# A modified subformula property for the modal logic K5

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# A Natural Deduction for Classical Logic, in which there exists a proof with the Subformula Property

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In this note, we will pose a new system, named $\mu$–calculus. $\lambda$–calculus can be simulated in $\mu$–calculus. While the type assignment system $TA_\lambda$ gives a natural deduction for implicational intuitionistic logic, the type assignment system $TA_\mu$ gives a natural deduction for classical implicational logic. Moreover for any classical implicational theorem $\alpha$ there exists a proof of $\alpha$ in $TA_\mu$ enjoying the subformula property.

Ryo Kashima has posed a Natural deduction system for the classical implicational logic. His system has three rules, the elimination of the implication, the introduction of the implication and the case rule. The author has noticed that the introduction of the implication is derivable from the case rule and the weakening. So we have gotten the system $TA_\mu$ from Kashima's system by replacing the introduction of implication by the weakening.

## 1 The type-free $\mu$-calculus and its type assignment

**DEFINITION 1.1 ($\mu$-terms)** An infinite sequence of term-variables is assmed to be given. Then linguistic expression called $\mu$-terms are defined thus:

1. each term-variable is a $\mu$-terms, called an atom or atomic term;
2. if $M$ and $N$ are $\mu$-terms then $(MN)$ is a $\mu$-term called an application;
3. if $M$ is a $\mu$-term then $KM$ is a $\mu$-term called a weakening;
4. if $x$ is a term-variable and $M$ and $N$ are $\mu$-terms then $(\mu x.M * N)$ is a $\mu$-term called an abstract or a $\mu$-abstract.

**DEFINITION 1.2 ($\beta$-reduction)** Any $\mu$-term of form $KMN$, $(\mu x.M{*}N)R$, $\mu x.[xMN/z]P{*}Q$ (where $z \in FV(P)$ and $x \notin FV(MP)$) or $\mu x.xM * N$ (where $x \notin FV(M)$) is called a $\beta$-redex. $\beta$-contracting an occurrence of a redex in a term $U$ means replacing one occurence of

| | | | |
|---|---|---|---|
| $KMN$ | by | $M$ , | |
| $(\mu x.M * N)R$ | by | $\mu x.MR * NR$ | (we $\alpha$-convert $(\mu x.M * N)$ if necessary), |
| $\mu x.[xMN/z]P * Q$ | by | $\mu x.[xM/z]P * Q$ | (where $z \in FV(P)$ and $x \notin FV(MP)$), |
| $\mu x.xM * N$ | by | $[M/x]N$ | (where $x \notin FV(M)$). |

If this changes $U$ to $V$, we say U $\beta$-contracts to $V$, or

$$U \rhd_{1\beta} V.$$

We say $U$ $\beta$-reduces to $V$, or

$$U \rhd_{\beta} V,$$

iff $V$ is obtained from $U$ by a finite (perhaps empty) series of $\beta$-contractions and changes of bound variables.

**CONJECTURE 1.3 (or FACT (Church-Rosser Theorem for $\beta$))** *If $M \rhd_\beta P$ and $N \rhd_\beta Q$ then there exist $T$ such that*

$$P \rhd_\beta T, \quad Q \rhd_\beta T.$$

We can consider different types of reductions that have logical reasons, for example,

$$R(\mu x.M * N) \rhd_1 \mu x.RM * RN.$$

But the reduction obtained from $\beta$-reduction adding the above reduction does not enjoy the Church-Rosser property.

**ABBREVIATION 1.4** $(\lambda x.M) \equiv (\mu x.x * KM)$.

**FACT 1.5 ($\mu$-calculus simulates $\lambda$-calculus)**

$$(\lambda x.M)N \rhd_\beta [M/x]N.$$

**DEFINITION 1.6 ($\eta$-reduction)** Any $\mu$-term of form $\mu x.x*K(Nx)$ or $\mu x.M*N$ (where $x \notin FV(N)$) is called a $\eta$-redex. $\eta$-Contracting an occurrence of a $\eta$-redex in a term $U$ means replacing one occurence of

$$\begin{array}{lll} \mu x.x * K(Nx) & \text{by} & \mu x.x * N \ , \\ \mu x.M * N & \text{by} & N \qquad\qquad (\text{where } x \notin FV(N)) \ . \end{array}$$

If this changes $U$ to $V$, we say U $\eta$-contracts to $V$, or

$$U \rhd_{1\eta} V.$$

We say $U$ $\eta$-reduces to $V$, or

$$U \rhd_\eta V,$$

iff $V$ is obtained from $U$ by a finite (perhaps empty) series of $\eta$-contractions and changes of bound variables.

