assuming the law of residuation. It might be necessary here to consider once again what is a proper definition of substructural logics.

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Alternatives to the relevant logic R

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We study two relevant logics ${}^{\cup}\mathbf{R}_{+}$ and **RNND**, which are proper extensions of the well-known relevant logic **R**. The logic ${}^{\cup}\mathbf{R}_{+}$ ("semilattice relevant logic") has connectives $\to$ (implication), $\wedge$ (conjunction), and $\vee$ (disjunction), and is characterized by semilattice models in which $\wedge$ and $\vee$ are interpreted in a natural way. The logic **RNND** ("Relevant Normal Natural Deduction") has connectives $\to$, $\wedge$, $\vee$, and $\neg$ (negation), and is formulated by a natural deduction system which is quite simple and easy to understand as compared with the usual natural deduction system of **R**. We try to show some nice properties of ${}^{\cup}\mathbf{R}_{+}$ and **RNND** — decidability, variable sharing, etc.

1 Semilattice Relevant Logic

In this note, the formulas of the relevant logic **R** are constructed by $\{\to, \wedge, \vee, \neg\}$. If c is a list of connectives in $\{\to, \wedge, \vee, \neg\}$, then $\mathcal{F}_c$ will denote the set of formulas whose connectives are in c, and $\mathbf{R}_c$ will denote the c-fragment of **R**.

A *semilattice model* is a 4-tuple $\langle I, \cdot, \mathbf{e}, \models\rangle$ that satisfies the following.

- $\langle I, \cdot, \mathbf{e}\rangle$ is an idempotent commutative monoid (a semilattice with identity $\mathbf{e}$). That is,

  I is a non-empty set, $\cdot$ is a binary operator on I, $\mathbf{e} \in I$.
  $(\alpha\cdot\beta)\cdot\gamma = \alpha\cdot(\beta\cdot\gamma)$, $\alpha\cdot\beta = \beta\cdot\alpha$, $\alpha\cdot\alpha = \alpha$, $\alpha\cdot\mathbf{e} = \alpha$.

- $\models$ is a relation on $I \times \mathcal{F}_{\to\wedge\vee}$ such that

  $\alpha \models A\to B \iff \forall\beta \in I[(\beta \models A) \Rightarrow (\alpha\cdot\beta \models B)]$,
  $\alpha \models A\wedge B \iff [(\alpha \models A) \text{ and } (\alpha \models B)]$,
  $\alpha \models A\vee B \iff [(\alpha \models A) \text{ or } (\alpha \models B)]$.

We say that a formula A is *valid in* this model if $\mathbf{e} \models A$. (The structure $\langle I, \cdot, \mathbf{e}\rangle$ is considered to be a "structure of information"—I is a set of pieces of information, $\cdot$ is a binary operator which combines two pieces of information, and $\mathbf{e}$ is an empty piece of information; and the notion $\alpha \models A$ is considered to be a fact "a formula A holds according to a piece α of information".)

The "semilattice relevant logic" ${}^{\cup}\mathbf{R}_{+}$ is defined to be the set

$$\{A \in \mathcal{F}_{\to\wedge\vee} \mid A \text{ is valid in any semilattice model}\}.$$

Proposition 1.1 ([6]) *${}^{\cup}\mathbf{R}_{+}$ is a proper superset of the set of provable formulas in $\mathbf{R}_{\to\wedge\vee}$ while the $\to$-fragment of ${}^{\cup}\mathbf{R}_{+}$ coincides with $\mathbf{R}_{\to}$. For example,*

$$[(A\to(B\vee C))\wedge(B\to D)]\to(A\to(D\vee C)) \in {}^{\cup}\mathbf{R}_{+}, \notin \mathbf{R}_{\to\wedge\vee}.$$

If α is a finite set of natural numbers and A is a formula, then the expression $\alpha{:}A$ is called a *labelled formula* (α is called a *label*). If Γ and Δ are finite multisets of labelled formulas, then the expression $\Gamma \Rightarrow \Delta$ is called a *labelled sequent*. A labelled sequent calculus $\mathcal{L}{}^{\cup}\mathbf{R}_{+}$ is defined as follows.

[Axioms]

$$\Gamma,\ \alpha{:}A \Rightarrow \alpha{:}A,\ \Delta$$

[Inference rules]

$$\frac{\Gamma,\ \alpha{:}A,\ \alpha{:}A \Rightarrow \Delta}{\Gamma,\ \alpha{:}A \Rightarrow \Delta}\ (\text{contraction left}) \qquad \frac{\Gamma \Rightarrow \alpha{:}A,\ \alpha{:}A,\ \Delta}{\Gamma \Rightarrow \alpha{:}A,\ \Delta}\ (\text{contraction right})$$

$$\frac{\Gamma \Rightarrow \beta{:}A,\ \Delta \quad \Gamma,\ \alpha{\cup}\beta{:}B \Rightarrow \Delta}{\Gamma,\ \alpha{:}A{\to}B \Rightarrow \Delta}\ (\to \text{left})$$

$$\frac{\Gamma,\ \{a\}{:}A \Rightarrow \alpha{\cup}\{a\}{:}B,\ \Delta}{\Gamma \Rightarrow \alpha{:}A{\to}B,\ \Delta}\ (\to \text{right}) \text{ with a condition:}$$

(Label Condition) *The natural number a does not appear in the labels of the lower sequent.*

$$\frac{\Gamma,\ \alpha{:}A,\ \alpha{:}B \Rightarrow \Delta}{\Gamma,\ \alpha{:}A{\wedge}B \Rightarrow \Delta}\ (\wedge \text{ left}) \qquad \frac{\Gamma \Rightarrow \alpha{:}A,\ \Delta \quad \Gamma \Rightarrow \alpha{:}B,\ \Delta}{\Gamma \Rightarrow \alpha{:}A{\wedge}B,\ \Delta}\ (\wedge \text{ right})$$

$$\frac{\Gamma,\ \alpha{:}A \Rightarrow \Delta \quad \Gamma,\ \alpha{:}B \Rightarrow \Delta}{\Gamma,\ \alpha{:}A{\vee}B \Rightarrow \Delta}\ (\vee \text{ left}) \qquad \frac{\Gamma \Rightarrow \alpha{:}A,\ \alpha{:}B,\ \Delta}{\Gamma \Rightarrow \alpha{:}A{\vee}B,\ \Delta}\ (\vee \text{ right})$$

Proposition 1.2 ([4](cf.[5])) *$A \in {}^{\cup}\mathbf{R}_+$ if and only if $\mathcal{L}^{\cup}\mathbf{R}_+ \vdash \Rightarrow \emptyset{:}A$.*

The system $\mathcal{L}^{\cup}\mathbf{R}'_+$ is obtained from $\mathcal{L}^{\cup}\mathbf{R}_+$ by replacing "$\to$ left" rule by

$$\frac{\Gamma \Rightarrow \beta{:}A \quad \Gamma,\ \alpha{\cup}\beta{:}B \Rightarrow \Delta}{\Gamma,\ \alpha{:}A{\to}B \Rightarrow \Delta.}\ (\to \text{left}')$$

We can (probably) prove:

Theorem 1.3 *$\mathcal{L}^{\cup}\mathbf{R}_+$ and $\mathcal{L}^{\cup}\mathbf{R}'_+$ are equivalent with respect to provable labelled sequents.*

By using this theorem, now I attack the following problems.

- Give a simple proof of the completeness of a Hilbert style axiomatization of ${}^{\cup}\mathbf{R}_+$. (A Hilbert style axiomatization of ${}^{\cup}\mathbf{R}_+$ was given by Charlwood ([2]), but the proof is somewhat complicated, and I cannot follow the detail.)
- Decision problem of ${}^{\cup}\mathbf{R}_+$. (I try to make a decision procedure for $\mathbf{R}_\to$ using the $\to$-fragment of $\mathcal{L}^{\cup}\mathbf{R}_+$. This and the above Theorem 1.3 may offer a decision procedure for ${}^{\cup}\mathbf{R}_+$.)
- Prove (or disprove) the variable sharing (and the interpolation) property for ${}^{\cup}\mathbf{R}_+$.

Remark

- A decision procedure for "contractionless ${}^{\cup}\mathbf{R}_+$" was given by Giambrone and Kron ([3]).
- Giambrone and Urquhart ([4]) introduced a natural deduction system and "proved" that the system is equivalent to $\mathcal{L}^{\cup}\mathbf{R}_+$. However, the proof seems to be wrong—$(A{\vee}B)_a \vdash A_a, B_a$ is (probably) a counterexample to Fact 6 in [4].

2 Relevant Normal Natural Deduction

The natural deduction system of $\mathbf{R}_\to$ is quite simple:

$$\frac{A{\to}B \quad A}{B}\ (\to\text{E}) \qquad \frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A{\to}B}\ (\to\text{I}) \text{ with a condition:}$$

(Relevant Condition) *We must cancel at least one occurrence of the assumption A.*

But introducing other connectives makes the situation problematic. For example, an irrelevant formula $B \to (A \to A)$ is provable by the proof

$$\dfrac{\dfrac{\dfrac{[A]^1 \quad [B]^2}{A \wedge B}}{A}\,1}{A \to A}\,2$$

which satisfies the Relevant Condition. To avoid such a situation, Yoshiura and Yonezaki ([7]) introduced a natural deduction system in which only "normal proofs" are permitted. This is a nice idea, but Yoshiura and Yonezaki's system is not elegant in the treatment of negation. Here we combine Yoshiura and Yonezaki's idea with Andou's "regular and normal proofs" ([1]), and we make an elegant natural deduction system of a relevant logic.

Remark

- The above proof of $B \to (A \to A)$ is not normal, and the normal form of it violates the Relevant Condition.
- Usually natural deduction systems of **R** have a restriction on $\wedge$I rule—*two premises must have the same assumption set.* This excludes the above proof of $B \to (A \to A)$.

Andou's regular NK for $\mathcal{F}_{\to \wedge \vee \neg}$

$$\dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \to B}\ (\to\text{I}) \qquad \dfrac{A \to B \quad A}{B}\ (\to\text{E}) \qquad \dfrac{A \quad B}{A \wedge B}\ (\wedge\text{I}) \qquad \dfrac{A \wedge B}{A}\ (\wedge\text{E}) \qquad \dfrac{A \wedge B}{B}\ (\wedge\text{E})$$

$$\dfrac{A}{A \vee B}\ (\vee\text{I}) \qquad \dfrac{B}{A \vee B}\ (\vee\text{I}) \qquad \dfrac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\ (\vee\text{E}) \qquad \dfrac{\begin{array}{c}[A]\\ \vdots \\ \bot\end{array}}{\neg A}\ (\neg\text{I}) \qquad \dfrac{\neg A \quad A}{\bot}\ (\neg\text{E})$$

$$\dfrac{\dfrac{\dfrac{[\neg A] \quad \begin{array}{c}\vdots \\ A\end{array}}{\bot}}{\begin{array}{c}\vdots \\ \bot\end{array}}}{A}\ (\bot\text{C})$$

The $\bot$C rule means that *an assumption which is cancelled by $\bot C$ must be the major premise of a $\neg E$.*

Let $\langle X_1, \cdots, X_n \rangle$ be a path in a proof of regular **NK** (X_1 is an assumption and X_n is the last formula), and $\langle Y_1, \cdots, Y_m \rangle$ be a subsequence of it (i.e., $\langle Y_1, \cdots, Y_m \rangle = \langle X_{i_1}, \cdots, X_{i_m} \rangle$ where $i_1 < \cdots < i_m$). We say that the sequence $\langle Y_1, \cdots, Y_m \rangle$ is *redundant* if it satisfies the following conditions.

- Y_1 is the conclusion of an "introduction rule" ($\to$I, $\wedge$I, $\vee$I, or $\neg$I).
- The position of Y_k and Y_{k+1} is one of the following.

$$\dfrac{\begin{array}{c}\vdots \\ A \vee B\end{array} \quad \begin{array}{c}[A]\\ \vdots \\ Y_k\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{\begin{array}{c}Y_{k+1}\\ \vdots\end{array}}\ (\vee\text{E}) \qquad \dfrac{\begin{array}{c}\vdots \\ A \vee B\end{array} \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ Y_k\end{array}}{\begin{array}{c}Y_{k+1}\\ \vdots\end{array}}\ (\vee\text{E}) \qquad \dfrac{\dfrac{[\neg A] \quad \begin{array}{c}\vdots \\ Y_k\end{array}}{\begin{array}{c}\vdots \\ \bot\end{array}}}{\begin{array}{c}Y_{k+1}\\ \vdots\end{array}}\ (\bot\text{C})$$

- Y_m is the major premise of an "elimination rule" ($\to$E, $\wedge$E, $\vee$E, or $\neg$E).

A proof is said to be *normal* if it contains no redundant sequence, and to be *relevant* if the Relevant Condition on cancelling assumptions holds (i.e., in the applications of the rules $\to$I, $\vee$E, $\neg$I, and $\bot$C, there must be an occurrence of each cancelled assumption indicated by $[A]$, $[B]$, and $[\neg A]$ in the above list of the rules).

The logic **RNND** (Relevant Normal Natural Deduction) is defined by relevant normal proofs; that is, a formula A is said to be a theorem of **RNND** if *there is a regular* **NK** *proof of* A *which is relevant and normal.*

Now I attack the problem to show the following nice properties of **RNND**. (For Yoshiura and Yonezaki's system, these have been shown by [7], [8] and [9].)

- Variable sharing (and interpolation).
- Decidability.
- **RNND** is strictly stronger than **R**. (There are some "natural" formulas which are provable in **RNND** and not provable in **R**; e.g., $A\to(B\to(A\wedge B))$. On the other hand, I do not know whether the following holds: *If* $\mathbf{R}\vdash A$ *then* $\mathbf{RNND}\vdash A$. This is a nontrivial problem because **RNND** is not closed under modus ponens.)

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A Kripke Type Semantics for Substructural Logics with Strong Negation (Abstract)

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Abstract

We introduce a Kripke type semantics for propositional substructural logics with strong negation, and show completeness theorems for these logics by using extended Ishihara's canonocal model construction method. The present paper's framework can deal with broard range of substructural logics with strong negation including Nelson's logic N^-, Wansing's logic COSPL (and its extensions), extended Visser's basic propositional logic with strong negation and extended positive relevant logics with strong negation.

1 Introduction

First, we will present a short servey of Kripke type semantics for substructural logics and logics with strong negation. Second, we will show the organization of this paper.

Examples of substructural logics discussed in this paper are as follows: positive relevant logics in philosophical motivation, intuitionistic linear logic in computer science and BCK-logic closely related to BCK-algebras. A historical servey of Kripke type semantics for these logics is as follows. In 1972, Routley and Meyer introduced a Kripke type semantics for positive relevant logics, which semantics was characterized with three-termed accessibility relation. (See [2]). In 1985, Ono and Komori gave us a Kripke type semantics for FL-family including BCK-logic, intuitionistic linear logic and full Lambek logic FL, which semantics was characterized with semilattice-orderd monoid. (see [10]). In 1988, Došen introduced a Kripke type semantics, named groupoid semantics, for broard range of substructural logics including the logics mentioned above (see [4][5]). Došen's semantics is essentially the same as Ono and Komori's one, but the semantics was independently proposed. In 2000, Ishihara introduced a canonical model construction method for the semantics of Ono and Komori (see [6]). This method can deal with broard range of substructural logics including some Došen's logics, Corsi's logic F and the standard substructural logics mentioned above. A class of the logics characterized by this method is called, by Ishihara, *normal substructural logic*.

Next we explain a servey of logics with strong negation (or constructive negation) and their Kripke type semantics. The strong negation was first proposed by Nelson (1949) and Markov (1950) independently. The negation was introduced to express "constructive falsity" in recursive realizability (see [8]). Nelson's logics such as N and N^- have been studied by many logicians (see, e.g., [1] [14]). These logics are, roughly speaking, slight extensions or modifications of the intuitionistic logic. Kripke type semantics for Nelson's logics are different from that for the intuitionistic logic. These semantics have two kinds of valuation: $\models^+$ (correspond to provability) and $\models^-$ (correspond to refutability). In 1969, Thomason introduced a Kripke type semantics for a predicate logic with strong negation, which semantics was characterized with binary accessibility relation similar to the binary relation of Kripke type semantics for the predicate intuitionistic logic (see [12]). In 1993, Wansing proposed propositional substructural logics with strong negation such as COSPL-family including some Nelson's logics (roughly speaking, COSPL-family is obtained from FL-family by adding strong negation). Wansing gave us a Kripke type semantics and its informational interpretation for these logics (see [14]). Wansing's semantics is a natural extention of Došen's one.

In the present paper, we extend Ishihara's result to substructural logics with strong negation. As a result, we can deal with broad range of logics with strong negation such as Nelson's N^- and Wansing's COSPL (and its extensions). Thus, the result is also an extension of Wansing's one.

Next we present the orgnization of the paper.

In Section 2, we present Ishihara's basic normal substructural logic L and its extensions, and introduce new normal substructural logics such as Visser's basic propositional logic BPL in [13] (a Hilbert style system for BPL is proposed in [9] [11]), relevant logics RMO^+, EMO^+, Corsi's logics FT, FR, FRT in [3] and Méndez's logics T°_+, E°_+, R°_+, RMO°_+, EMO°_+, $S4^\circ_+$ in [7]. (We can prove the compleateness theorem for the new logics by using Ishihara's canonical model construction method.) Next we introduce extended basic normal substructural logic with strong negation $L^\sim$ and its extensions. These extensions include Nelson's logic N^- and Wansing's logic COSPL (and its extensions). Further we define a new family of normal substructural logics with strong negation. The new logics are as follows: extended Došen's logic $DE^\sim$, extended positive relevant logics $B^\sim$, $T^\sim$, $E^\sim$, $R^\sim$, $RMO^\sim$, $EMO^\sim$, extended strict implication $S4^\sim$, extended Corsi's logics $F^\sim$, $FT^\sim$, $FR^\sim$ $FRT^\sim$, extended Visser's logic $BPL^\sim$, and extended Méndez's logics $T^\sim_\circ$, $E^\sim_\circ$, $R^\sim_\circ$, $EMO^\sim_\circ$, $RMO^\sim_\circ$, $S4^\sim_\circ$.

In Section 3, we introduce a Kripke type semantics for the normal substructural logics with strong negation mentioned above and show soundness theorem and a correspondence between axiom schemes and conditions on frames.

In Section 4, we prove the completeness theorems for the logics by using extended Ishihara's canonical model construction method. In the sequel, we can show the completeness theorem for any logic obtained from $L^\sim$ by adding arbitrary combinations of the axiom schemes introduced in Section 2.

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Kripke completeness of predicate extensions of cofinal subframe logics

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1 The problem.

In the study of non-classical logics, it is a natural and fundamental question to ask the completeness of predicate extensions of Kripke complete propositional logics.

For the subframe logics containing **S4**, we have the following characterization for the completeness of the predicate extensions with constant domains.

Theorem 1 (Shimura [5], Tanaka-Ono [7], Chagrov-Zakharyashev [1]) *Let* $\mathbf{L}$ *be a subframe logic containing* **S4**. *The following are equivalent:*

- $\mathbf{L}$ *is canonical.*
- $\mathbf{L}$ *has the finite embedding property.*
- $\mathbf{Q}$-$\mathbf{L} + \mathrm{BF}$ *is Kripke complete.*

In particular, we have the following corollary.