**CONJECTURE 1.7 (or FACT (Church-Rosser Theorem for $\beta\eta$))** *If $M \rhd_{\beta\eta} P$ and $N \rhd_{\beta\eta} Q$ then there exist $T$ such that*

$$P \rhd_{\beta\eta} T, \quad Q \rhd_{\beta\eta} T.$$

Types are defined as usual.

**DEFINITION 1.8 (The type-assignment system $TA_\mu$)** $TA_\mu$ is a Natural Deduction system. Its formulas are arbitrary expressions $M : \alpha$, where $M$ is a $\mu$-term and $\alpha$ is a type.

$$\frac{\begin{array}{cc} \Pi_1 & \Pi_2 \\ M : \alpha \to \beta & N : \alpha \end{array}}{MN : \beta} \ (\to E), \qquad \frac{\begin{array}{c} \Pi \\ M : \beta \end{array}}{KM : \alpha \to \beta} \ (Weakening),$$

$$\frac{\begin{array}{cc} \overset{k}{x : \alpha \to \beta} & \overset{k}{x : \alpha} \\ \Pi_1 & \Pi_2 \\ M : \gamma & N : \gamma \end{array}}{\mu x.M * N : \gamma} \ k(Case).$$

**FACT 1.9 ($TA_\mu$ is a proof of classical implicational logic)** *$\alpha$ is a classical implicational theorem if and only if there exists a closed $\mu$-term $M$ such that $\vdash_\mu M : \alpha$.*

## 2 The type-free $\lambda A$-calculus and its type assignment

We will next introduce an auxiliary type type-assignment system $TA_{\lambda A}$ to prove the subformula property of the system $TA_{\mu}$.

**DEFINITION 2.1 ($\lambda A$-terms)** An infinite sequence of term-variables is assmed to be given. Then linguistic expression called $\lambda A$-terms are defined thus:

1. each term-variable is a $\lambda A$-terms, called an atom or atomic term;
2. if $M$ and $N$ are $\lambda A$-terms then $(MN)$ is a $\lambda A$-term called an application;
3. if $M$ is a $\lambda A$-term then $AM$ is a $\lambda A$-term called a r-weakening;
4. if $x$ is a term-variable and $M$ is a $\lambda A$-terms then $(\lambda x.M)$ is a $\lambda A$-term called an abstract or a $\lambda$-abstract.

**DEFINITION 2.2 ($A\beta$-reduction)** Any $\lambda A$-term of form $AMN$ and $(\lambda x.M)N$ is called a redex. Contracting an occurrence of a redex in a term $U$ means replacing one occurence of

$$\begin{array}{lll} AMN & \text{by} & AM, \\ (\lambda x.M)N & \text{by} & [N/x]M. \end{array}$$

If this changes $U$ to $V$, we say U $A\beta$-contracts to $V$, or

$$U \rhd_{1A\beta} V.$$

We say $U$ $A\beta$-reduces to $V$, or

$$U \rhd_{A\beta} V,$$

iff $V$ is obtained from $U$ by a finite (perhaps empty) series of $A\beta$-contractions and changes of bound variables.

**FACT 2.3 (Church-Rosser Theorem for $A\beta$)** *If $M \rhd_{A\beta} P$ and $N \rhd_{A\beta} Q$ then there exist $T$ such that*

$$P \rhd_{A\beta} T, \quad Q \rhd_{A\beta} T.$$

A $\perp$-type is a type constructed from variables and a type constant '$\perp$'.

**DEFINITION 2.4 (The type-assignment system $TA_{\lambda A}$)** $TA_{\lambda A}$ is a Natural Deduction system. Its formulas are arbitrary expressions $M : \alpha$, where $M$ is a $\lambda A$-term and $\alpha$ is a $\perp$-type.

$$\frac{\begin{array}{cc}\Pi_1 & \Pi_2 \\ M : \alpha \to \beta & N : \alpha\end{array}}{MN : \beta}\ (\to E), \qquad \frac{\begin{array}{c}\overset{k}{x : \alpha} \\ \Pi \\ M : \beta\end{array}}{\lambda x.M : \alpha \to \beta}\ k(\to I),$$

$$\frac{\begin{array}{c}\Pi \\ M : \perp\end{array}}{AM : \alpha}\ (R - Weakening).$$