Corollary 1 (Shimura [5], cf. Shimura [3]) (1). *Let* $\{\mathbf{M}_i\}$ *be a set of finite and rooted Kripke frames without non-trivial clusters.*

Then, $\mathbf{QS4} + \{\alpha(\mathbf{M}_i)\} + \mathrm{BF}$ *is Kripke complete.*

(2). *Let* $\mathbf{M}$ *be a finite and rooted Kripke frame with a non-trivial cluster.*

Then, $\mathbf{QS4} + \alpha(\mathbf{M}) + \mathrm{BF}$ *is Kripke incomplete.*

If we consider a similar characterization problem for cofinal subframe logics, it is natural to settle the following question. (The necessity of axioms like $T(K)$ is discussed in the talk [5]).

Question 1 *Find a necessary and sufficient condition for the Kripke completeness of* $\mathbf{QS4} + \{\alpha(\mathbf{M}_i, \bot)\} + T(K) + \mathrm{BF}$.

Note that the canonicity of $\mathbf{L}$ is not sufficient for the completeness by the result of Skvortsov [6]. So, the answer of the above question seems much difficult than the subframe case.

Although we do not have the answer, we give a partial answer corresponding to Corollary 1.

Theorem 2 *We consider the following conditions for finite and rooted* (**S4**-) *Kripke frame.*

(C1) *There is no non-trivial inner clusters.*

(C2) *The set of the inner points has a common upper bound c such that $\{c\}$ is a trivial cluster.*

Then, we have the following.

(1). *Let $\{\mathbf{M}_i\}$ be a set of finite and rooted Kripke frames satisfying above conditions. Then, $\mathbf{QS4} + \{\alpha(\mathbf{M}_i, \bot)\} + T(K) + \mathrm{BF}$ is Kripke complete.*

(2). *Let $\mathbf{M}$ be a finite and rooted Kripke frame. If $\mathbb{M}$ does not satisfy one of the above conditions, $\mathbf{QS4} + \alpha(\mathbf{M}, \bot) + T(K) + \mathrm{BF}$ is Kripke incomplete.*

2 The sketch of the proof of Theorem 2 (1).

To axiomatize the cofinal subframe logics, we use a variant $\neg C_{\mathbf{M}_i}$ of Fine's subframe formula $\neg B_{\mathbf{M}_i}$ and frame formula $\neg F_{\mathbf{M}_i}$, where $B_{\mathbf{M}_i}$, $F_{\mathbf{M}_i}$ and $C_{\mathbf{M}_i}$ are the following:

$$B_{\mathbf{M}_i} = p_r \ (\ r \text{ is the root of } \mathbf{M}_i) \wedge \bigwedge_{a \neq b} \Box\neg(p_a \wedge p_b)$$
$$\wedge \bigwedge_{aRb} \Box(p_a \supset \Diamond p_b) \wedge \bigwedge_{a \not R b} \Box(p_a \supset \neg\Diamond p_b)$$

$$F_{\mathbf{M}_i} = \Box(\bigvee_{a \in M_i} p_a) \wedge B_{\mathbf{M}_i},$$

$$C_{\mathbf{M}_i} = \Box\Diamond(\bigvee_{a \in M_i} p_a) \wedge B_{\mathbf{M}_i}.$$

The main tool of the completeness proof is the same as in [3], that is, the method of canonical models. Let $\mathbf{N}$ be the canonical frame for $\mathbf{QS4} + \{\alpha(\mathbf{M}_i, \bot)\} + T(K) + \mathrm{BF}$. We show a contradiction from the assumption that there is a cofinal subreduction from $\mathbf{N}$ to some $\mathbf{M}_i$. Let $C' = \{c\}$ be a maximal cluster of $\mathbf{M}_i$ such that c is a common upper bound of the inner points.

We show that there is a 'definable' cofinal subreduction from $\mathbf{N}$ to $\mathbf{M}_i$. A function h from a Kripke model to a Kripke frame $\mathbf{M}_i$ is said to be difinable if there are formulas F_a $(a \in M_i)$ such that

$$u \in h^{-1}(a) \Leftrightarrow u \models F_a.$$

The difficulty arises from the fact that the family $\{u|u \models F_a\}$ (a is maximal) must forms a partition of the maximal points of $\mathbf{N}$. To avoid this difficulty, we construct the map in two steps. Firstly, we construct a disjoint family $\{u|u \models F_a\}$ (a is maximal, $a \neq c$). Secondly, we define the map so that the rest maximal points are mapped to c. By virtue of the condition (C2), the map constructed turns to be a cofinal subreduction.

Lemma 1 *There is a finite subframe $\mathbf{N}'$ of $\mathbf{N}$ with the following conditions.*

- *There is a p-morphism g from $\mathbf{N}'$ to $\mathbf{M}_i$.*
- *Each maximal cluster of $\mathbf{N}'$ is contained by some maximal cluster of $\mathbf{N}$.*

Proof) This is because of the conditions that $T(K)$ is valid $\mathbf{N}$ and $\mathbf{M}_i$ contains no non-trivial inner clusters. ■

Lemma 2 (1). *For each $a \in N'$, there exists a formula F_a satisfying the following:*

$$a \models F_a, \quad a \not\models F_b \ (b \neq a).$$

(2). *For each maximal cluster C of $\mathbf{N}'$, there exists a formula G_C satisfying the following:*

$$a \models \Box G_C \ (a \in C), \quad a \not\models \Diamond G_C \ (a \text{ is maximal}, a \notin C).$$

(3). *For each inner point $a \in N'$, there exists a formula H_a satisfying the following:*

$$a \not\models \Box\Diamond\Box H_a \supset H_a.$$

Definition 1

$$H = \bigwedge \Box(\Box\Diamond\Box H_a \supset H_a)$$

$$P_a = \begin{cases} H \wedge \Box G_C \wedge \bigwedge_{b \in C} \Box\Diamond F_b \wedge F_a & (a \text{ is maximal}, a \neq c, a \in C) \\ H \wedge \bigwedge_{b \neq c, b \text{ is maximal}} \Box\neg P_b & (a = c) \\ \neg(\Box\Diamond\Box H_a \supset H_a) \wedge F_a & \\ \qquad \wedge \bigwedge_{aR\not b} \Box\neg F_b \wedge \bigwedge_{b \neq a} \neg F_b \wedge \bigwedge_{aRb} \Diamond P_b & (a \text{ is an inner point.}) \end{cases}$$

Lemma 3 (1). *$N_0 = \{u \in N | u \models P_a$ for some $a\}$ is a cofinal subframe of $\mathbf{N}$.*

(2). *For each $u \in N_0$, there exists exactly one $a \in N'$ with $u \models P_a$. The map f defined by $f(u) = g(a)$ is a cofinal subreduction from $\mathbf{N}$ to $\mathbf{M}_i$.*

(3). *$u \models P_0$ implies $u \models C$ for some instance C of $C_{\mathbf{M}_i}$.*

3 The outline of the proof of Theorem 2 (2).

We treat the case dviding into three cases.

Lemma 4 *If* $\mathbf{M}$ *does not satisfy the condition* (C1), $\mathbf{QS4} + \alpha(\mathbf{M}, \bot) + T(K) + \mathrm{BF}$ *is Kripke incomplete.*

Proof) The incompleteness proof for $\mathbf{QS4} + \alpha(\mathbf{M}) + \mathrm{BF}$ works in this case. ■

Lemma 5 *If the set of the inner points of* $\mathbf{M}$ *does not have a common upper bound,* $\mathbf{QS4} + \alpha(\mathbf{M}, \bot) + T(K) + \mathrm{BF}$ *is Kripke incomplete.*

Lemma 6 *If the inner points of* $\mathbf{M}$ *has a common upper bound but no trivial clusters are u* $\mathbf{QS4} + \alpha(\mathbf{M}, \bot) + T(K) + \mathrm{BF}$ *is Kripke incomplete.*

Since the proofs of Lemma 5, 6 are quite complicated, we show the outline usinge typical examples.

The example for Lemma 6 is treated by Skvortsov [6] for intermediate predicate logics. But, his proof seems difficult to generalize since he uses Kripke bundle semantics.

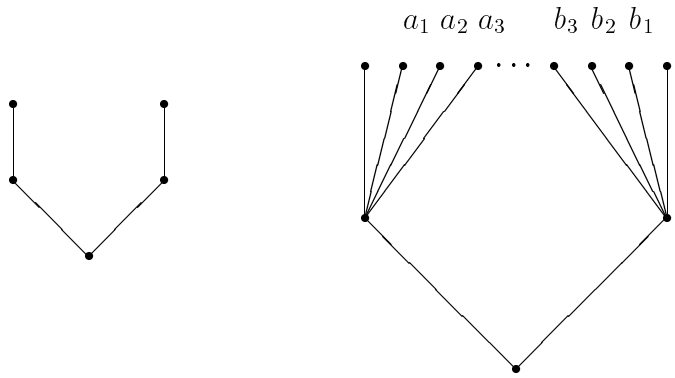
We give a proof using a Kripke model. For each formula, the model behaves as a finite model, but it behaves as an infinite model for the whole formulas. The idea is described in Hughes-Cresswell [2].

Example 1 *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be the following frames. There exist a formula A and a model* $\langle \mathbf{N}, \omega, \models \rangle$ *with the following conditions.*

(1). *A is not separable from* $\mathbf{L} = \mathbf{QS4} + \alpha(\mathbf{M}, \bot) + T(K) + \mathrm{BF}$ *by Kripke frame semantics.*

(2). *Every instance of* $\mathbf{L}$ *is true in* $\langle \mathbf{N}, \models \rangle$.

(3). *A is refuted in* $\langle \mathbf{N}, \models \rangle$.



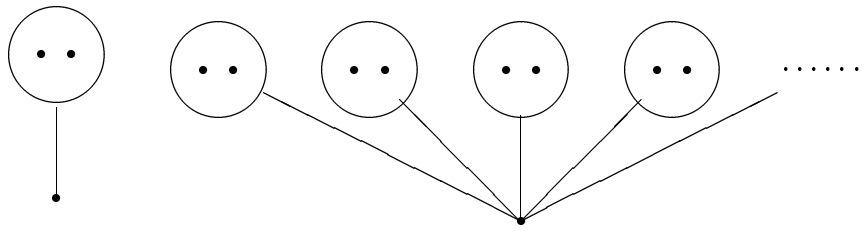
For each language with finitely many constants, some a_n and b_n has completely same behaver under the valuation $\models$.

Example 2 *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be the following frames. There exist a formula A and a model* $\langle \mathbf{N}, \omega, \models \rangle$ *with the following conditions.*

(1). *A is not separable from* $\mathbf{L} = \mathbf{QS4}+\alpha(\mathbf{M}, \bot)+T(K)+\mathrm{BF}$ *by Kripke frame semantics.*

(2). *Every instance of* $\mathbf{L}$ *is true in* $\langle \mathbf{N}, \models \rangle$.

(3). *A is refuted in* $\langle \mathbf{N}, \models \rangle$.



For each language with finitely many constant symboles, some cluster behaves as a trivial cluster under $\models$.

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On Komori Algebras

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Humberstone([2]) has defined the notion of Komori algebras and proved a generalization of Lemma 2 of [3]. In this note, we will define Komori Algebras as a broader class of algebras than Komori algebras by Humberstone. And we will make the result of Humberstone more general.

We begin with the definition of Komori algebras. A *Komori algebra* is an algebra $\langle A, \to, 1\rangle$ of type $\langle 2, 0\rangle$ in which a quasi-identity (Ko) are satisfied:

$$\text{(Ko)} \quad \text{if } x \to y = y \to x = 1, \text{ then } x = y.$$

Henceforth Komori algebras defined by Humberstone are called Humberstone algebras. A Humberstone algebras is a Komori algebras which is satisfying the following three identities (M),(I) and (K):

$$\begin{array}{lrcl} \text{(M)} & 1 \to x & = & x \\ \text{(I)} & x \to x & = & 1 \\ \text{(K)} & x \to 1 & = & 1. \end{array}$$

A BIK^+-algebra is a Humberstone algebra which is satisfying the following identity (B):

$$\text{(B)} \quad (y \to z) \to (x \to y) \to x \to z = 1.$$

A BIK^+-algebra had been called a BCC-algebra in [3]. In that paper, we proved that the quasivariety of all BIK^+-algebras is not a variety. A BCK-algebra ia a BIK^+-algebra which is satisfying the following identity (C):

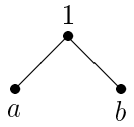
$$\text{(C)} \quad x \to (y \to z) = y \to (x \to z).$$

Wronski([5]) proved that the quasivarieties of all BCK-algebras is not a variety.

Now we define two finite algebras **NK** and **K4**. The algebra **NK** is closely related to the partition of the set of terms constructed out of the variables x, y (and the constant 1) defined in [3]. The algebras **NK** and **K4** are defined by Hasse diagrams Figure 1 and Figure 2 with the definition of a binary operation $\to$, respectively. The algebra **NK** is not a Komori algebra. The algebra **K4** is a BIK^+-algebra, but not a BCK-algebra because $c \to (a \to b) = 1 \neq a \to (c \to b) = c$. Let f be a function from **K4** onto **NK** such that $f(1) = f(c) = 1, f(a) = a$ and $f(b) = b$. Then f is a homomorphism from **K4** onto **NK**. So we have the following.

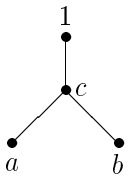
Theorem 1 *If a class K of Komori algebras contains* **K4**, *then K is not a variety.*

Corollary 2 *The class of all BIK^+-algebras is not a variety.*



$$\begin{cases} 1 \to x = x, \\ x \to y = 1 \text{ if } x \leq y, \\ a \to b = b \to a = 1. \end{cases}$$

Figure 1: **NK**



$$\begin{cases} 1 \to x = x, \\ x \to y = 1 \text{ if } x \leq y, \\ a \to b = b \to a = c, \\ c \to a = a, \quad c \to b = b. \end{cases}$$

Figure 2: **K4**

Let **WR** be the algebra defined by Wronski to prove that the class of all BCK-algebras is not a variety.

Theorem 3 *If a class K of Komori algebras contains* **WR**, *then K is not a variety.*

Corollary 4 *The class of all BCK-algebras is not a variety.*

The above corollaries have been proved in [3] and [5], respectively. Corollary 2 follows from Corollary 4 becase a class K of Komori algebras is not a variety if $Id(K) \subseteq \Sigma$ and $\{\mathbf{A} | \Sigma \subseteq Id(\mathbf{A})$ and $\mathbf{A}$ is a Komori algebra$\}$ is not a variety, in general. Here $Id(K)$ is the set of identities holding in all algebras in K and $Id(\mathbf{A}) = Id(\{\mathbf{A}\})$. Theorem 1 and Theorem 3 are mutually independent becasue $x \to (x \to y) = x \to (x \to (x \to y)) \in Id(\mathbf{K4})$ and the identity $\notin Id(\mathbf{WR})$. We call such algebras in the above theorems criterions.

Definition 5 A Komori algebra $\mathbf{A}$ is a *criterion* if any class of Komori algebras containing $\mathbf{A}$ is not variety.

For any Komori algebras $\mathbf{A}$ and $\mathbf{B}$, if $Id(\mathbf{A}) \subseteq Id(\mathbf{B})$ and $\mathbf{B}$ is a criterion, then $\mathbf{A}$ is a criterion. For any criterion $\mathbf{A}$, by Zorn's lemma, there exists a criterion $\mathbf{B}$ such that $Id(\mathbf{A}) \subseteq Id(\mathbf{B})$ and $\mathbf{C}$ is not a criterion if $Id(\mathbf{B}) \subsetneq Id(\mathbf{C})$. We call such criterion $\mathbf{B}$ maximal.

Conjecture 6 *The Komori algebras* **K4** *and* **WR** *are both maximal, that is, any Komori algebra $\mathbf{A}$ is not a criterion if $Id(\mathbf{K4}) \subsetneq Id(\mathbf{A})$ or $Id(\mathbf{WR}) \subsetneq Id(\mathbf{A})$.*

Now we describe a partition in [3] of the set of terms constructed out of the variables x, y (and the constant 1), whose blocks are called $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{1}$. The partition in question is defined by induction on the construction of these x, y-terms:

1. $x \in \mathbf{X}$, $y \in \mathbf{Y}$, and $1 \in \mathbf{1}$,
2. if $s \in \mathbf{X}$ or $s \in \mathbf{Y}$, then $s \to t \in \mathbf{1}$,

3. if $s \in \mathbf{1}$, then $s \to t$ belongs to the block which t belongs to.

Let $\mathbf{XY}$ be the set of identities $\{s = t | s \in \mathbf{X} \text{ and } t \in \mathbf{Y}\}$.

Lemma 7 *Let K be a class of Komori algebras. If x, y-terms s_1 and s_2 lie in the different block of the partition $\{\mathbf{X}, \mathbf{Y}, \mathbf{1}\}$ and $s_1 = s_2 \in Id(K)$, then*

$$Id(K) \cap \mathbf{XY} \neq \emptyset \ .$$

Let K be a class of Komori algebras and $\mathbf{F}_K(2)$ be the K-free algebra on two generators(x and y). Suppose that $Id(\mathbf{K}) \cap \mathbf{XY} = \emptyset$. Let f be a function from $\mathbf{F}_K(2)$ onto $\mathbf{NK}$ defined below:

$$f([t]) = \begin{cases} 1 & \text{if t } \in \mathbf{1}, \\ a & \text{if t } \in \mathbf{X}, \\ b & \text{if t } \in \mathbf{Y}. \end{cases}$$

By Lemma 7, the function f is well-defined. Remark that Lemma 7 and $Id(K) \cap \mathbf{XY} = \emptyset$ are used only for establishing the well-definedness of the function f. The function f is a homomorphism from $\mathbf{F}_K(2)$ onto $\mathbf{NK}$. So the algebra $\mathbf{F}_K(2)$ is a criterion.

Theorem 8 *Let K be a class of Komori algebras. If $Id(K) \cap \mathbf{XY} = \emptyset$, then the class K is not a variety.*

The above theorem is a generalization of Lemma 2 of [3] and Proposition 3 of [2]. Remark that a identity in the set $\mathbf{XY}$ is not always a Komori identity (in the sence of [1]) for the quasi-variety of all Komori algebras though the set $\mathbf{XY}$ is a complete set of Komori identies for the quasi-variety of all Humberstone algebras.

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An Extension of Martin-Löf's Type Theory with an Evolution Relation

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In this note, we will extend Martin-Löf's type theory by introducing an evolution relation between types and also expressions (similar discussions are found in [1]). The purpose is to formalize the software evolution problem proposed by T. Katayama in [2] under the constructive type theory [3,4].

1 Introduction

As a method of maintaining and/or producing a large scale of software rapidly, the software evolution problem was proposed by T. Katayama in [2]. Here *software evolution* means to the mechanism that software can change its structure and functions to adapt changes of its requirements and environment.

Let $\mathbb{S} = \langle \mathcal{S}, \sqsubseteq \rangle$ and $\mathbb{P} = \langle \mathcal{P}, \sqsubseteq \rangle$ be the structures of specification and program, respectively. Then in general, *software evolution problem* is defined as finding a new program P' of a new specification S', in symbols, $P' \vdash S'$ under the assumption that the program P of S is given, $P \vdash S$ and the specification S changes to S', in symbols $S \sqsubseteq S'$. Here, if the change $S \sqsubseteq S'$ is arbitrary, then we should make P' newly and software evolution problem does not arise. In [2], there are explanations of two typical evolutional patterns:

(1) $P \vdash S$ and $S \sqsubseteq S'$: if we calculate the difference of specifications S' and S, in symbols, $\Delta S = \mathrm{Diff}(S', S)$ and derive the program ΔP corresponding to this difference ΔS, $\Delta P \vdash \Delta S$ then the desired program P' is obtained by merging two programs P and ΔP, in symbols, $P' = \mathrm{Sup}(P, \Delta P)$.

(2) $P \vdash S$, $T \sqsubseteq S$ and $T \sqsubseteq S'$: if $\Delta S = \mathrm{Diff}(S', T)$, $\Delta P \vdash \Delta S$ and derive the program Q of T by degenerating the original program P with ΔP, in symbols, $Q = \mathrm{Diff}(P, \Delta P)$ then the desired program P' is obtained by applying pattern (1).

2 Martin-Löf's Type Theory

In this section we will introduce Martin-Löf's type theory (**MTT** in short) [3,4]. Since **MTT** is one of the predicative type theory, in general **MTT** has a universe type $\mathbf{U}_n$ ($n = 0, 1, 2, \ldots$). But in this investigation we will omit to include universes type $\mathbf{U}_n$ (and also well orderings type $\mathbf{W}$) for the sake of simplicity. **MTT** has its two major syntactical categories, that is **MTT**-type and **MTT**-expression. Here the **MTT**-types are words over the alphabet and Greek letters such that (1) constants : finite set type $\mathbf{N}_n$ ($n = 0, 1, \ldots$) and natural number set type $\mathbf{N}$ (here we abbreviate empty set type as $\bot$ ($= \emptyset = \mathbf{N}_0$) and singleton set type as $\top$ ($= \mathbf{N}_1$)), (2) symbols : type constructors $\Pi, \Sigma, +, \mathrm{I}$, types $A, B, C, \ldots$, objects $a, b, c, \ldots$, object variables $x, y, z, \ldots$, parentheses $(,)$ and membership relation $\in$.

Definition 2.1 *The set of* **MTT***-types* $\mathcal{T}$ *is defined inductively as follows:*

(i) *all constants* $\mathbf{N}_n$ ($n = 0, 1, \ldots$), $\mathbf{N}$, $\bot$ *and* $\top$ *are* **MTT***-types* (*called* atomic types),

(ii) *if* A *and* B *are any* **MTT***-types and* $x \in A$ *is any object variable, then* $(\Pi x \in A)B(x)$ *is a* **MTT***-type* (*called* dependent product type),

(iii) *if* A *and* B *are any* **MTT***-types and* $x \in A$ *is any object variable, then* $(\Sigma x \in A)B(x)$ *is a* **MTT***-type* (*called* dependent sum type),

(iv) *if* A *and* B *are any* **MTT***-types, then* $A + B$ *is a* **MTT***-type* (*called* direct sum type),

(v) *if* A *is any* **MTT***-type and* $a, b \in A$ *are any objects, then* $\mathrm{I}(A, a, b)$ *is a* **MTT***-type* (*called* propositional equality type).

On the other hand, the **MTT**-expressions are words over the alphabet and Greek letters such that (1) constant : elements of finite set m_n $(n = 0, 1, \ldots ; m = 0, 1, \ldots, n-1)$, natural number 0, here ϵ and t $(= 0_1)$ are abbreviation of empty element and singleton element, respectively, (2) variables : $x, y, z, \ldots$, (3) symbols : form constructors $\mathrm{Ap}, \mathrm{E}, \mathrm{D}, \mathrm{J}, \mathrm{R}_n$ $(n = 0, 1, \ldots), \mathrm{R}$, functional abstraction λ, indices of set i, j, successor function succ, objects $a, b, c, \ldots$, parentheses or ordered pair $(,)$.

Definition 2.2 *The set of* **MTT***-expressions* $\mathcal{E}$ *is defined inductively as follows* (*notice that* $[,]$ *denotes the assumptions*) :

(i) *all variables* $x, y, z, \ldots$ *and constants* m_n $(n = 0, 1, \ldots ; m = 0, 1, \ldots, n-1)$, 0, ϵ *and* t *are* **MTT***-expresssions* (*called* atomic expressions),

(ii) *if* $a \in A$, $b(x) \in B(x)$ $[x \in A]$ *and* $c \in (\Pi x \in A)B(x)$ *are any* **MTT***-expressions, then* $(\lambda x)b(x) \in (\Pi x \in A)B(x)$ *and* $\mathrm{Ap}(c, a) \in B(a)$ *are* **MTT***-expressions* (*called* canonical *and* noncanonical *expressions of the type* $(\Pi x \in A)B(x)$, *respectively*),

(iii) *if* $a \in A$, $b \in B(a)$, $c \in (\Sigma x \in A)B(x)$ *and* $d(x, y) \in C((x, y))$ $[x \in A, y \in B(x)]$ *are any* **MTT***-expressions, then* $(a, b) \in (\Sigma x \in A)B(x)$ *and* $\mathrm{E}(c, d(x, y)) \in C(c)$ *are* **MTT***-expressions* (*called* canonical *and* noncanonical *expressions of the type* $(\Sigma x \in A)B(x)$, *respectively*),

(iv) *if* $a \in A$, $b \in B$, $c \in A + B$, $d(x) \in C(i(x))$ $[x \in A]$ *and* $e(y) \in C(j(y))$ $[y \in B]$ *are any* **MTT***-expressions, then* $i(a), j(b) \in A + B$ *and* $\mathrm{D}(c, d(x), e(y)) \in C(c)$ *are* **MTT***-expressions* (*called* canonical *and* noncanonical *expressions of the type* $A + B$, *respectively*),

(v) *if* $a, b \in A$, $c \in \mathrm{I}(A, a, b)$ *and* $d \in C(r)$ *are any* **MTT***-expressions, then* $r \in \mathrm{I}(A, a, b)$ *and* $\mathrm{J}(c, d) \in C(c)$ *are* **MTT***-expressions* (*called* canonical *and* noncanonical *expressions of the type* $\mathrm{I}(A, a, b)$, *respectively*),

(vi) *if* $c, m_n \in \mathbf{N}_n$ $(n = 0, 1, \ldots ; m = 0, 1, \ldots, n-1)$ *and* $c_m \in C(m_n)$ *are any* **MTT***- expressions, then* $m_n \in \mathbf{N}_n$ *and* $\mathrm{R}_n(c, c_0, c_1, \ldots, c_{n-1}) \in C(c)$ *are* **MTT***-expressions* (*called* canonical *and* noncanonical *expressions of the type* $\mathbf{N}_n$ $(n = 0, 1, \ldots))$, *respectively*,

(vii) *if* $0, a, c \in \mathbf{N}$, $d \in C(0)$ *and* $e(x, y) \in C(\mathrm{succ}(x))$ $[x \in \mathbf{N}, y \in C(x)]$ *are any* **MTT***-expressions, then* $0, \mathrm{succ}(a) \in \mathbf{N}$ *and* $\mathrm{R}(c, d, e(x, y)) \in C(c)$ *are* **MTT***-expressions* (*called* canonical *and* noncanonical *expressions of the type* $\mathbf{N}$, *respectively*).

The theory of type **MTT** can evaluate each **MTT**-expression like a *reduction* of term in λ-calculus. One reduction (i.e., evaluation) of **MTT**-expression is performed by *simultaneously substitutions*. After evaluating **MTT**-expressions, if a **MTT**-expression in some **MTT**-type has itself as a value of substitution, then we will call this expression to *canonical form*, and otherwise to *noncanonical form* of this type. For any expressions $a \in A$ and $b \in B$, an expression a is *equal to* b, in symbols, $a = b \in A$ if two expressions a, b have the same canonical form. Moreover, for any types A and B, a type A is *equal to* B, in symbols, $A = B$ if for two canonical forms $c_A \in A$ and $c_B \in B$, it holds that $c_A = c_B \in A$.

Under the following judgments, the theory of type **MTT** is axiomatized as Gentzen's natural deduction style. We will omit to show each individual inference rule now (see [3,4]).

(1) A type : A is *a type*.

(2) $a \in A$: a is *an object of type* A.

(3) $a = b \in A$: a and b are *equal objects of type* A.

(4) $A = B$: A and B are *equal types*.

3 Evolution relation and its related operations

In this section we will introduce the notion of evolution relation $\sqsubseteq$ between **MTT**-types and also expressions. Then we will define some basic operations, i.e., supremum, infimum and difference operations related to this order. Here our idea is that we can order every similar types (generated by the same type constructor) in **MTT** from the viewpoint of an evolutional relationship by replacing some atomic types in these types with the unit type T, since only $\mathbf{N}_n$ $(n = 0, 1, \ldots)$ and $\mathbf{N}$ are atomic types in **MTT** and the type $\mathrm{T} = (\mathbf{N}_1)$, which denotes a singleton set, has a noncanonical form $\mathrm{R}_1(c, \mathrm{t})$ that is corresponding to NOP operation in the programs.

Any **MTT**-expression b is *degenerated* according to the following rule : $b[\mathrm{t}, \ldots, \mathrm{t}\,/b_1, \ldots, b_n]$, where this denotes the result of *simultaneously replacements* of some atomic expressions $b_1, \ldots, b_n$ in b by the singleton element $\mathrm{t} \in \mathrm{T}$. For any expressions $a \in A$ and $b \in B$, an expression a is *evolved to* b, in symbols, $a \sqsubseteq b \in A$ if for some atomic expressions $b_1, \ldots, b_n \in \mathrm{At}(b)$, it holds that $b[\mathrm{t}, \ldots, \mathrm{t}\,/b_1, \ldots, b_n] = a \in A$. Moreover, for any types A and B, a type A is *evolved to* B, in symbols, $A \sqsubseteq B$ if for two canonical forms $c_A \in A$ and $c_B \in B$, it holds that $c_A \sqsubseteq c_B \in A$. According to the above definition of an evolution relation, we will define the evolution relation of **MTT**-types and expressions individually below under the assumption that each type and form constructors are *continuous*.

Definition 3.1 *For any types A and B in $\mathcal{T}$, the* evolution relation $A \sqsubseteq B$ *of type, is defined inductively as follows:*

(i) *For any interpretations $C : \mathbf{N}_k \to \mathcal{T}$ and $D : \mathbf{N}_l \to \mathcal{T}$, if $\bigcup_{m=0}^{k-1} C(m_k) \subseteq \bigcup_{m=0}^{l-1} D(m_l)$, then $\mathbf{N}_k \sqsubseteq \mathbf{N}_l$ $(k, l = 0, 1, \ldots; k \leq l)$.*

(ii) $\mathbf{N} \sqsubseteq \mathbf{N}$ *and* $\bot \sqsubseteq \mathrm{T}$.

(iii) *If $A_1 \sqsubseteq A_2$ and $B_1(x_1) \sqsubseteq B_2(x_2)$ $[x_1 \in A_1, x_2 \in A_2]$, then $(\Pi x_1 \in A_1)B_1(x_1) \sqsubseteq (\Pi x_2 \in A_2)B_2(x_2)$.*

(iv) *If $A_1 \sqsubseteq A_2$ and $B_1(x_1) \sqsubseteq B_2(x_2)$ $[x_1 \in A_1, x_2 \in A_2]$, then $(\Sigma x_1 \in A_1)B_1(x_1) \sqsubseteq (\Sigma x_2 \in A_2)B_2(x_2)$.*

(v) *If $A_1 \sqsubseteq A_2$ and $B_1 \sqsubseteq B_2$, then $A_1 + B_1 \sqsubseteq A_2 + B_2$.*

(vi) *If $A_1 \sqsubseteq A_2$ and $a_1 \sqsubseteq a_2 \in A_1$ and $b_1 \sqsubseteq b_2 \in A_1$, then $\mathrm{I}(A_1, a_1, b_1) \sqsubseteq \mathrm{I}(A_2, a_2, b_2)$.*

Definition 3.2 *For any expressions $a \in A$ and $b \in B$ in $\mathcal{E}$, the* evolution relation $a \sqsubseteq b \in A$ *of expression, is defined inductively as follows:*

(i) *For any object variables $x \in A$ and $y \in B$ in $\mathcal{E}$ such that $A \sqsubseteq B$, $x \sqsubseteq y \in A$.*

(ii) *For any $m_k \in \mathbf{N}_k$ and $m_l \in \mathbf{N}_l$ $(k, l = 0, 1, \ldots; k \leq l)$, if $C(m_k) \subseteq D(m_l)$, then $m_k \sqsubseteq m_l \in \mathbf{N}_k$.*

(iii) $0 \sqsubseteq 0 \in \mathbf{N}$ *and* $\epsilon \sqsubseteq \mathrm{t} \in \bot$.

(iv) *If $A_1 \sqsubseteq A_2$ and $b_1(x_1) \sqsubseteq b_2(x_2)$ $[x_1 \in A_1, x_2 \in A_2]$, then $(\lambda x_1)b_1(x_1) \sqsubseteq (\lambda x_2)b_2(x_2) \in (\Pi x_1 \in A_1)B_1(x_1)$. Moreover, if $c_1 \sqsubseteq c_2 \in (\Pi x_1 \in A_1)B_1(x_1)$ and $a_1 \sqsubseteq a_2 \in A_1$, then $\mathrm{Ap}(c_1, a_1) \sqsubseteq \mathrm{Ap}(c_2, a_2) \in B_1(a_1)$.*

(v) *If $a_1 \sqsubseteq a_2 \in A_1$ and $b_1(a_1) \sqsubseteq b_2(a_2) \in B_1(a_1)$ $[a_1 \in A_1, a_2 \in A_2]$, then $(a_1, b_1) \sqsubseteq (a_2, b_2) \in (\Sigma x_1 \in A_1)B_1(x_1)$. Moreover, if $c_1 \sqsubseteq c_2 \in (\Sigma x_1 \in A_1)B_1(x_1)$ and $d_1(x_1, y_1) \sqsubseteq d_2(x_2, y_2) \in C_1((x_1, y_1))$, then $\mathrm{E}(c_1, d_1(x_1, y_1)) \sqsubseteq \mathrm{E}(c_2, d_2(x_2, y_2)) \in C_1(c_1)$.*

(vi) *If $a_1 \sqsubseteq a_2 \in A_1$ and $b_1 \sqsubseteq b_2 \in B_1$, then $i(a_1) \sqsubseteq i(a_2)$, $j(b_1) \sqsubseteq j(b_2) \in A_1 + B_1$. Moreover, if $c_1 \sqsubseteq c_2 \in A_1 + B_1$, $d_1(x_1) \sqsubseteq d_2(x_2) \in C_1(i(x_1))$ $[x_1 \sqsubseteq x_2 \in A_1]$ and $e_1(y_1) \sqsubseteq e_2(y_2) \in C_1(j(y_1))$ $[y_1 \sqsubseteq y_2 \in B_1]$, then $\mathrm{D}(c_1, d_1(x_1), e_1(y_1)) \sqsubseteq \mathrm{D}(c_2, d_2(x_2), e_2(y_2)) \in C_1(c_1)$.*

(vii) *If $a_1 = b_1 \in A_1$, $a_2 = b_2 \in A_2$, $a_1 \sqsubseteq a_2 \in A_1$ and $b_1 \sqsubseteq b_2 \in A_1$, then $r_1 \sqsubseteq r_2 \in \mathrm{I}(A_1, a_1, b_1)$. Moreover, if $c_1 \sqsubseteq c_2 \in \mathrm{I}(A_1, a_1, b_1)$ and $d_1 \sqsubseteq d_2 \in C_1(r_1)$, then $\mathrm{J}(c_1, d_1) \sqsubseteq \mathrm{J}(c_2, d_2) \in C_1(c_1)$.*

(viii) *If $c_1 \sqsubseteq c_2 \in \mathbf{N}_k$ and $a_m \sqsubseteq b_{m'} \in C_1(m_k)$ $(m = 0, 1, \ldots, k-1; m' = 0, 1, \ldots, l-1)$, then $\mathrm{R}_k(c_1, a_0, a_1, \ldots, a_{k-1}) \sqsubseteq \mathrm{R}_l(c_2, b_0, b_1, \ldots, b_{l-1}) \in C_1(c_1)$.*

(ix) $a \sqsubseteq a \in \mathbf{N}$ *implies* $\mathrm{succ}(a) \sqsubseteq \mathrm{succ}(a) \in \mathbf{N}$. *Moreover, if* $c_1 \sqsubseteq c_2 \in \mathbf{N}$, $d_1 \sqsubseteq d_2 \in C_1(0)$ *and* $e_1(x_1, y_1) \sqsubseteq e_2(x_2, y_2) \in C_1(\mathrm{succ}(x_1))$ $[x_1, x_2 \in \mathbf{N}, y_1 \in C_1(x_1), y_2 \in C_2(x_2)]$, *then* $\mathrm{R}(c_1, d_1, e_1(x_1, y_1)) \sqsubseteq \mathrm{R}(c_2, d_2, e_2(x_2, y_2)) \in C_1(c_1)$.

Proposition 3.3 *For each structures* $\mathbb{T} = \langle \mathcal{T}, \sqsubseteq \rangle$ *and* $\mathbb{E} = \langle \mathcal{E}, \sqsubseteq \rangle$, *the evolution relation* $\sqsubseteq$ *is a partial order.*

Next we will define both supremum and infimum operations between every **MTT**-types w.r.t. the above evolution relation under the assumption that every type constructors are continuous.

Definition 3.4 *For any types* A *and* B *in* $\mathcal{T}$, *the* supremum $\mathrm{Sup}(A, B)$ *and* infimum $\mathrm{Inf}(A, B)$ *functions between them w.r.t. the evolution relation* $\sqsubseteq$ *is defined inductively as follows:*

(i) $\mathrm{Sup}(\mathbf{N}_k, \mathbf{N}_l) = \mathbf{N}_r$ *and* $\mathrm{Inf}(\mathbf{N}_k, \mathbf{N}_l) = \mathbf{N}_s$ $(k, l = 0, 1, \ldots; k \leq l)$, *where* $r = \left|\bigcup_{m=0}^{k-1} C(m_k) \cup \bigcup_{m=0}^{l-1} D(m_l)\right|$ *and* $s = \left|\bigcup_{m=0}^{k-1} C(m_k) \cap \bigcup_{m=0}^{l-1} D(m_l)\right|$.

(ii) $\mathrm{Sup}(\mathbf{N}, \mathbf{N}) = \mathbf{N}$, $\mathrm{Inf}(\mathbf{N}, \mathbf{N}) = \mathbf{N}$, $\mathrm{Sup}(\bot, \mathrm{T}) = \mathrm{T}$ *and* $\mathrm{Inf}(\bot, \mathrm{T}) = \bot$.