**FACT 2.5 ($TA_{\lambda A}$ is a proof of intutionistic logic)** *$\alpha$ is a theorem of intutionistic $\{\to, \perp\}$-fragment logic if and only if there exists a closed $\lambda A$-term $M$ such that $\vdash_{\lambda A} M : \alpha$.*

**FACT 2.6 (Normalization Theorem for $TA_{\lambda A}$)** *Every $TA_{\lambda A}$-typable term has a $A\beta$-normal form.*

**FACT 2.7 (Subformula Property for $TA_{\lambda A}$)** *Let $\lambda A$-term $M$ be a $A\beta$-normal form and $\Delta$ a $TA_{\lambda A}$-deduction of $M : \alpha$. Then every type in $\Delta$ has an occurence in $\alpha$ or in a predicate of undischarged assumptions.*

# 3 Transelation of $\lambda A$-terms to $\mu$-terms

In this section we will define a translation $\mathscr{T}$ of $\lambda A$-terms to $\mu$-terms. At first, we define a simple translation $\mathscr{T}^*$ of $\lambda A$-terms without occurences of $A$ to $\mu$-terms.

**DEFINITION 3.1 (Translation $\mathscr{T}^*$)** We define recursively the translation $\mathscr{T}^*$ of $\lambda A$-terms, in which no $A$ occurs, to $\mu$-terms, as follows:

1. $\mathscr{T}^*(x) \equiv x$,
2. $\mathscr{T}^*(MN) \equiv (\mathscr{T}^*(M)\,\mathscr{T}^*(N))$,
3. $\mathscr{T}^*(\lambda x.M) \equiv (\mu x.x * K\,\mathscr{T}^*(M))$.

**DEFINITION 3.2 (Translation $\mathscr{T}$)** We define recursively the translation $\mathscr{T}$ of $\lambda A$-terms of the form $AM$ to $\mu$-terms, as follows:

1. $\mathscr{T}(AM) \equiv \mathscr{T}^*(M)$ if no $A$ occurs in $N$,
2. $\mathscr{T}(AM) \equiv \mathscr{T}(AN)$ if $M$ is of the form $(\cdots AN\cdots)$ where this $A$ is the leftmost occurence and no free variable in $N$ is bounded in $M$,
3. $\mathscr{T}(AM) \equiv \mu x.\mathscr{T}(AN) * \mathscr{T}(AR)$ if $M$ is of the form $(\cdots(\lambda x.\cdots AQ\cdots)\cdots)$ where this $A$ is the leftmost occurence and this '$\lambda x$' is the leftmost occurence which bounds a free variable in $Q$ and $N$ is obtained from $M$ by replacing $(\lambda x.\cdots AQ\cdots)$ by $x$ (we $\alpha$-convert $M$ if necessary) and $R \equiv (\cdots(\lambda z.\cdots AQ\cdots)\cdots)$ and the variable $z$ is new.

# 4 Proof of Subformula Property

**LEMMA 4.1** *Let $M$ be a $A\beta$-normal term without occurences $A$, $\Gamma \vdash_{\lambda A} M : \alpha$ and $g$ be a new type variable. Then*

1. $[g/\bot]\Gamma \vdash_\mu \mathscr{T}^*(M) : [g/\bot]\alpha$,
2. *every type in $TA_\mu$-deduction of $\mathscr{T}^*(M) : [g/\bot]\alpha$ has an occurence in $[g/\bot]\alpha$ or in a type in $[g/\bot]\Gamma$.*

**LEMMA 4.2** *Let $AM$ be a $A\beta$-normal term, $\Gamma \vdash_{\lambda A} M : \bot$ and $g$ be a new type variable. Then*

1. $[g/\bot]\Gamma \vdash_\mu \mathscr{T}(AM) : g$,
2. *every type in $TA_\mu$-deduction of $\mathscr{T}(AM) : g$ is $g$ or has an occurence in a type in $[g/\bot]\Gamma$.*

**LEMMA 4.3** *Let $\alpha$ be a implicational formula and $g$ be the rightmost variable in $\alpha$. Then $\alpha$ is a classical teorem if and only if $[\bot/g]\alpha$ is an intuitionistic theorem.*

**THEOREM 4.4 (Subformula Property of $TA_\mu$)** *If a type $\alpha$ is a classical implicational theorem, then there exists a closed $\mu$-term $M$ such that*