(iii) $\mathrm{Sup}((\Pi x_1 \in A_1)B_1(x_1), (\Pi x_2 \in A_2)B_2(x_2)) = (\Pi x \in \mathrm{Sup}(A_1, A_2))\,\mathrm{Sup}(B_1, B_2)(x)$,
$\mathrm{Inf}((\Pi x_1 \in A_1)B_1(x_1), (\Pi x_2 \in A_2)B_2(x_2)) = (\Pi x \in \mathrm{Inf}(A_1, A_2))\,\mathrm{Inf}(B_1, B_2)(x)$.

(iv) $\mathrm{Sup}((\Sigma x_1 \in A_1)B_1(x_1), (\Sigma x_2 \in A_2)B_2(x_2)) = (\Sigma x \in \mathrm{Sup}(A_1, A_2))\,\mathrm{Sup}(B_1, B_2)(x)$,
$\mathrm{Inf}((\Sigma x_1 \in A_1)B_1(x_1), (\Sigma x_2 \in A_2)B_2(x_2)) = (\Sigma x \in \mathrm{Inf}(A_1, A_2))\,\mathrm{Inf}(B_1, B_2)(x)$.

(v) $\mathrm{Sup}(A_1 + B_1, A_2 + B_2) = \mathrm{Sup}(A_1, A_2) + \mathrm{Sup}(B_1, B_2)$,
$\mathrm{Inf}(A_1 + B_1, A_2 + B_2) = \mathrm{Inf}(A_1, A_2) + \mathrm{Inf}(B_1, B_2)$.

(vi) $\mathrm{Sup}(\mathrm{I}(A_1, a_1, b_1), \mathrm{I}(A_2, a_2, b_2)) = \mathrm{I}(\mathrm{Sup}(A_1, A_2), \mathrm{Sup}(a_1, a_2), \mathrm{Sup}(b_1, b_2))$,
$\mathrm{Inf}(\mathrm{I}(A_1, a_1, b_1), \mathrm{I}(A_2, a_2, b_2)) = \mathrm{I}(\mathrm{Inf}(A_1, A_2), \mathrm{Inf}(a_1, a_2), \mathrm{Inf}(b_1, b_2))$.

(vii) *For any types* A *and* B *which are not matched any cases of the above* (i)–(vi), $\mathrm{Sup}(A, B) = \mathcal{T}$ *and* $\mathrm{Inf}(A, B) = \bot$.

Moreover, we can also define both supremum and infimum functions between every **MTT**-expressions w.r.t. the evolution relation by the similar manner.

Lemma 3.5 (i) *For any types* A *and* B *in* $\mathcal{T}$, $\mathrm{Sup}(A, B)$ *and* $\mathrm{Inf}(A, B)$ *are the l.u.b. and g.l.b. of* A *and* B *w.r.t. the evolution relation* $\sqsubseteq$, *respectively.*

(ii) *Similarly, for any expressions* a *and* b *in* $\mathcal{E}$, $\mathrm{Sup}(a, b)$ *and* $\mathrm{Inf}(a, b)$ *are the l.u.b. and g.l.b. of* a *and* b *w.r.t. the evolution relation* $\sqsubseteq$, *respectively.*

(iii) *Both structures* $\mathbb{T} = \langle \mathcal{T}, \sqsubseteq \rangle$ *and* $\mathbb{E} = \langle \mathcal{E}, \sqsubseteq \rangle$ *are the complete lattices w.r.t. the evolution relation* $\sqsubseteq$.

Theorem 3.6 (i) *For any expressions* $a \in A$ *and* $b \in B$, *it holds that* $\mathrm{Sup}(a, b) \in \mathrm{Sup}(A, B)$.

(ii) *For any expressions* $a \in A$ *and* $b \in B$, *it holds that* $\mathrm{Inf}(a, b) \in \mathrm{Inf}(A, B)$.

For any types A and B in $\mathcal{T}$ such that $A \sqsubseteq B$, we will say that $\mathrm{Diff}(B, A)$ is a *difference* between B and A if this operation satisfies the following conditions:

(1) $\mathrm{Sup}(A, \mathrm{Diff}(B, A)) = B$,

(2) $\mathrm{Inf}(A, \mathrm{Diff}(B, A)) = \bot$.

Definition 3.7 *For any types* A *and* B *in* $\mathcal{T}$, *the* difference $\mathrm{Diff}(B, A)$ *function between* B *and* A *w.r.t. the evolution relation* $\sqsubseteq$ *is defined inductively as follows:*

(i) $\mathrm{Diff}(\mathbf{N}_l, \mathbf{N}_k) = \mathbf{N}_v$ $(k, l = 0, 1, \ldots; k \leq l)$, *where* $v = \left|\bigcup_{m=0}^{l-1} D(m_l) \setminus \bigcup_{m=0}^{k-1} C(m_k)\right|$.

(ii) $\mathrm{Diff}(\mathbf{N}, \mathbf{N}) = \bot$ *and* $\mathrm{Diff}(\mathrm{T}, \bot) = \mathrm{T}$.

(iii) *If* $(\Pi x_1 \in A_1)B_1(x_1) \sqsubseteq (\Pi x_2 \in A_2)B_2(x_2)$, *then*

$$\begin{array}{ll} & \mathrm{Diff}((\Pi x_2 \in A_2)B_2(x_2), (\Pi x_1 \in A_1)B_1(x_1)) \\ = & (\Pi x \in \mathrm{Diff}(A_2, A_1))B_1(x) \cup (\Pi x_1 \in A_1)\,\mathrm{Diff}(B_2, B_1)(x_1) \cup \\ & (\Pi x \in \mathrm{Diff}(A_2, A_1))\,\mathrm{Diff}(B_2, B_1)(x). \end{array}$$

(iv) *If* $(\Sigma x_1 \in A_1)B_1(x_1) \sqsubseteq (\Sigma x_2 \in A_2)B_2(x_2)$, *then*

$$\begin{array}{ll} & \mathrm{Diff}((\Sigma x_2 \in A_2)B_2(x_2), (\Sigma x_1 \in A_1)B_1(x_1)) \\ = & (\Sigma x \in \mathrm{Diff}(A_2, A_1))B_1(x) \cup (\Sigma x_1 \in A_1)\,\mathrm{Diff}(B_2, B_1)(x_1) \cup \\ & (\Sigma x \in \mathrm{Diff}(A_2, A_1))\,\mathrm{Diff}(B_2, B_1)(x). \end{array}$$

(v) *If* $A_1 + B_1 \sqsubseteq A_2 + B_2$, *then*

$$\begin{array}{ll} & \mathrm{Diff}(A_2 + B_2, A_1 + B_1) \\ = & \mathrm{Diff}(A_2, A_1) + \mathrm{Diff}(B_2, B_1). \end{array}$$

(vi) *If* $\mathrm{I}(A_1, a_1, b_1) \sqsubseteq \mathrm{I}(A_2, a_2, b_2)$, *then*

$$\begin{array}{ll} & \mathrm{Diff}(\mathrm{I}(A_2, a_2, b_2), \mathrm{I}(A_1, a_1, b_1) \\ = & \mathrm{I}(\mathrm{Diff}(A_2, A_1), a_1, b_1) \cup \mathrm{I}(A_2, \mathrm{Diff}(a_2, a_1), \mathrm{Diff}(b_2, b_1)) \cup \\ & \mathrm{I}(\mathrm{Diff}(A_2, A_1), \mathrm{Diff}(a_2, a_1), \mathrm{Diff}(b_2, b_1)). \end{array}$$

(vii) *For any types* A *and* B *which are not matched any cases of the above* (i)–(vi),

$$\mathrm{Diff}(A, B) = \bot.$$

By the similar manner, we can introduce the difference between expressions $\mathrm{Diff}(b, a)$ for any expressions a and b in $\mathcal{E}$ such that $a \sqsubseteq b$.

Lemma 3.8 (i) *For any types* A *and* B *in* $\mathcal{T}$ *such that* $A \sqsubseteq B$, $\mathrm{Diff}(B, A)$ *is a difference between* B *and* A.

(ii) *Similary, for any expressions* $a \in A$ *and* $b \in B$ *in* $\mathcal{E}$ *such that* $a \sqsubseteq b \in A$, $\mathrm{Diff}(b, a)$ *is a difference between* b *and* a.

Theorem 3.9 *For any expressions* $a \in A$ *and* $b \in B$, *if an expression* a *is evolved to* b *in a type* A, $a \sqsubseteq b \in A$, *then it holds that* $\mathrm{Diff}(b, a) \in \mathrm{Diff}(B, A)$.

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Fuzzy relations and fuzzy mappings between fuzzy subsets

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Introduction

In fuzzy literature, fuzzy relations and fuzzy mappings are usually treated as those between crisp sets. With our natural interpretation in a Heyting valued model for intuitionistic set theory, we are to define and study fuzzy relations and fuzzy mappings between fuzzy sets, especially between fuzzy subsets of crisp sets. By using membership functions we can characterize fuzzy relations, fuzzy mappings, and equivalence and order relations.

1 The Heyting valued model

Let H be a complete Heyting algebra with operations and constants $\wedge, \vee, \bigwedge, \bigvee, \to, \neg, \mathbf{0}, \mathbf{1}$, and $\leq$. We construct an H-valued model for the *extended intuitionistic set theory*, which is a first order intuitionistic logic with predicates $\in, =$ and E (called an *existence predicate*) together with axioms of intuitionistic set theory. Let V be the class of all crisp sets and On be the class of all ordinals.

Definition 1.1. The *H-valued model* V^H is constructed as follows.
For every ordinal α, V_α^H is defined by induction:

$$V_0^H = \phi, \qquad V_\alpha^H = \bigcup_{\beta<\alpha} V_\beta^H \quad (\text{if } \alpha \text{ is a limit ordinal}),$$

$$V_{\alpha+1}^H = \big\{\, u = \langle |u|, Eu\rangle \,;\, |u| : \mathcal{D}u \longrightarrow H,\ \mathcal{D}u \subseteq V_\alpha^H,\ Eu \in H,\ |u|(x) \leq Eu \wedge Ex \ \text{ for all } x \in \mathcal{D}u \big\}.$$

Then $V^H = \bigcup_{\alpha \in On} V_\alpha^H$.

For a sentence φ of V^H, the *Heyting value* $\|\varphi\| \in H$ is defined as follows.

Definition 1.2. Let $u, v \in V^H$ and $\varphi, \psi, \varphi(a)$ be formulas of V^H.
For atomic formulae,

$$\|\mathrm{E}u\| = Eu, \quad \text{and by simultaneous induction,}$$
$$\|u \in v\| = \bigvee_{y \in \mathcal{D}v} \big(v(y) \wedge \|u = y\|\big), \ \text{and}$$
$$\|u = v\| = \bigwedge_{x \in \mathcal{D}u} \big(u(x) \to \|x \in v\|\big) \wedge \bigwedge_{y \in \mathcal{D}v} \big(v(y) \to \|y \in u\|\big) \wedge Eu \wedge Ev.$$

For compound formulae, by induction (on the number of logical symbols),

$$\|\varphi \wedge \psi\| = \|\varphi\| \wedge \|\psi\|, \qquad \|\varphi \vee \psi\| = \|\varphi\| \vee \|\psi\|,$$
$$\|\varphi \to \psi\| = \|\varphi\| \to \|\psi\|, \qquad \|\neg\varphi\| = \neg\|\varphi\|,$$
$$\|\forall x \varphi(x)\| = \bigwedge_{u \in V^H} \big(Eu \to \|\varphi(x)\|\big), \ \text{and} \quad \|\exists x \varphi(x)\| = \bigvee_{u \in V^H} \big(Eu \wedge \|\varphi(x)\|\big).$$

An element of the model is called a *set in* V^H. For a sentence φ of V^H, φ is *valid in* V^H if $\|\varphi\| = \mathbf{1}$. We say u is a *subset of* v *in* V^H ($u \sqsubseteq v$) if $\|u \subseteq v\| = \bigwedge_{x \in \mathcal{D}u} \big(u(x) \to \|x \in v\|\big) = \mathbf{1}$. If $u \sqsubseteq v$ and $v \sqsubseteq u$, we say u and v are *similar* ($u \sim v$). If $u \sim v$ and $Eu = Ev$, u and v are said to be *equivalent* ($u \approx v$).

Definition 1.3. For every crisp set x in V, $\check{x} \in V^H$ is defined recursively by:

$$\mathcal{D}(\check{x}) = \{\check{y}; y \in x\}, \quad E\check{x} = \mathbf{1}, \quad \check{x} : \check{y} \longmapsto \mathbf{1}.$$

We call $\check{x}$ the *check set of* x.

The *canonical embedding* which assigns to each crisp set its check set plays an important role in our interpretation. For a bounded formula $\varphi(a_1, \cdots, a_n)$ and crisp sets $x_1, \cdots, x_n$, $\varphi(x_1, \cdots, x_n)$ holds iff $\|\varphi(\check{x}_1, \cdots, \check{x}_n)\| = \mathbf{1}$, and $\neg\varphi(x_1, \cdots, x_n)$ holds iff $\|\varphi(\check{x}_1, \cdots, \check{x}_n)\| = \mathbf{0}$.

Definition 1.4. Let $u, v \in V^H$.

(1) $\mathcal{D}\{u, v\}^H = \{u, v\}$, $E\{u, v\}^H = Eu \wedge Ev$, and $\{u, v\}^H : x \longmapsto \mathbf{1}$.
(2) $\langle u, v\rangle^H = \{\{u, u\}^H, \{u, v\}^H\}^H$.
(3) $\mathcal{D}(u \overset{H}{\times} v) = \{\langle x, y\rangle^H ; x \in \mathcal{D}u,\ y \in \mathcal{D}v\}$, $E(u \overset{H}{\times} v) = Eu \wedge Ev$, and $u \overset{H}{\times} v : \langle x, y\rangle^H \longmapsto u(x) \wedge v(y)$.
(4) $\mathcal{D}(u \cap^H v) = \mathcal{D}u \cup \mathcal{D}v$, $E(u \cap^H v) = Eu \wedge Ev$, and $u \cap^H v : x \longmapsto \|x \in u\| \wedge \|x \in v\|$.
(5) $\mathcal{D}(u \cup^H v) = \mathcal{D}u \cup \mathcal{D}v$, $E(u \cup^H v) = Eu \vee Ev$, and $u \cup^H v : x \longmapsto \|x \in u\| \vee \|x \in v\|$.
(6) $\mathcal{D}(u \setminus^H v) = \mathcal{D}u \cup \mathcal{D}v$, $E(u \setminus^H v) = Eu \vee Ev$, and $u \setminus^H v : x \longmapsto \|x \in u\| \wedge \neg\|x \in v\|$.

The superscripts and commas may be omitted. We can also define power sets, infinite sums, and so on. Hence all axioms of (extended) intuitionistic set theory are valid in V^H.

A *relation in* V^H is a subset of a cartesian product in V^H. For $R, u, v \in V^H$, R is a *relation from* u *to* v *in* V^H if R is a subset of $u \overset{H}{\times} v$ in V^H. We often write xRy instead of $\langle xy\rangle \in R$.

For each set u in V^H, the *identity relation* I_u *on* u is defined by:

$$\mathcal{D}(I_u) = \{\langle xx\rangle; x \in \mathcal{D}u\}, \quad E(I_u) = Eu, \text{ and } I_u : \langle xx\rangle \longmapsto u(x).$$

Definition 1.5. For R, S in V^H, the *composition* $S \circ R$ and the *inverse relation* R^{-1} are defined by:

$$\mathcal{D}(S \circ R) = \{\langle xz\rangle; x, z \in V_\alpha^H\},\ E(S \circ R) = ER \wedge ES,\ S \circ R : \langle xz\rangle \longmapsto \|\exists y(xRy \wedge ySz)\|, \text{ and}$$
$$\mathcal{D}(R^{-1}) = \{\langle xy\rangle; x, y \in V_\alpha^H\},\ E(R^{-1}) = ER,\ R^{-1} : \langle xy\rangle \longmapsto \|yRx\|,$$

where α is a suitable ordinal which satisfies $\mathcal{D}R \cup \mathcal{D}S \subseteq V_\alpha^H$.

The definitions are not affected by the choice of the ordinal α. If R and S are relations from u to v and from v to w respectively, then $S \circ R$ is a relation from u to w and R^{-1} is a relation from v to u.

Definition 1.6. Let R be a relation from u to v in V^H.
(1) For $A \in V^H$, the *image* $R(A) \in V^H$ is defined by:

$$\mathcal{D}(R(A)) = \mathcal{D}v, \quad E(R(A)) = ER \wedge EA, \quad R(A) : y \longmapsto \|(\exists x \in A)xRy\|.$$

(2) For $B \in V^H$, the *inverse image* $R^{-1}(B) \in V^H$ is defined by:

$$\mathcal{D}(R^{-1}(B)) = \mathcal{D}u, \quad E(R^{-1}(B)) = ER \wedge EB, \quad R^{-1}(B) : x \longmapsto \|(\exists y \in B)xRy\|.$$

A relation R from u to v in V^H is called *total* if $(\forall x \in u)(\exists y \in v)(xRy)$ is valid, and is *surjective* if $(\forall y \in v)(\exists x \in u)(xRy)$ is valid. Similarly, R is *injective* if $(\forall x, y \in u)(\forall z \in v)(xRz \wedge yRz \to x = y)$ is valid, and is *univalent* if $(\forall x \in u)(\forall y, z \in v)(xRy \wedge xRz \to y = z)$ is valid. Then R is total iff $I_u \sqsubseteq R^{-1} \circ R$, R is surjective iff $I_v \sqsubseteq R \circ R^{-1}$, R is injective iff $R^{-1} \circ R \sqsubseteq I_u$, and R is univalent iff $R \circ R^{-1} \sqsubseteq I_v$.

A relation R on u (that is, a relation from u to u) is called *reflexive* if $(\forall x \in u)(xRx)$ is valid. Similarly R is *symmetric* if $\forall x \forall y(xRy \to yRx)$ is valid, is *transitive* if $\forall x \forall y \forall z(xRy \wedge yRz \to xRz)$ is valid, is *antisymmetric* if $\forall x \forall y(xRy \wedge yRx \to x = y)$ is valid, and is *connected* if $(\forall x \in u)(\forall y \in u)(xRy \vee yRx)$ is valid. Then R is reflexive iff $I_u \sqsubseteq R$, is symmetric iff $R^{-1} \sqsubseteq R$ iff $R^{-1} \approx R$, is transitive iff $R \circ R \sqsubseteq R$, is antisymmetric iff $R \cap R^{-1} \sqsubseteq I_u$, and is connected iff $u \times u = R \cup R^{-1}$.

An *equivalence relation* is a reflexive, symmetric, and transitive relation, an *order relation* is a reflexive, antisymmetric, and transitive relation, and a *total order* is a connected order relation. Hence a relation R on u is an equivalence relation iff $I_u \sqsubseteq R$, $R^{-1} \sqsubseteq R$, and $R \circ R \sqsubseteq R$. If R is a relation $R^{-1} \circ R$ is symmetric, and in addition if R is total $R^{-1} \circ R$ is reflexive, and if it is univalent $R^{-1} \circ R$ is transitive.

A *mapping in* V^H is a total and univalent relation in the model. For $u, v, f \in V^H$, f is called *a mapping from u to v in V^H* if f is a total and univalent relation from u to v in V^H. We write $f : u \longrightarrow v$ *in* V^H if f is a mapping from u to v. We often write $f(x) = y$ instead of $\langle xy \rangle \in f$ or xfy. A relation f from u to v is a mapping iff $I_u \sqsubseteq f^{-1} \circ f$ and $f \circ f^{-1} \sqsubseteq I_v$.

An *injection* (resp. a *surjection*) is an injective (resp. surjective) mapping, and a *bijection* is an injective and surjective mapping. A composition of injections (resp. surjections or bijections) is also an injection (resp. a surjection or a bijection). For a relation f from u to v, f is an injection iff $I_u \approx f^{-1} \circ f$ and $f \circ f^{-1} \sqsubseteq I_v$, is a surjection iff $I_u \sqsubseteq f^{-1} \circ f$ and $f \circ f^{-1} \approx I_v$, and f is a bijection iff $I_u \approx f^{-1} \circ f$ and $f \circ f^{-1} \approx I_v$. Hence for a mapping $f : u \longrightarrow v$ *in* V^H, f is injective iff $I_u \approx f^{-1} \circ f$ and is surjective iff $f \circ f^{-1} \approx I_v$. For a mapping f in V^H, $f^{-1} \circ f$ is an equivalence relation.

We can define images and inverse images of mappings similarly to those of relations.

2 Relations and mappings between fuzzy subsets

In the following, let X, Y, Z be crisp sets. We call every set in the model V^H an *H-fuzzy set*.

Definition 2.1. For an H-fuzzy set A, the *membership function of A on X* is the mapping

$$\mu_A = \mu_A^X : X \longrightarrow H;\ x \longmapsto \|\check{x} \in A\|.$$

An *H-fuzzy subset* of X is a subset of $\check{X}$ in V^H, that is, for $A \in V^H$,

$$A \text{ is an } H\text{-fuzzy subset of } X \text{ iff } A \sqsubseteq \check{X}.$$

Lemma 2.1. *Every mapping from X to H is the membership function of some H-fuzzy subset of X.*

Hence there is a natural correspondence between H-fuzzy subsets of X and mappings from X to H.

Definition 2.2. Let μ, ν be mappings from X to H.

(1) $\mu \leq \nu$ iff $\mu(x) \leq \nu(x)$ for all $x \in X$.

(2) The mappings $\mu \wedge \nu$, $\mu \vee \nu$, $\neg\, \mu : X \longrightarrow H$ are defined by:

$$\mu \wedge \nu : x \longmapsto \mu(x) \wedge \nu(x),\ \mu \vee \nu : x \longmapsto \mu(x) \vee \nu(x),\ \text{ and } \neg\, \mu : x \longmapsto \neg\, \mu(x)\ \text{ (for all } x \in X).$$

The natural correspondence preserves inclusion (order) and the basic set operations.

Proposition 2.2. *Let A, B be H-fuzzy sets and all the membership functions below be on X.*

(1) *If A and B are H-fuzzy subsets of X, then $A \sqsubseteq B$ iff $\mu_A \leq \mu_B$.*

(2) $\mu_{A \cap B} = \mu_A \wedge \mu_B$, $\mu_{A \cup B} = \mu_A \vee \mu_B$, *and* $\mu_{\check{X} \setminus A} = \neg\, \mu_A$.

Every relation in the model is called an *H-fuzzy relation.*

Definition 2.3. For $R, u, v \in V^H$, R is an *H-fuzzy relation from u to v* if it is a relation from u to v in V^H. For every $R \in V^H$, the *membership function of R from X to Y* is the membership function of R on $X \times Y$, hence it is the mapping

$$\mu_R : X \times Y \longrightarrow H;\ \langle xy \rangle \longmapsto \|\langle xy \rangle^\vee \in R\|.$$

An *H-fuzzy relation from X to Y* is an H-fuzzy subset of $X \times Y$.

If A, B are H-fuzzy subsets of X, Y respectively, then obviously an H-fuzzy relation from A to B is an H-fuzzy relation from X to Y.

Lemma 2.3. *Let A, B be H-fuzzy subsets of X, Y respectively and R be a set in V^H. Then*

$$R \text{ is an } H\text{-fuzzy relation from } A \text{ to } B \quad \text{iff} \quad \mu_R\langle xy \rangle \leq \mu_A(x) \wedge \mu_B(y) \ \text{ for all } x \in X,\ y \in Y,$$

where μ_A, μ_B, μ_R be the membership functions on $X, Y, X \times Y$ respectively.

Proposition 2.4. *Let A, B, C be H-fuzzy subsets of X, Y, Z respectively, R be an H-fuzzy relation from A to B, S be an H-fuzzy relation from B to C, and $\mu_R, \mu_S, \mu_{S \circ R}, \mu_{R^{-1}}$ be their membership functions on $X \times Y, Y \times Z, X \times Z, Y \times X$ respectively.*

(1) *The composition $S \circ R$ is an H-fuzzy relation from A to C, and*

$$\mu_{S \circ R}\langle xz \rangle = \bigvee_{y \in Y} \left(\mu_R\langle xy \rangle \wedge \mu_S\langle yz \rangle \right) \quad \text{for all } x \in X, z \in Z.$$

(2) *The inverse relation R^{-1} is an H-fuzzy relation from B to A, and*

$$\mu_{R^{-1}}\langle yx \rangle = \mu_R\langle xy \rangle \ \text{ for all } x \in X, y \in Y.$$

These are extensions of the defining equations of max-min composition and those of inverse for ordinary fuzzy relations.

Proposition 2.5. *Let A, B be H-fuzzy subsets of X, Y respectively, R be an H-fuzzy relation from A to B, and μ_R be its membership function on $X \times Y$.*

(1) *For every $C \in V^H$, the image $R(C)$ is a subset of B in V^H, and*

$$\mu_{R(C)}(y) = \bigvee_{x \in X} \left(\mu_C(x) \wedge \mu_R\langle xy \rangle \right) \ \text{ for all } y \in Y,$$

where μ_C and $\mu_{R(C)}$ are the membership functions on X and on Y respectively.

(2) *For every $D \in V^H$, the inverse image $R^{-1}(D)$ is a subset of A in V^H, and*

$$\mu_{R^{-1}(D)}(x) = \bigvee_{y \in Y} \left(\mu_D(y) \wedge \mu_R\langle xy \rangle \right) \ \text{ for all } x \in X,$$

where μ_D and $\mu_{R^{-1}(D)}$ are the membership functions on Y and on X respectively.

Proposition 2.6. *Let A, B be H-fuzzy subsets of X, Y respectively, R be an H-fuzzy relation from A to B, and μ_A, μ_B, μ_R be their membership functions on $X, Y, X \times Y$ respectively.*

(1) *R is total iff $\bigvee_{y \in Y} \mu_R\langle xy \rangle = \mu_A(x)$ for all $x \in X$.*

(2) *R is surjective iff $\bigvee_{x \in X} \mu_R\langle xy \rangle = \mu_B(y)$ for all $y \in Y$.*

(3) *R is injective iff $\mu_R\langle xz \rangle \wedge \mu_R\langle yz \rangle > \mathbf{0}$ implies $x = y$ for all $x, y \in X$ and all $z \in Y$.*

(4) *R is univalent iff $\mu_R\langle xy \rangle \wedge \mu_R\langle xz \rangle > \mathbf{0}$ implies $y = z$ for all $x \in X$ and all $y, z \in Y$.*

Theorem 1. *Let A be an H-fuzzy subset of X, R be an H-fuzzy relation on A, and μ_A, μ_R be their membership functions on $X, X \times X$ respectively. Then the followings hold.*

(1) *R is reflexive iff $\mu_R\langle xx\rangle = \mu_A(x)$ for all $x \in X$.*

(2) *R is symmetric iff $\mu_R\langle xy\rangle = \mu_R\langle yx\rangle$ for all $x, y \in X$.*

(3) *R is transitive iff $\mu_R\langle xy\rangle \wedge \mu_R\langle yz\rangle \leq \mu_R\langle xz\rangle$ for all $x, y, z \in X$*

(4) *R is antisymmetric iff $\mu_R\langle xy\rangle \wedge \mu_R\langle yx\rangle > \mathbf{0}$ implies $x = y$ for all $x, y \in X$.*

(5) *R is connected iff $\mu_R\langle xy\rangle \vee \mu_R\langle yx\rangle = \mu_A(x) \wedge \mu_A(y)$ for all $x, y \in X$.*

Every mapping in V^H is called an *H-fuzzy mapping.*

Definition 2.4. A mapping from u to v in V^H is called an *H-fuzzy mapping from u to v*. An *H-fuzzy mapping from X to Y* is a mapping from $\check{X}$ to $\check{Y}$ in V^H.

Obviously an H-fuzzy mapping from u to v is a total and univalent H-fuzzy relation from u to v.

Theorem 2. *Let A, B be H-fuzzy subsets of X, Y respectively, and μ_A, μ_B be their membership function on X, Y respectively. Suppose $f \in V^H$ and $\psi = \mu_f$ is its membership function on $X \times Y$. Then f is an H-fuzzy mapping from A to B iff ψ satisfies the following three conditions:*

(R) $\psi\langle xy\rangle \leq \mu_A(x) \wedge \mu_B(y)$ *for all $x \in X$, $y \in Y$,*

(E) $\bigvee_{y \in Y} \psi\langle xy\rangle = \mu_A(x)$ *for all $x \in X$,*

(U) $\psi\langle xy\rangle \wedge \psi\langle xz\rangle > \mathbf{0}$ *implies $y = z$ for all $x \in X$, $y, z \in Y$.*

Conversely, if a crisp mapping $\psi : X \times Y \longrightarrow H$ satisfies the three conditions, there is an H-fuzzy mapping f from A to B whose membership function on $X \times Y$ is identical with ψ.

Proposition 2.7. *Let A, B be H-fuzzy subsets of X, Y respectively, f be an H-fuzzy mapping from A to B, and μ_f be its membership function on $X \times Y$.*

(1) *For every $C \in V^H$, the image $f(C)$ is a subset of B in V^H, and*

$$\mu_{f(C)}(y) = \bigvee_{x \in X} (\mu_C(x) \wedge \mu_f\langle xy\rangle) \quad \text{for all } y \in Y,$$

where μ_C and $\mu_{f(C)}$ are the membership functions on X and on Y respectively.

(2) *For every $D \in V^H$, the inverse image $f^{-1}(D)$ is a subset of A in V^H, and*

$$\mu_{f^{-1}(D)}(x) = \bigvee_{y \in Y} (\mu_D(y) \wedge \mu_f\langle xy\rangle) \quad \text{for all } x \in X,$$

where μ_D and $\mu_{f^{-1}(D)}$ are the membership functions on Y and on X respectively.

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Theorem Provers with Counter Models from xpe

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Abstract

Xpe is a grammar editor for proof figure on TEX. It makes you easier to draw proof figure. And theorem provers called from xpe help you to research and write papers.

1 Introduction

How do you draw proof figure on TEX? Most of us must use 'proof.sty' macros. But we feel confusion to draw and fix one because it has a structural problem that tree structure is expressed by so many brackets. We would desire another method to draw it. To that solution, I produce xpe here.

Xpe (**X** window system **P**roof **E**ditor) is a grammar editor for proof figure. It interprets and displays TEXcommand, for example \Gamma$\cdots\Gamma$,\Delta$\cdots\Delta$,\supset$\cdots\supset$, and so on. It supports cut and paste formulae. The file format for input and output is compatible with proof.sty. And it is easy to transfer from editor to xpe, and opposite. We can get WYSIWYG(What You See Is What You Get) environment to draw proof figure.

In addition, some theorem provers are supported. They are propositional logics:

- classical logic
- intuitionistic logic
- modal logics K, KT, S4, S5
- lambda type assign system

At this time, these theorem provers and proof checker are supported:

- modal logics K, KT with counter models (called K+, KT+)
- modal logics K5, KD5
- cut elimination checker

Most of this paper refers to System K+ and KT+ that are theorem provers of K or KT with counter modes.

By the way, xpe is a free software. Everybody can download from above my homepage.

2 System K+

Modal logic K is defined by adding ($\Box$) rule to classical logic LK of sequent calculus system. On decision procedure, it is defined by adding ($\Box$) rule including weakening to Wang's system (see [4],[1]). The rules are shown at figure 1.

$$\frac{\Gamma \to A}{\Box\Gamma \to \Box A}\ (\Box) \qquad \frac{\Gamma \to A}{\Delta, \Box\Gamma \to \Box A, \Sigma}\ (\Box)$$

Figure 1: $\Box$ rules for LK and Wang's system

If we generate a counter model for a unprovable sequent, we need take the System K+ (figure 2). We consider both sides of a sequent as multi–sets. It is enough to consider as sets, too. But to make it more easily to understand decision procedure in the theorem prover, both sides of sequent are treated as multi–sets.

$$\Gamma, p \to p, \Delta$$

$$\frac{\Gamma \to \Delta, A \quad \Gamma, B \to \Delta}{\Gamma, A \supset B \to \Delta}\ (\supset L) \qquad \frac{\Gamma, A \to \Delta, B}{\Gamma \to \Delta, A \supset B}\ (\supset R)$$

$$\frac{\Gamma, A, B \to \Delta}{\Gamma, A \wedge B \to \Delta}\ (\wedge L) \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}\ (\wedge R)$$

$$\frac{\Gamma, A \to \Delta \quad \Gamma, B \to \Delta}{\Gamma, A \vee B \to \Delta}\ (\vee L) \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}\ (\vee R)$$

$$\frac{\Gamma, A \to \Delta}{\Gamma \to \Delta, \neg A}\ (\neg R) \qquad \frac{\Gamma, A \to \Delta}{\Gamma \to \Delta, \neg A}\ (\neg R)$$

$$\frac{\Gamma \to A_1 \quad \cdots \quad \Gamma \to A_m}{p_1, \cdots, p_n, \Box\Gamma \to \Box A_1, \cdots, \Box A_m, q_1, \cdots q_l}\ (\Box)$$

Figure 2: System K+

I am explaining the procedure to generate a counter model for an unprovable sequent. Here, consider $\Box C \vee \Box B \wedge D \to \Box(\neg A \vee \Box C) \vee \Box B$. The theorem prover of K+ outputs a decision tree and a counter modes as follows:

$$\dfrac{\dfrac{\dfrac{

3 System KT+

Modal logic KT is defined by adding (T) rule to K. On decision procedure, it is defined by adding (T) rule including contraction to K+ system. However, we adopt a black marked (T) rule, because (T) rule including contraction has loop. If we mark to $\Box$ whenever we apply (T), it has no loop.

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (T) \qquad \frac{\Box A, A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (T) \qquad \frac{\blacksquare A, A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (T)$$

Figure 5: T rules for KT and KT+

Generaging a counter model for a unprovable sequent for KT+ is similar to K+. At first, the procedure reduces logical connectives $\wedge, \vee, \wedge$ and $\supset$ to easier formulae, and transfers from $\Box$ formulae to $\blacksquare$ formulae by (T) rule. And then it determines that each consequence is an initial sequents or not. If no more rule is applied, it tries to apply following $(\Box)$ rule. Finally each top sequent is a form like $p_1, \cdots, p_n, \blacksquare\Gamma \to q_1, \cdots, q_n$. The sequent corresponds to the formulae set $\{p_1, \cdots, p_n\}$.

$$\frac{\Gamma \to A_1 \quad \cdots \quad \Gamma \to A_m}{p_1, \cdots, p_n, \blacksquare\Gamma \to \Box A_1, \cdots, \Box A_m, q_1, \cdots q_l}\ (\Box)$$

Figure 6: $\Box$ rule for KT+

And $(\Box)$ rule corresponds a frame $\begin{array}{ccccc} \{\Gamma, \overset{\curvearrowright}{\neg A_1}\} & & \cdots & & \{\Gamma, \overset{\curvearrowright}{\neg A_m}\} \\ & \nwarrow & \curvearrowright & \nearrow & \\ & & \{p_1, \cdots, p_n\} & & \end{array}$.

Now, consider the formula $\Box\Box B, C \to \Box(\neg A \vee \Box C)$. The theorem prover KT+ outputs a decision tree and a counter model as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{B \to C}{\blacksquare B, B, B, A \to \Box C}\ (\Box)}{\Box B, B, A \to \Box C}\ (T)}{\Box B, B \to \neg A, \Box C}\ (\neg R)}{\Box B, B \to \neg A \vee \Box C}\ (\vee R)}{\blacksquare\Box B, \blacksquare B, B, C \to \Box(\neg A \vee \Box C)}\ (\Box)}{\blacksquare\Box B, \Box B, C \to \Box(\neg A \vee \Box C)}\ (T)}{\Box\Box B, C \to \Box(\neg A \vee \Box C)}\ (T) \qquad \begin{array}{c} \{B\} \\ \vdots \\ \{B, B, A\} \\ \vdots \\ \{B, C\} \end{array}$$

Figure 7: Decision tree and counter model by KT+

Similarly to K+, we can find correspondence between decision trees and formula sets as follows:

$$B \to C \quad \cdots \{B\}$$

$$\dfrac{\dfrac{\dfrac{\blacksquare B, B, B, A \to \Box C}{\Box B, B, A \to \Box C}\,(T)}{\Box B, B \to \neg A, \Box C}\,(\neg R)}{\Box B, B \to \neg A \vee \Box C}\,(\vee R) \quad \cdots \{B, B, A\}$$

$$\dfrac{\dfrac{\blacksquare \Box B, \blacksquare B, B, C \to \Box(\neg A \vee \Box C)}{\blacksquare \Box B, \Box B, C \to \Box(\neg A \vee \Box C)}\,(T)}{\Box \Box B, C \to \Box(\neg A \vee \Box C)}\,(T) \quad \cdots \{C, B\}$$

Figure 8: Relationship between tree and set

4 Conclusion

Most of theorem provers in the world reply only whether a formula is valid or not. However, I want a proof if it is provable, and a counter model if it is unprovable. And it is hard to learn how to use a theorem prover. So I take TEX style as user interface. Because most of logicians know TEX, it is easy to learn how to use xpe. Especially, as they usually use xpe to draw proof figure, they will become familiar to xpe. My primary motivation to create xpe is the easy use.

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Initial segments of Rudin-Frolík order

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Abstract

The Rudin-Frolík order is originally defined on $\beta(\omega)$ [3, 7]. We defined Rudin-Frolík order on the ultrafilters of the complete Boolean algebras [5]. It is well-known that an initial segment of Rudin-Frolík order on $\beta(\omega)$ is totally ordered. The main theorem is Theorem 1.1. When the complete Boolean algebra satisfies countably chain condition (ccc), an initial segment of Rudin-Frolík order is totally ordered.

1 Rudin-Frolík order

Definition 1.1 (Rudin-Frolík order). Let p be an ultrafilter of a complete Boolean algebra $\mathcal{A}$ and let q be an ultrafilter of a complete Boolean algebra $\mathcal{B}$. The *Rudin-Frolík order* $\sqsubseteq$ is the binary relation, which is given by $p \sqsubseteq q$, if there are $a \in p$, $b \in q$, a complete homomorphism $h\colon \mathcal{A} \to \mathcal{B}\restriction b$ and a homomorphism $f\colon \mathcal{B} \to \mathcal{A}\restriction a$ such that $f \circ h(x) = x$ for all $x \in \mathcal{A}\restriction a$ and $f^{-1}\text{“}p = q$.

Precisely saying, Rudin-Frolík order is preorder, which dose not satisfy anti-symmetric law. We consider the equivalence class, the equivalence relation is defined by $p \sqsubseteq q \wedge q \sqsubseteq p$, implicitly.

Theorem 1.1. *Let q be an ultrafilter of a complete Boolean algebra $\mathcal{B}$ with countably chain condition (ccc). Let p_1 be an ultrafilter of a complete Boolean algebra $\mathcal{A}_1$ and let p_2 be an ultrafilter of a complete Boolean algebra $\mathcal{A}_2$. If $p_1 \sqsubseteq q$ and $p_2 \sqsubseteq q$, then $p_1 \sqsubseteq p_2$ or $p_2 \sqsubseteq p_1$.*

2 Nonstandard universes

Given a set X, we define the *iterated power set* $V_n(X)$ over X recursively by $V_0(X) = X$ and $V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X))$. The *superstructure* $V(X)$ is the union $\bigcup_{n<\omega} V_n(X)$. The set X is said to be a *base set* if $\varnothing \notin X$ and each element of X is disjoint from $V(X)$. From now on, we reserve the letter X for infinite base set. In a substructure W of a superstructure $V(X)$, the elements of $W \setminus X$ are called *sets relative to* W. We denote the model $\langle V(X), \in\rangle$ for the language $\mathcal{L}_\in = \{\in\}$ of set theory by the same symbol $V(X)$. A *Δ_0-formula* of $\mathcal{L}_\in$ is a formula obtained from atomic formulae using connectives and bounded quantifiers such as $\forall x \in y$ and $\exists x \in y$. Let $\mathfrak{M} = \langle M, \epsilon_M\rangle$ and $\mathfrak{N} = \langle N, \epsilon_N\rangle$ be models for $\mathcal{L}_\in$. We say that $\mathfrak{M}$ is a *bounded elementary submodel* of $\mathfrak{N}$ if $\mathfrak{M}$ is a submodel of $\mathfrak{N}$ and for any Δ_0-formula $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L}_\in$ and for any $a_1, \ldots, a_n \in M$, $\mathfrak{M} \models \varphi(a_1, \ldots, a_n)$ if and only if $\mathfrak{N} \models \varphi(a_1, \ldots, a_n)$. A map $e\colon M \to N$ is said to be a *bounded elementary embedding* of $\mathfrak{M}$ into $\mathfrak{N}$ if e is an isomorphic embedding of $\mathfrak{N}$ onto a bounded elementary submodel of $\mathfrak{M}$.

A *nonstandard universe* is a triple $\langle V(X), V(Y), \star\rangle$ such that:

(1) X and Y are infinite base sets.

(2) **(Transfer Principle)** The map $\star$ is a bounded elementary embedding of $V(X)$ into $V(Y)$.

(3) ${}^\star X = Y$.

(4) For every infinite subset of A of X, $\{{}^\star a \mid a \in A\}$ is a proper subset of ${}^\star A$.

Ignoring the condition (4), we call the triple a *pre-nonstandard universe*. By the condition (3), we can denote by $\langle V(X), V({}^\star X), {}^\star\rangle$ a pre-nonstandard universe.

We call an element $a \in V({}^\star X)$ *standard* if there is an $x \in V(X)$ such that $a = {}^\star x$. We call an element $a \in V({}^\star X)$ *internal* if there is an $x \in V(X)$ such that $a \in {}^\star x$. We denote by ${}^\star V(X)$ the set of all internal elements in $V({}^\star X)$. The model ${}^\star V(X)$ is a transitive submodel of $V({}^\star X)$. Then, we can simply denote by single ${}^\star V(X)$ pre-nonstandard universe. The *norm (of standardness)* $\mathrm{nos}(a)$ of an internal element a is a cardinal defined by

$$\mathrm{nos}(a) = \min\left\{|x| \;\middle|\; a \in {}^\star x\right\}.$$

The *radius* of ${}^\star V(X)$ is a cardinal defined by

$$\mathrm{rad}({}^\star V(X)) = \min\left\{\kappa \;\middle|\; \forall y \in {}^\star V(X)\ \ \mathrm{nos}(y) < \kappa\right\}.$$

We call an element of $V({}^\star X)$ *external* if it is not internal. For the general theory of nonstandard universes, we refer to [1].

Definition 2.1 (Standardizable bounded elementary embedding). Let ${}^\star V(X)$ and ${}^\bullet V(X)$ be pre-nonstandard universes. A bounded elementary embedding $e\colon {}^\star V(X) \to {}^\bullet V(X)$ is *standardizable* if $e^{-1}\text{“}\, y \in {}^\star V(X)$ for any set y relative to ${}^\bullet V(X)$. We call $e^{-1}\text{“}\, y$ *the standardization of* y *by* e if $e^{-1}\text{“}\, y \in {}^\star V(X)$.

Definition 2.2 (End extension). Let $\langle N, <\rangle$ be a structure inside $V(X)$ which is isomorphic to $\langle \omega, <\rangle$. A bounded elementary embedding $e\colon {}^\star V(X) \to {}^\bullet V(X)$ is an *end extension* if $\{e(x) \mid x < \nu\} = \{y \mid y < e(\nu)\}$ for any $\nu \in {}^\star N$.

Lemma 2.1. *A standardizable bounded elementary embedding of pre-nonstandard universe is an end extension.*

Proof. Easy. □

3 Boolean valued superstructures

Definition 3.1 (Bounded Boolean ultrapower). Let $\mathcal{A}$ be a complete Boolean algebra. We denote

$$\widehat{V}(X)^{\langle\mathcal{A}\rangle} = \left\{ u\colon V(X) \to \mathcal{A} \;\middle|\; \begin{array}{l} u(x) \wedge u(y) = \mathbf{0} \text{ for } x \neq y, \\ \bigvee u\text{“}\, V_n(X) = \mathbf{1} \text{ for some } n < \omega \end{array} \right\}.$$

Let p be an ultrafilter of $\mathcal{A}$. The *bounded Boolean ultrapower* $\widehat{V}(X)^{\langle\mathcal{A}\rangle}/p$ *modulo* p *of* $\mathcal{A}$ is the set of equivalence class by p with the membership relation.

We can regard a bounded Boolean ultrapower as a pre-nonstandard universe by Mostowski collapsing. The bounded elementary embedding of $V(X)$ into $\widehat{V}(X)^{\langle\mathcal{A}\rangle}/p$ is given by $\check{x}/p$ where $\check{x} = V(X) \times \{\mathbf{1}_{\mathcal{A}}\}$.

Definition 3.2 (Bounded ultrasheaf). Let $\mathcal{A}$ be a complete Boolean algebra. We denote

$$\begin{aligned} V_0(X)^{(\mathcal{A})} &= \left\{ u\colon \bigcup X \to \mathcal{A} \;\middle|\; \bigvee_{x \in X} \left(\bigvee u\text{“}\, x \wedge \neg \bigwedge u\text{“}\, \left(\bigcup X \setminus x\right)\right) = \mathbf{1} \right\}, \\ V_{n+1}(X)^{(\mathcal{A})} &= \{\, u \mid \mathrm{dom}\, u \subseteq V_n(X),\ \mathrm{rng}\, u \subseteq \mathcal{A} \,\}, \\ \widehat{V}(X)^{(\mathcal{A})} &= \bigcup_{n<\omega} V_n(X)^{(\mathcal{A})}. \end{aligned}$$

Then $\widehat{V}(X)^{(\mathcal{A})}$ is equipped with $\mathcal{A}$-valued membership relation. Let p be an ultrafilter of $\mathcal{A}$. The *bounded Boolean ultrasheaf* $\widehat{V}(X)^{(\mathcal{A})}/p$ *modulo* p *of* $\mathcal{A}$ is the set of equivalence class by p with the membership relation.

We can regard a bounded ultrasheaf as a pre-nonstandard universe by Mostowski collapsing. Then we can regard the bounded Boolean ultrapower $\widehat{V}(X)^{\langle\mathcal{A}\rangle}/p$ is a substructure of the bounded ultrasheaf $\widehat{V}(X)^{(\mathcal{A})}/p$. For the general theory of bounded ultrasheaves, we refer to [6].

Bounded Boolean ultrapower is constructed by equivalence relation "=". As the Boolean valued universe $V^{(\mathcal{B})}$ of set theory, bounded ultrasheaf is constructed by membership relation "$\in$". Membership relation "$\in$" is stronger than equivalence relation "=".

Definition 3.3 (Standardizable bounded elementary embedding of ultrasheaves). Let $\widehat{V}(X)^{(\mathcal{A})}/p$ and $\widehat{V}(X)^{(\mathcal{B})}/q$ be ultrasheaves. A bounded elementary embedding $e\colon \widehat{V}(X)^{(\mathcal{A})}/p \to \widehat{V}(X)^{(\mathcal{B})}/q$ is *standardizable* if $e^{-1}\text{“}y \in \widehat{V}(X)^{(\mathcal{A})}/p$ for any set y relative to $\widehat{V}(X)^{(\mathcal{B})}/q$. We call $e^{-1}\text{“}y$ *the standardization of* y *by* e if $e^{-1}\text{“}y \in \widehat{V}(X)^{(\mathcal{A})}/p$.

Theorem 3.1 (Characterization theorem of the Rudin-Frolík order [5]). *Let p be an ultrafilter of a complete Boolean algebra $\mathcal{A}$ and let q be an ultrafilter of a complete Boolean algebra $\mathcal{B}$. Suppose $\mathcal{A}$, $\mathcal{B} \in V(X) \setminus X$. Then, $p \sqsubseteq q$ if and only if there is a standardizable bounded elementary embedding $e\colon \widehat{V}(X)^{(\mathcal{A})}/p \to \widehat{V}(X)^{(\mathcal{B})}/q$ of ultrasheaves.*

Proof. See [5, Theorem 5.3]. □

Lemma 3.2. *Let $e\colon \widehat{V}(X)^{(\mathcal{A})}/p \to \widehat{V}(X)^{(\mathcal{B})}/q$ be a standardizable bounded elementary embedding. Then $e{\restriction}\widehat{V}(X)^{\langle\mathcal{A}\rangle}/p$ is an end extension.*

Proof. As easy as that of Lemma 2.1. □

Theorem 3.3. *Let $e_0\colon \widehat{V}(X)^{\langle\mathcal{A}\rangle}/p \to \widehat{V}(X)^{\langle\mathcal{B}\rangle}/q$ be a bounded elementary embedding. Suppose $|\mathcal{A}|$, $|\mathcal{B}| < |V(X)| = \beth_\omega(|X|)$, then e_0 can be extended to a unique bounded elementary embedding $e\colon \widehat{V}(X)^{(\mathcal{A})}/p \to \widehat{V}(X)^{(\mathcal{B})}/q$ of ultrasheaves.*

I am sorry to omit the proof of the theorem above, because it is not enough space to state it — it costs at least 12 pages. The theorem seems to be deep, which suggests that every pre-nonstandard universe has the generic extension *a priori*. If you are interested in the theorem, please email me.

4 Proof of Theorem 1.1

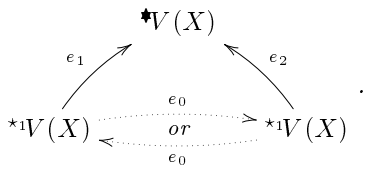
Lemma 4.1. *Let $\mathcal{B}$ be a complete Boolean algebra with ccc. Let p be an ultrafilter of a complete Boolean algebra $\mathcal{A}$ and let q be an ultrafilter of $\mathcal{B}$. Suppose $p \sqsubseteq q$. Then there is $a \in p$ such that $\mathcal{A}{\restriction}a$ satisfies ccc.*

Proof. Choose $a \in p$, $b \in q$, a complete homomorphism $h\colon \mathcal{A} \to \mathcal{B}{\restriction}b$ and an epimorphism $f\colon \mathcal{B} \to \mathcal{A}{\restriction}a$ satisfying that $f \circ h(x) = x$ for all $x \in \mathcal{A}{\restriction}a$ and $f^{-1}\text{“}p = q$. Let P be a partition of unity of $\mathcal{A}{\restriction}a$. Since $h\text{“}P$ is a partition of unity of $\mathcal{B}$, we have $|P| \leq \aleph_0$. □

Lemma 4.2. *Let $\mathcal{A}$ be a complete Boolean algebra with ccc and let p be an ultrafilter of $\mathcal{A}$. Then the radius of the bounded Boolean ultrapower is at most $\aleph_1$.*

Proof. For $u/p \in \widehat{V}(X)^{\langle\mathcal{A}\rangle}$, put $I = \{x \in V(X) \mid u(x) \neq \mathbf{0}_{\mathcal{A}}\}$, then I is a set relative to $V(X)$. Since $u\text{“}I$ is a partition of unity of $\mathcal{A}$, $|I|$ is at most countable. Then $[\![u \in \check{I}]\!] = \mathbf{1}_{\mathcal{A}} \in p$. Thus u/p is an element of $\check{I}/p$ where $|I|$ is at most countable. □

Lemma 4.3. *Let ${}^{\star_1}V(X)$ and ${}^{\star_2}V(X)$ be pre-nonstandard universes with the radii at most $\aleph_1$. Let $e_1\colon {}^{\star_1}V(X) \to {}^{\bullet}V(X)$ and $e_2\colon {}^{\star_2}V(X) \to {}^{\bullet}V(X)$ be end extensions. Then either $\operatorname{rng} e_1 \subseteq \operatorname{rng} e_2$ or $\operatorname{rng} e_2 \subseteq \operatorname{rng} e_1$. When $\operatorname{rng} e_1 \subseteq \operatorname{rng} e_2$, there is a bounded elementary embedding $e_0\colon {}^{\star_1}V(X) \to {}^{\star_2}V(X)$ such that $e_1 = e_2 \circ e_0$. When $\operatorname{rng} e_2 \subseteq \operatorname{rng} e_1$, there is a bounded elementary embedding $e_0\colon {}^{\star_2}V(X) \to {}^{\star_1}V(X)$ such that $e_2 = e_1 \circ e_0$.*



Proof. Let $\langle N, <\rangle$ be a structure inside $V(X)$ which is isomorphic to $\langle \omega, <\rangle$. Since e_1 and e_2 are end extensions, we have either $e_1 \text{“} {}^{\star_1}N \subseteq e_2 \text{“} {}^{\star_2}N$ or $e_2 \text{“} {}^{\star_2}N \subseteq e_1 \text{“} {}^{\star_1}N$. Suppose $e_1 \text{“} {}^{\star_1}N \subseteq e_2 \text{“} {}^{\star_2}N$. We shall show $\operatorname{rng} e_1 \subseteq \operatorname{rng} e_2$. Let $a \in {}^{\star_1}V(X)$. Since the radius of ${}^{\star_1}V(X)$ is at most $\aleph_1$, there is a countable set I relative to $V(X)$ such that $a \in {}^{\star_1}I$. Let $\tilde{a}\colon N \to I$ be a bijection. Note that $\tilde{a} \in V(X)$ and that ${}^{\star_1}\tilde{a}\colon {}^{\star_1}N \to {}^{\star_1}I$ is a bijection. Then there are $\nu_1 \in {}^{\star_1}N$ and $\nu_2 \in {}^{\star_2}N$ such that $a = \big({}^{\star_1}\tilde{a}\big)(\nu_1)$ and $e_1(\nu_1) = e_2(\nu_2)$. We obtain that $e_1(a) = e_2\big(\big({}^{\star_2}\tilde{a}\big)(\nu_2)\big)$. In fact, since $e_1({}^{\star_1}\tilde{a}) = e_2({}^{\star_2}\tilde{a})$ and $e_1(\nu_1) = e_2(\nu_2)$, we have

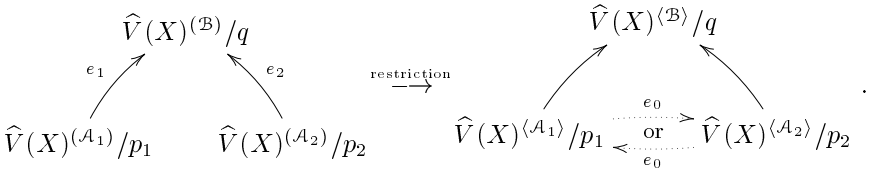
$$\begin{aligned} \big(e_1({}^{\star_1}\tilde{a})\big)\big(e_1(\nu_1)\big) &= \big(e_2({}^{\star_2}\tilde{a})\big)\big(e_2(\nu_2)\big), \\ e_1(a) = e_1\big(\big({}^{\star_1}\tilde{a}\big)(\nu_1)\big) &= e_2\big(\big({}^{\star_2}\tilde{a}\big)(\nu_2)\big). \end{aligned}$$

Thus we have obtained that $\operatorname{rng} e_1 \subseteq \operatorname{rng} e_2$. Now we can define the map $e_0\colon {}^{\star_1}V(X) \to {}^{\star_2}V(X)$ such that $e_1 = e_2 \circ e_0$. Let $\varphi(x_1, \ldots, x_n)$ be a Δ_0-formula of $\mathcal{L}_\in$, and let $a_1, \ldots, a_n \in {}^{\star_1}V(X)$. Since both e_1 and e_2 are bounded elementary embeddings, we get

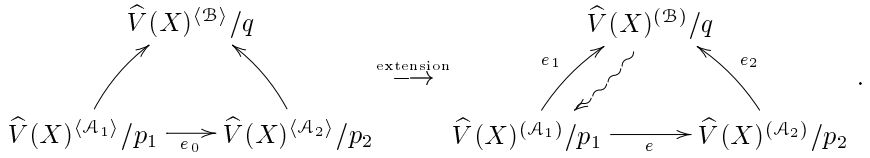
$$\begin{aligned} {}^{\bullet}V(X) \models \varphi\big(e_1(a_1), \ldots, e_1(a_n)\big) &\iff {}^{\bullet}V(X) \models \varphi\big(e_2(e_0(a_1)), \ldots, e_2(e_0(a_n))\big), \\ {}^{\star_1}V(X) \models \varphi(a_1, \ldots, a_n) &\iff {}^{\star_2}V(X) \models \varphi\big(e_0(a_1), \ldots, e_0(a_n)\big). \end{aligned}$$

Thus we e_0 is a bounded elementary embedding. Exchanging $\star_1$ for $\star_2$, we have the proof in case $e_2 \text{“} {}^{\star_2}N \subseteq e_1 \text{“} {}^{\star_1}N$. □

Proof of Theorem 1.1. From Lemma 4.1, we can assume both $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy ccc. Choose a base set X such that $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B} \in V(X) \setminus X$. By the characterization theorem of the Rudin-Frolík order, there are standardizable bounded elementary embeddings of ultrasheaves $e_1\colon \widehat{V}(X)^{(\mathcal{A}_1)}/p_1 \to \widehat{V}(X)^{(\mathcal{B})}/q$ and $e_2\colon \widehat{V}(X)^{(\mathcal{A}_2)}/p_2 \to \widehat{V}(X)^{(\mathcal{B})}/q$. Then, from Lemma 3.2, both $e_1 | \widehat{V}(X)^{\langle \mathcal{A}_1 \rangle}/p_1$ and $e_2 | \widehat{V}(X)^{\langle \mathcal{A}_2 \rangle}/p_2$ are end extensions.



Since both $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy ccc, the radii of the bounded Boolean ultrapowers $\widehat{V}(X)^{\langle \mathcal{A}_1 \rangle}/p_1$ and $\widehat{V}(X)^{\langle \mathcal{A}_2 \rangle}/p_2$ are at most $\aleph_1$ by Lemma 4.2. Then, from Lemma 4.3, there is a bounded elementary embedding $e_0\colon \widehat{V}(X)^{\langle \mathcal{A}_1 \rangle}/p_1 \to \widehat{V}(X)^{\langle \mathcal{A}_2 \rangle}/p_2$ or $e_0\colon \widehat{V}(X)^{\langle \mathcal{A}_2 \rangle}/p_2 \to \widehat{V}(X)^{\langle \mathcal{A}_1 \rangle}/p_1$ that makes the diagram above commute. Suppose that there is $e_0\colon \widehat{V}(X)^{\langle \mathcal{A}_1 \rangle}/p_1 \to \widehat{V}(X)^{\langle \mathcal{A}_2 \rangle}/p_2$ such that $e_1 = e_2 \circ e_0$. From Theorem 3.3, let $e\colon \widehat{V}(X)^{(\mathcal{A}_1)}/p_1 \to \widehat{V}(X)^{(\mathcal{A}_2)}/p_2$ be a unique extension of e_0. Since $e_2 \circ e$ is a bounded elementary embedding of ultrasheaves, $e_2 \circ e$ must coincide with e_1.



Then e is standardizable. In fact, let y be a set relative to $\widehat{V}(X)^{(\mathcal{A}_2)}/p_2$ and let $s \in \widehat{V}(X)^{(\mathcal{A}_1)}/p_1$ be the standardization of $e_2(y)$ by e_1. We have that, for any $x \in \widehat{V}(X)^{(\mathcal{A}_1)}/p_1$,

$$\begin{aligned} e(x) \in y &\iff e_1(x) = e_2(e(x)) \in e_2(y), \\ x \in e^{-1} \text{“} y &\iff x \in {e_1}^{-1} \text{“} e_2(y) = s. \end{aligned}$$

Thus s is the standardization of y by e. We have shown e is standardizable. Again by the characterization theorem of the Rudin-Frolík order, we have $p_1 \sqsubseteq p_2$. In case there is $e_0\colon \widehat{V}(X)^{\langle \mathcal{A}_2 \rangle}/p_2 \to \widehat{V}(X)^{\langle \mathcal{A}_1 \rangle}/p_1$, we conclude $p_2 \sqsubseteq p_1$ similarly. □

5 Conjectures

We have proved that the initial segments of Rudin-Frolík order of the ultrafilters of the complete Boolean algebras with ccc are totally ordered. Now we consider eliminating conditions completeness or ccc.

Against completeness, we have to consider the general nonstandard universes and extend the definition of Rudin-Frolík order. This means the order completion of Rudin-Frolík order. I think that eliminating the condition completeness must be possible.

Against ccc, I state a conjecture here.

Conjecture 5.1. *Under the first measurable cardinal, the initial segments Rudin-Frolík order are totally ordered.*

In the conjecture above, the first measurable cardinal is derived from that every σ-complete homomorphism of Boolean algebras of sets is complete.

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Atomic incompleteness

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1 Introduction

A standard counterexample in modal logic looks somewhat like this: first conctruct a fancy Kripke frame, then take (some variation on) the algebra of finite and cofinite sets on this frame. It works. Algebraically, one outstanding feature of the algebra produced that way is that it is atomic. This may suggest the following, rather informally stated, conjecture: For any property P that does not hold of all modal logics, P can be falsified on an atomic modal algebra. True to the spirit of the great diagonalisation tradition we will exhibit a property P such that (1) if it does not hold of all modal logics then, by its very nature, it cannot be falsified on an atomic modal algebra, and (2) it indeed does not hold of all modal logics.

Let Λ be a modal logic, ϕ a formula. Further, let $\mathcal{V}(\Lambda)$ stand for the variety of modal algebras corresponding to Λ. Consider the following property:

(P) if $\phi \notin \Lambda$, then there is an atomic algebra $\mathbf{A}$ in $\mathcal{V}(\Lambda)$, with $\mathbf{A} \not\models \phi$.

Clearly, if there is a modal logic Λ that does not posses (P), then any atomic modal algebra is either not in $\mathcal{V}(\Lambda)$, or else it satisfies ϕ, hence (1) above indeed holds for our (P). All that remains is to show that (P) is not true of all modal logics Λ. Notice, that (P) expressed algebraically reads: $\mathcal{V}(\Lambda)$ is generated by its atomic members. Thus, we need to show that there is a variety of modal algebras not generated by its atomic members. This we will do in two stages. First, we will explicitly construct a variety $\mathcal{V}$ of tense algebras not generated by its atomic members. Then, we employ Kracht and Wolter's simulation technique to produce a variety $\mathcal{V}^S$ of modal algebras with the same property.

2 Construction

We now proceed to construct a variety $\mathcal{V}$ of tense algebras which is not generated by its atomic members. Our $\mathcal{V}$ will be generated by a single algebra that is best described in terms of the frame defined below.

For each $j \in \omega$ and $i \in \{0, \dots, 2^j - 1\}$ define $U_{i,j}$ to be $\{k : k \equiv i \pmod{2^j}\}$. Let $\mathbb{Z}$ be the usual set of integers and ∞ an object not in $\mathbb{Z}$. Put $\mathbb{U} = \{\infty\} \cup \bigcup_{i,j \in \mathbb{Z}} U_{i,j}$. The set of points of our frame will be $W = \mathbb{U} \cup \mathbb{Z}$. Then define the accessibility relation $\lhd$ as follows:

- $k \lhd U_{i,j}$ iff $k \in U_{i,j}$ and $i \leq (2^j - 1)/2$,
- $U_{i,j} \lhd k$ iff $k \in U_{i,j}$ and $i > (2^j - 1)/2$,
- $U_{i,j} \lhd \infty$, for all $U_{i,j}$,
- $U_{i,j} \lhd U_{k,l}$ iff $U_{i,j} \subset U_{k,l}$.

Put $\mathfrak{f} = \langle W, \lhd, \lhd^{-1}, I \rangle$, with the family I of internal sets defined as follows: $A \in I$ iff $A = B \cup C$, where B is a finite or cofinite subset of $\mathbb{U}$ and C is a finite union of some $U_{i,j}$. Note the double role that sets $U_{i,j}$ play: they are both points of the frame (as members of $\mathbb{U}$) and internal sets (as subsets of $\mathbb{Z}$).

Let $\mathbf{I}$ stand for the algebra of internal sets of the frame above. By means of a general algebraic argument, applicable to many more cases, we can prove:

Theorem 1 *The variety $\mathcal{V}(\mathbf{I})$ is a minimal variety of tense algebras. It is not generated by its atomic members, in fact, it has none.*

3 Lifting

Now, we lift the result from the previous section to the case of modal algebras. To do so, we make use of the simulation technique, originated in [1], [2] and developed in [3]. Intuitively, the technique works by associating with each polymodal frame a unimodal frame that consists of as many copies of the original frame as there were modal operators on it. These copies (with some additional points) form the universe of the new *simulation frame*, and the modal operator is so defined that: (a) it links the copies in a special way that allows to distinguish between them by constant formulas, (b) it behaves on each copy in exactly the way one of the original operators behaved on the original frame.

The same construction can be performed on any polymodal algebra $\mathbf{A}$ yielding a *simulation algebra* $\mathbf{A}^S$. Simulation algebras form a rather well-behaved class. They are equationally definable, and given a unimodal simulation algebra $\mathbf{B}$, the polymodal algebra $\mathbf{A}$ of which $\mathbf{B}$ is a simulation can be read off the structure of $\mathbf{B}$.

For any class $\mathcal{K}$ of algebras, let $H(\mathcal{K})$, $S(\mathcal{K})$, $P_U(\mathcal{K})$ stand, respectively, for the classes of homomorphic images, subalgebras, and ultraproducts of algebras from $\mathcal{K}$. The key to our lifting procedure is the following slight strengthening of [4] Corollary 9.6:

Lemma 1 *If $\mathcal{K}$ is a class of simulation algebras, so is $SP_U(\mathcal{K})$.*

Let $\mathcal{V}^S$ stand for the variety generated by the class of all simulation algebras of algebras from $\mathcal{V}$, in other words: $\mathcal{V}^S = HSP(\{\mathbf{A}^S : \mathbf{A} \in \mathcal{V}\})$. With the help of the lemma above, we prove:

Theorem 2 *The variety $\mathcal{V}^S(\mathbf{I})$ is not generated by its atomic members.*

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On Quantales

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Introduction

Propositional system and Hilbert space

We shall call a **question** every experiment leading to alternative of which the terms are "yes" and "no". When the physical system has been prepared in such a way that physicist may affirm that in the event of an experiment the result will be "yes", we shall say that **the question is true**. For certain pairs of questions β, γ, one may have following property: "If the physical system is prepared in such a way that β is ture, then one is sure that γ is ture." We shall denote it as $\beta < \gamma$. By definition, two questions β and γ are equivalent if one has $\beta < \gamma$ and $\gamma < \beta$. We shall name a **proposition**, and denote by b, the equivalent class containing the quesiton β. Then, the set of propositions $\mathcal{L}$ is a lattice which is complete, orthocomplemented, and weakly moduar. We call it a **propositional system**. In propositional system $\mathcal{L}$, two propositions b and c are said to be **compatible** if the sublattice generated by $\{b, b^{\perp}, c, c^{\perp}\}$ is distributive.

Let $\mathbb{P}(H)$ be the set of projection of Hilbert space H. If $P_1, P_2 \in \mathbb{P}(H)$and $K = P_1(H), L = P_2(H)$, then we define

1. $P_1 < P_2$ if and only if $K \subset L$
2. $P_1 \vee P_2$ is projection of H onto $K \vee L$.
3. $P_1 \wedge P_2$ is projection of H onto $K \cap L$.

Then, $\mathbb{P}(H)$ is complete lattice, and it is easy to verify the weak modularity relation. In $\mathbb{P}(H)$, **commutativity** is completely equivalent to compatibility. Thus, $\mathbb{P}(H)$ is the lattice which is which is complete, orthocomplemented, and weakly moduar.

By generalizing these results, the propositional systems can always realized by the lattice of closed subspaces of a generalized Hilbert space.

A Quantale

A **quantale** is a complete lattice Q with an associative binary operation & which is distribution over arbitrary joins:

$$a\& \bigvee S = \bigvee \{a\&s \mid s \in S\} \quad (\bigvee S)\&a = \bigvee \{s\&a \mid s \in S\}$$

Now, we consider a quantale $Q(\mathbb{P}(H))$ where

$$Q(\mathbb{P}(H)) = \{\varphi : \mathbb{P}(H) \to \mathbb{P}(H) \mid (\bigvee_i P_i)\varphi = \bigvee_i (P_i)\varphi, \forall \{P_i\} \subset \mathbb{P}(H)\}$$

and we shall show that $\mathbb{P}(H)$ is embedded in $Q(\mathbb{P}(H))$. The multiplication operater & on $Q(\mathbb{P}(H))$ is interpreted as " **and then** ".

We review the quantales in C.J.Mulvey and J.W.Pelletier[4] in this paper.

Quantales

Definition 1 A complete lattice Q with an associative binary operation $\&$ is called a **quantale** if

$$\begin{aligned} a\&(\bigvee b_i) &= \bigvee(a\&b_i) \\ (\bigvee a_i)\&b &= \bigvee(a_i\&b) \\ & \forall a, b, a_i, b_i \in Q. \end{aligned}$$

We denote the top and the bottom elements of Q by 1 and 0 respectively,

$$\bigvee Q = 1, \quad \bigvee \emptyset = 0.$$

Definition 2 A quantale Q is said to be **unital** if there exits a unit element $e \in Q$ satisfying

$$e\&a = a = a\&e \quad \forall a \in Q.$$

Definition 3 A quantale Q with a unary operation *, $\langle Q, * \rangle$ is said to be **involutive** if

$$\begin{aligned} a^{**} &= a \\ (a\&b)^* &= b^*\&a^* \\ (\bigvee a_i)^* &= \bigvee a_i^*. \end{aligned}$$

Definition 4 An element $a \in Q$ of quantale Q is said to be **right-sided** if

$$\forall b \in Q(a\&b \leq a).$$

Similary, $a \in Q$ of quantale Q is said to be **left-sided** if $b\&a \leq a$ for all $b \in Q$. We denote $R(Q)$ and $L(Q)$, respectively the subsets of right-sided and of left-sided elements of Q.

Definition 5 A unital and involutive quantale $\langle Q, \&, *, e \rangle$ is called a **Gelfand quntale** if

$$a\&a^*\&a = a \quad \forall a \in R(Q).$$

The right(left)-adjoint of $\&$ is denoted by $\underset{r}{\to}$ ($\underset{l}{\to}$):

$$(a \underset{r}{\to} b) = \bigvee\{x \in R(Q) \mid a\&x \leqslant b\}$$

$$(a \underset{l}{\to} b) = \bigvee\{x \in L(Q) \mid x\&a \leqslant b\}$$

Then we have

$$a\&(a \underset{r}{\to} b) \leqslant b; \quad (a \underset{l}{\to} b)\&a \leqslant b.$$

If $a \in R(Q)$, then $a^* \underset{r}{\to} 0$ is $R(Q)$ and denoted by $a^{\perp}$; if $a \in L(Q)$, then $a^* \underset{l}{\to} 0$ is $L(Q)$ and denoted by ${}^{\perp}a$.

lemma 6 ([4]) For any Gelfand quantale Q, the $R(Q)$ of right-sided elements is a quantale, and has the following properties:

1. $a\&a = a$

2. $a \wedge b \leq a^* \& b$

3. $a \wedge a^\perp = 0$

for every $a, b \in Q$.

Proof

quantale: For any $a_i \in R(Q) \forall i$ and any $c \in Q$, then

$$\begin{aligned} (\bigvee_i a_i)\&c &= \bigvee_i (a_i \& c) \\ &\leq \bigvee_i a_i. \end{aligned}$$

Then, $\bigvee_i a_i \in R(Q)$. Similary, $\bigwedge_i a_i \in R(Q)$. Thus, $R(Q)$ is a complete lattice. Since $R(Q) \subseteq Q$, $R(Q)$ is clearly a quantale.

a&a=a: For any $a \in R(Q)$, we have $a\&a \leq a = a\&a^*\&a \leq a\&a$.

a$\wedge$b$\leq$ a*&b: Since $a^*\&a$ is left-sided, $a = a\&a^*\&a \leq 1\&a^*\&a \leq a^*\&a$. Thus,

$$a \wedge b \leq (a \wedge b)^* \& (a \wedge b) \leq a^* \& b.$$

a$\wedge$a$^\perp$ =0:

$$a \wedge a^\perp \leq a^* \& a^\perp = a^* \& (a^* \underset{r}{\to} 0) = 0$$

□

Theorem 7 ([4]) For some Hilbert space H, $Q(\mathbb{P}(H))$ is the quantale of sup-preserving mappings from sup-lattice $\mathbb{P}(H)$ to itself.

$$Q(\mathbb{P}(H)) = \{\varphi : \mathbb{P}(H) \to \mathbb{P}(H) \mid (\bigvee_i P_i)\varphi = \bigvee_i (P_i)\varphi, \forall \{P_i\} \subset \mathbb{P}(H)\}$$

forms a Gelfand quantale, where

$$\begin{aligned} \varphi \leq \psi &\overset{\text{def}}{\iff} (M)\varphi \leq (M)\psi \quad \text{for all} M \in \mathbb{P}(H) \\ (M)(\bigvee_i \varphi_i) &\overset{\text{def}}{=} \bigvee_i (M)\varphi_i \quad \text{for all } M \in \mathbb{P}(H) \\ (M)\varphi\&\psi &\overset{\text{def}}{=} ((M)\varphi)\psi \quad \text{for all } M \in \mathbb{P}(H) \\ (M)\varphi^* &\overset{\text{def}}{=} (\bigvee \{N \in \mathbb{P}(H) \mid (N)\varphi \leq M^\perp\})^\perp \quad \text{for all } M \in \mathbb{P}(H) \\ (M)e &\overset{\text{def}}{=} M \quad \text{for all } M \in \mathbb{P}(H). \end{aligned}$$

Proof We define λ_M, for $M \in \mathbb{P}(H)$ by

$$(N)\lambda_M = \begin{cases} 0 & \text{if } N \leq M \\ H & \text{otherwise.} \end{cases}$$

It is clear that λ_M is a sup-preserving mapping from $\mathbb{P}(H)$ to $\mathbb{P}(H)$ and that

$$\lambda_M \& \varphi \leq \lambda_M$$

for all $\varphi \in Q(\mathbb{P}(H))$. Moreover, if φ is any right-sided element of $Q(\mathbb{P}(H))$, we must

$$\varphi \& 1 \leq \varphi$$

where

$$(N)1 = \begin{cases} 0 & \text{if } N = 0 \\ H & \text{otherwise.} \end{cases}$$

Then, clearly φ must be of the form λ_M, where

$$M = \bigvee\{N \in \mathbb{P}(H) \mid (N)\varphi = 0\}.$$

Thus, sup-preserving mappings of the form λ_M are the right-sided elemnts of $Q(\mathbb{P}(H))$. Similary, sup-preserving mappings of the form κ_M are the left-sided elemnts of $Q(\mathbb{P}(H))$, where

$$(N)\kappa_M = \begin{cases} 0 & \text{if } N = 0 \\ M & \text{otherwise.} \end{cases}$$

Now we shall show that $Q(\mathbb{P}(H))$ is Gelfand quantale, that is, that

$$\lambda_M \& \lambda_M^* \& \lambda_M = \lambda_M$$

for all $M \in \mathbb{P}(H)$, turns on the fact $\lambda_M^* = \kappa_{M^\perp}$. For, by definition

$$(N)\lambda_M^* = (\bigvee_{(L)\lambda_M \leq N^\perp} L)^\perp.$$

Since $(L)\lambda_M \leq N^\perp$ when $N = 0$ and L is any projection on H or when $N \neq 0$ and $L \leq M$, we see that for $N = 0$

$$(0)\lambda_M^* = H^\perp = 0$$

and for $N \neq 0$

$$(N)\lambda_M^* = M^\perp$$

which shows that

$$\lambda_M^* = \kappa_{M^\perp}.$$

This completes the proof of the thorem, since it is easy to see that $\lambda_M \& \kappa_{M^\perp} \& \lambda_M = \lambda_M$. □

From this proof, the operation * is an isomorphism from $R(Q(\mathbb{P}(H)))$ onto $L(Q(\mathbb{P}(H)))$.

Theorem 8 ([4]) For any Hilbert space H, the mapping

$$\chi : \mathbb{P}(H) \to R(Q(\mathbb{P}(H)))$$

defined by

$$(M)\chi = \lambda_{M^\perp}$$

is an isomorphism of orthocomplemented sup-lattice.

Proof First, we shall show surjective. From previous theorem's proof, if $\varphi \in R(Q(\mathbb{P}(H)))$, then

$$\begin{aligned} \varphi &= \lambda_M \\ M &= \bigvee\{N \in \mathbb{P}(H) \mid (N)\varphi = 0\}. \end{aligned}$$

Since $\mathbb{P}(H)$ is orthocomplemented sup-lattice, we have $M, M^\perp \in \mathbb{P}(H)$. That is,

$$\exists M^\perp \in \mathbb{P}(H)(M^\perp)\chi = \lambda_{M^{\perp\perp}} = \lambda_M.$$

Therefore, mapping χ is surjective. It is injective, because if $M \leq N$ and $M \neq N$, then

$$(M^\perp)\lambda_{M^\perp} = 0, \quad (M^\perp)\lambda_{N^\perp} = 1$$

and if $M \not\leq N$, then

$$(N^\perp)\lambda_{M^\perp} = 1, \quad (N^\perp)\lambda_{N^\perp} = 0.$$

The mapping χ preserves order, since

$$M \leq N \Rightarrow N^{\perp} \leq M^{\perp} \Rightarrow \lambda_{M^{\perp}} \leq \lambda_{N^{\perp}} \Rightarrow (M)\chi \leq (N)\chi.$$

Finally, we note that

$$(M^{\perp})\chi = \lambda_{M^{\perp\perp}} = (\lambda_{M^{\perp}})^{\perp} = ((M)\chi)^{\perp}.$$

□

From this theorem, it follows that $\mathbb{P}(H)$ is extended to Gelfand quantale $Q(\mathbb{P}(H))$. The quantale valued logic is the logic for propositions represented by Gelfand quantale $Q(\mathbb{P}(H))$.

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Quantale valued set theory

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Introduction

Observation of a physical system is described by using a Hilbert space. States of the physical system are represented as vectors of the Hilbert space H, and observables (physical quantities) are represented as self-adjoint operators on H. A proposition in quantum theory, which is considered as a property of states, are represented as a projection p on H such that

"The proposition is true in state x" is certain if and only if x is in the range of p

Projections on a Hilbert space H form a complete orthomodular lattice with respect to the inclusion of their ranges. We denote the complete orthomodular lattice by $\mathbb{P}(H)$.

In [4] and [6], G.Takeuti developed the quantum set theory on the $\mathbb{P}(H)$-valued universe $V^{\mathbb{P}(H)}$.

$$\begin{aligned} V_\alpha^{\mathbb{P}(H)} &= \{u \mid \exists \beta < \alpha\, \exists \mathcal{D}u \subset V_\beta^{\mathbb{P}(H)} (u : \mathcal{D}u \to \mathbb{P}(H))\} \\ V^{\mathbb{P}(H)} &= \bigcup_{\alpha \in On} V_\alpha^{\mathbb{P}(H)} \end{aligned}$$

In his quantum set theory, he defined implication $\underset{q}{\to}$ by

$$\varphi \underset{q}{\to} \psi \overset{\text{def}}{\Longleftrightarrow} \varphi^\perp \vee (\varphi \wedge \psi),$$

and defined the interpretation of equality $\underset{q}{=}$ by

$$\begin{aligned} [\![u \underset{q}{=} v]\!] &= \bigwedge_{x \in \mathcal{D}u} (u(x) \underset{q}{\to} [\![x \underset{q}{\in} v]\!]) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \underset{q}{\to} [\![x \underset{q}{\in} u]\!]) \\ [\![u \underset{q}{\in} v]\!] &= \bigvee_{x \in \mathcal{D}v} [\![u \underset{q}{=} x]\!] \wedge v(x). \end{aligned}$$

The operation $\underset{q}{\to}$ is an implication in the sense that

$$[\![\varphi \wedge (\varphi \underset{q}{\to} \psi)]\!] \leqslant [\![\psi]\!].$$

But, unfortunately,

$$[\![\varphi \wedge \psi]\!] \leqslant [\![\xi]\!] \quad \text{does not imply} \quad [\![\varphi]\!] \leqslant [\![\psi \underset{q}{\to} \xi]\!],$$

because of non-distributivity of $\mathbb{P}(H)$. It follows that the transitivity of $\underset{q}{=}$ is not valid in the quantum set theory on $V^{\mathbb{P}(H)}$.

The aim of this paper is to review the concepts of implication and equality in the universe $V^{\mathbb{P}(H)}$. We extend the orthomodular lattice $\mathbb{P}(H)$ to a quantale. Quantale is a complete lattice Q with associative binary operation & which is distributive over arbitrary joins (Definition 1). Then we introduce the right-adjoint $\underset{r}{\to}$ of & as an implication on Q. The equality $\underset{r}{=}$ corresponding to & satisfies the transitivity. The

quantale Q is defined as an algebra consisting of sup-preserving mappings $\varphi : p \mapsto (p)\varphi$ from $\mathbb{P}(H)$ to itself.

$$Q = \{\varphi : \mathbb{P}(H) \to \mathbb{P}(H) \mid (\bigvee P)\varphi = \bigvee\{(p)\varphi \mid p \in P\} \quad \text{for arbitrary } P \subset \mathbb{P}(H)\}.$$

Q is a Gelfand quantale with respect to composition & of operations, and $\mathbb{P}(H)$ is embedded in Q as the subquantale $R(Q)$ of right-sided elements (Theorem 1),

$$\mathbb{P}(H) \cong R(Q) = \{\varphi \in Q \mid \varphi\&1 \leqslant \varphi\} \subset Q,$$

The operator & on Q is interpreted as "and then".

In [7] we formulated a lattice valued logic and a lattice valued set theory, by introducing a strong implication, which represents the lattice order $\leqslant$. The lattice valued logic is the logical counterpart of general complete lattices. The completeness of the lattice valued logic was proved by Takano [5]. For an arbitrary complete lattice L, we can construct a L-valued universe V^L. The lattice valued set theory on the universe V^L was formulated as a set theory based on the lattice valued logic.

Now we take Q as the complete lattice L, and consider the quantale valued set theory V^Q. Q-valued logic is obtained from a lattice valued logic by adding new logical operations such as &, *, $\underset{r}{\to}$, $\underset{l}{\to}$, and new inference rules for them.

Since the orthocomplemented sup-lattice $\mathbb{P}(H)$ can be embedded in Q, $V^{\mathbb{P}(H)}$ is considered as a subuniverse of V^Q. Similarly, $\mathbb{P}(H)$ can be embedded in Q as the subquantale $L(Q)$ of left-sided elements,

$$L(Q) = \{\varphi \in Q \mid 1\&\varphi \leqslant \varphi\}.$$

This means that there are two isomorphic subuniverses $V^{R(Q)}$ and $V^{L(Q)}$ of V^Q, which are mutually conjugate.

In section 1 we briefly review the quantales in C.J.Mulvey and J.W.Pelletier [2], and then, in section 2, discuss the quantale valued set theory on V^Q.

1 Quantales

Definition 1 A sup-lattice (complete lattice) Q with an associative binary operation & is called a *quantale*, if the operation & is distributive over arbitrary joins:

$$a\&\bigvee S = \bigvee\{a\&s \mid s \in S\}, \qquad (\bigvee S)\&a = \bigvee\{s\&a \mid s \in S\} \quad \text{for } S \subset Q.$$

We denote the top and the bottom elements of Q by 1 and 0, respectively.

$$\bigvee Q = 1, \qquad \bigvee \emptyset = 0.$$

Definition 2 A quantale Q is said to be *unital* if there exists a unit element e in Q such that

$$e\&a = a = a\&e \quad \text{for all } a \in Q$$

Definition 3 An element $a \in Q$ of a quantale Q is said to be *right-sided* (respectively, *left-sided*) if

$$a\&1 \leqslant a \quad (\text{respectively, } 1\&a \leqslant a).$$

The set of right-sided elements (respectively, left-sided elements) of Q is denoted by $R(Q)$ (respectively, $L(Q)$).

$$R(Q) \stackrel{\text{def}}{=} \{a \in Q \mid a\&1 \leqslant a\}, \quad L(Q) \stackrel{\text{def}}{=} \{a \in Q \mid 1\&a \leqslant a\}$$

$R(Q)$ and $L(Q)$ are subquantale of Q. The quantale Q is said to be *right-sided* (respectively, *left-sided*), if every $a \in Q$ is right-sided (respectively, *left-sided*).

The right(left)-adjoint of & is denoted by $\underset{r}{\to}$ ($\underset{l}{\to}$) :

$$(a \underset{r}{\to} b) \stackrel{\text{def}}{=} \bigvee\{x \in R(Q) \mid a\&x \leqslant b\}\ ; \quad (a \underset{l}{\to} b) \stackrel{\text{def}}{=} \bigvee\{x \in L(Q) \mid x\&a \leqslant b\}.$$

Definition 4 Unary operation * on a quantale Q is called an *involution* if

1. $a^{**} = a$
2. $(a\&b)^* = b^*\&a^*$
3. $(\bigvee_\alpha a_\alpha)^* = \bigvee_\alpha a_\alpha^*$.

A quantale with an involution * is called *involutive quantale*.

If $a \in R(Q)$, then $(a^* \underset{r}{\to} 0)$ is an element $R(Q)$ and denoted by $a^\perp$.

Definition 5 A unital and involutive quantale $Q = (Q, \bigvee, \&, ^*, e)$ is called *Gelfand quantale*, if

$$a\&a^*\&a = a \quad \text{for all } a \in R(Q)$$

Lemma 1.1 ([2]) *For any Gelfand quantale Q, the set $R(Q)$ of right-sided elements is a Gelfand quantale with respect to the operations of Q, and has the following properties:*

$$a\&a = a, \qquad a \wedge b \leqslant a^*\&b, \qquad a \wedge a^\perp = 0 \qquad \textit{for every } a, b \in R(Q).$$

Let $\mathcal{L}$ be an orthocomplemented sup-lattice, that is, $\mathcal{L} = (\mathcal{L}, \bigvee, ^\perp)$ is a sup-lattice with operation $^\perp$ such that, for all $x \in \mathcal{L}$ and $\{x_\alpha\} \subset \mathcal{L}$,

$$x^{\perp\perp} = x, \qquad (\bigvee_\alpha x_\alpha)^\perp = \bigwedge_\alpha x_\alpha^\perp, \qquad x \vee x^\perp = 1, \quad x \wedge x^\perp = 0.$$

Then the set $Q(\mathcal{L})$ of sup-preserving mappings from $\mathcal{L}$ to itself, $\varphi : x \mapsto (x)\varphi$,

$$Q(\mathcal{L}) = \{\varphi{:}\mathcal{L} \to \mathcal{L} \mid (\bigvee_\alpha x_\alpha)\varphi = \bigvee_\alpha (x_\alpha)\varphi \quad \text{for all } \{x_\alpha\} \subset \mathcal{L}\},$$

forms a Gelfand quantal with respect to $\leqslant$, $\bigvee$, &, *, e defined by

$$\begin{aligned}
\varphi \leqslant \psi \quad &\stackrel{\text{def}}{\Longleftrightarrow} \quad (x)\varphi \leqslant (x)\psi \quad \text{for all } x \in \mathcal{L} \\
(x)(\bigvee_\alpha \varphi_\alpha) \quad &\stackrel{\text{def}}{=} \quad \bigvee_\alpha (x)\varphi_\alpha \quad \text{for all } x \in \mathcal{L} \\
(x)(\varphi\&\psi) \quad &\stackrel{\text{def}}{=} \quad ((x)\varphi)\psi \quad \text{for all } x \in \mathcal{L} \\
(x)\varphi^* \quad &\stackrel{\text{def}}{=} \quad (\bigvee\{t \in \mathcal{L} \mid (t)\varphi \leqslant x^\perp\})^\perp \quad \text{for all } x \in \mathcal{L} \\
(x)e \quad &\stackrel{\text{def}}{=} \quad x \quad \text{for all } x \in \mathcal{L}
\end{aligned}$$

Theorem 1 ([2]) *An orthocomplemented sup-lattice $\mathcal{L}$ is embedded in the Gelfand quantale $Q(\mathcal{L})$ consisting of all sup-preserving mappings from $\mathcal{L}$ to itself :*

$$\mathcal{L} \cong R(Q(\mathcal{L})) \subset Q(\mathcal{L}).$$

Now, as $\mathcal{L}$ in the above Theorem 1, we take the orthocomplemented sup-lattice $\mathbb{P}(H)$ consisting of projections on a Hilbert space H. The lattice $\mathbb{P}(H)$ is extended to Gelfand quantale $Q(\mathbb{P}(H))$ by the Theorem 1. We denote the Gelfand quantale $Q(\mathbb{P}(H))$ simply by Q, and consider a quantale valued set theory on the Q-valued universe. Truth values of propositions in quantale valued set theory are considered as elements of Gelfand quantale Q.

On Q, we have three implications, $\underset{r}{\to}$, $\underset{l}{\to}$, and the basic implication $\to$ of lattice valued logic ;

$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b \\ 0 & \text{otherwise} \end{cases}.$$

For the right-implication $\underset{r}{\to}$, and for $a, b, c \in R(Q)$, we have :

$$(1) \quad a\&(a \underset{r}{\to} b) \leqslant b \quad \text{and} \quad (2) \quad a\&b \leqslant c \ \text{ implies } \ b \leqslant (a \underset{r}{\to} c)$$

$\underset{l}{\to}$ has the symmetric property.

2 Quantale valued set theory

2.1 Language of quantale valued set theory

Primitive symbols are :

1. individual variables $x, y, z, \cdots$
2. propositional constant E
3. predicate constants $=$, $\in$,
4. logical symbols $\wedge$, $\vee$, $\neg$, $\to$, $\forall$, $\exists$, $\&$, *, $\underset{r}{\to}$, $\underset{l}{\to}$.
5. parentheses (,)

Propositional constant E and expressions of the form $t_1 = t_2$ or $t_1 \in t_2$ with individual variables t_1, t_2 are *atomic formulas*. *Formulas* are constructed from atomic formulas, by using the logical symbols. To denote formulas, we use $\varphi, \psi, \xi, \cdots$, $\varphi(x), \cdots$.

2.2 Quantale valued universe V^Q

Let $Q = (Q, \bigvee, \&, ^*, e)$ be a Gelfand quantale. Q-valued universe V^Q is constructed by induction.

$$\begin{aligned} V_\alpha^Q &= \{u \mid \exists \beta < \alpha \, \exists \mathcal{D}u \subset V_\beta^Q (u : \mathcal{D}u \to Q)\} \\ V^Q &= \bigcup_{\alpha \in On} V_\alpha^Q \end{aligned}$$

The least ordinal α such that $u \in V_\alpha^Q$ is called the *rank* of u, and the truth value of a formula φ is denoted by $[\![\varphi]\!]$. $u = v$ and $u \in v$ for $u, v \in V^Q$ are interpreted by induction on rank of u, v.

$$\begin{aligned} [\![u = v]\!] &= \bigwedge_{x \in \mathcal{D}u} (u(x) \to [\![x \in v]\!]) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \to [\![x \in u]\!]) \\ [\![u \in v]\!] &= \bigvee_{x \in \mathcal{D}v} [\![u = x]\!] \wedge v(x) \end{aligned}$$

E is interpreted as the unit e of Q, $[\![E]\!] = e$. Logical operations $\wedge$, $\vee$, $\&$, $\to$, $\forall x$, $\exists x$, involution *, and right- and left-implications $\underset{r}{\to}$, $\underset{l}{\to}$ are interpreted as the corresponding operators on Q.

$$[\![\varphi_1 \wedge \varphi_2]\!] = [\![\varphi_1]\!] \wedge [\![\varphi_2]\!] \quad ; \quad [\![\varphi_1 \vee \varphi_2]\!] = [\![\varphi_1]\!] \vee [\![\varphi_2]\!] \quad ; \quad [\![\varphi_1 \to \varphi_2]\!] = [\![\varphi_1]\!] \to [\![\varphi_2]\!]$$
$$[\![\forall x \varphi(x)]\!] = \bigwedge_{u \in V^Q} [\![\varphi(u)]\!] \quad ; \quad [\![\exists x \varphi(x)]\!] = \bigvee_{u \in V^Q} [\![\varphi(u)]\!]$$
$$[\![\varphi_1 \& \varphi_2]\!] = [\![\varphi_1]\!] \& [\![\varphi_2]\!] \quad ; \quad [\![\varphi^*]\!] = [\![\varphi]\!]^*$$
$$[\![\varphi \underset{r}{\to} \psi]\!] = [\![\varphi]\!] \underset{r}{\to} [\![\psi]\!] \quad ; \quad [\![\varphi \underset{l}{\to} \psi]\!] = [\![\varphi]\!] \underset{l}{\to} [\![\psi]\!]$$

In the lattice valued set theory, $[\![u=v]\!]$ is 1 or 0, and hence, $[\![u=v]\!] \wedge \bigvee_k b_k = \bigvee_k [\![u=v]\!] \wedge b_k$ for $u, v \in V^Q$ and $\{b_k\}_k \subset Q$. It follows that equality axioms holds. Since Q is a complete lattice, all axioms of lattice valued set theory are valid on V^Q.

For the right-implications $\underset{r}{\to}$, the corresponding equality $\underset{r}{=}$ and membership relation $\underset{r}{\in}$ are defined by:

$$u \underset{r}{=} v \iff^{\text{def}} \forall x (x \in u \underset{r}{\to} x \underset{r}{\in} v) \wedge \forall x (x \in v \underset{r}{\to} x \underset{r}{\in} u)$$
$$u \underset{r}{\in} v \iff^{\text{def}} \exists x ((x \underset{r}{=} u) \& x \in v)$$

Lemma 2.1 *If $u, v \in V^{R(Q)}$, then $[\![u \underset{r}{=} v]\!]$ and $[\![u \underset{r}{\in} v]\!]$ are elements of $R(Q)$.*

Theorem 2 (Equality axioms) *If $u, v, w \in V^{R(Q)}$, then*

$$[\![u \underset{r}{=} v]\!] \& [\![v \underset{r}{=} w]\!] \leqslant [\![u \underset{r}{=} w]\!]$$
$$[\![u \underset{r}{=} v]\!] \& [\![v \underset{r}{\in} w]\!] \leqslant [\![u \underset{r}{\in} w]\!]$$
$$[\![u \underset{r}{\in} v]\!] \& [\![v \underset{r}{=} w]\!] \leqslant [\![u \underset{r}{\in} w]\!]$$

Definition 6 For $u \in V^Q$, we define the restriction $u|p \in V^Q$ of u to p by

$$\mathcal{D}(u|p) = \mathcal{D}u, \quad (u|p)(x) = [\![x \in u]\!] \& p$$

Theorem 3 *If p is a right-sided element of Q, i.e. $p \in R(Q)$, and $u, v \in V^{R(Q)}$, then*

$$p \leqslant [\![u \underset{r}{=} v]\!] \quad \textit{if and only if} \quad [\![u|p = v|p]\!] = 1$$

Proof By using Theorem 2 and distributiveity of & over $\bigvee$.

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