1. $\vdash_\mu M : \alpha$,
2. *for any $TA_\mu$-deduction $\Delta$ of $M : \alpha$, every type in $\Delta$ has an occurence in $\alpha$.*

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# Labelled Sequent Calculi and Completeness Theorems for Implicational Relevant Logics

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## 1 Introduction

A natural semantic treatment of relevant logics was given by Urquhart [10]. (See [1], [2] and [4] for information on relevant logics.) Let **E** be the implicational fragment of the relevant logic **E**. (In this paper, we treat only "$\to$" (implication) as a logical symbol; therefore, **E**/**R**/**S4**$\cdots$ will denote the implicational fragments of themselves.) An Urquhart's model for **E** (we call it an *E-model*) consists of two structures $\langle I, \cdot, \mathbf{e}\rangle$ and $\langle W, R\rangle$ where the former is an idempotent commutative monoid (i.e., semilattice with identity $\mathbf{e}$) and the latter is a quasi-ordered set (i.e., $R$ is a reflexive and transitive relation on $W$). The structure $\langle I, \cdot, \mathbf{e}\rangle$ is considered to be a "structure of information"—$I$ is a set of pieces of information, $\cdot$ is a binary operator which combines two pieces of information, and $\mathbf{e}$ is an empty piece of information; and the structure $\langle W, R\rangle$ is, like a Kripke's model for modal logics, considered to be a "structure of possible worlds"—$W$ is a set of worlds, and $R$ is an accessibility relation. The notion "$\alpha, x \models A$" (a formula $A$ holds according to a piece $\alpha$ of information at a world $x$) is inductively defined by the following.

$$\alpha, x \models B{\to}C \iff \forall \beta \in I, \forall y \in W[(xRy)\ \&\ (\beta, y \models B)\ \Rightarrow\ (\alpha{\cdot}\beta, y \models C)].$$

We say that a formula $A$ is *valid in* the model if $\mathbf{e}, x \models A$ for any $x \in W$. The completeness theorem, which was shown in [10] (a proof appeared in [2]), claims that a formula $A$ is provable in **E** if and only if $A$ is valid in any E-model.

The relevant logic **R** and the logic **S4** of "strict implication" are obtained from **E** by adding, respectively, the axiom schemes

C: $(A{\to}(B{\to}C)){\to}(B{\to}(A{\to}C))$, and
K$^{01}$: $\overrightarrow{A}{\to}(B{\to}\overrightarrow{A})$

where $\overrightarrow{A}$ is an abbreviation for $A_1{\to}A_2$; and the intuitionistic logic **Int** is obtained from **E** by adding both the schemes C and K$^{01}$ (or, equivalently, **Int** is obtained from **R** by adding the scheme K: $A{\to}(B{\to}A)$). (See, e.g., [1] and [9] for axiomatizations of these logics. The name K$^{01}$ comes from [9], where the superscripts represent certain restrictions by "over-arrows".) The completeness theorems for these logics were shown as follows. We define two conditions on E-models:

(Single World Condition) The set of possible worlds is a singleton.

(Hereditary Condition) If $\alpha, x \models A$, then $\alpha{\cdot}\beta, x \models A$.

Then, the logic (**R** / **S4** / **Int**) is complete with respect to the class of E-models satisfying the condition (Single World / Hereditary / both Single World and Hereditary, respectively). That is, for example, a formula $A$ is provable in **Int** if and only if $A$ is valid in any E-model that satisfies both the Single World and Hereditary Conditions.

Moreover, such completeness results were shown for other two relevant logics—called **E5** and **RM0** (the names come from [5] and [1]).

The logic **E5** is obtained from **E** by adding the axiom scheme

$C^{001}$: $(A\to(B\to\overrightarrow{C}))\to(B\to(A\to\overrightarrow{C}))$.

(In [5] and [10], the axiom scheme $B\to((B\to\overrightarrow{C})\to\overrightarrow{C})$ was adopted where this scheme and $C^{001}$ are mutually inferable over **E**.) The scheme $C^{001}$ is an instance of the scheme C, and therefore **E5** is an intermediate logic between **E** and **R**. It was shown that the logic **E5** is complete with respect to the class of E-models satisfying the following condition.

(Single Cluster Condition) The accessibility relation is universal, that is, $xRy$ for any $x, y \in W$.

Note that this condition is weaker than the Single World Condition.

The logic **RM0** is obtained from **R** by adding the axiom scheme

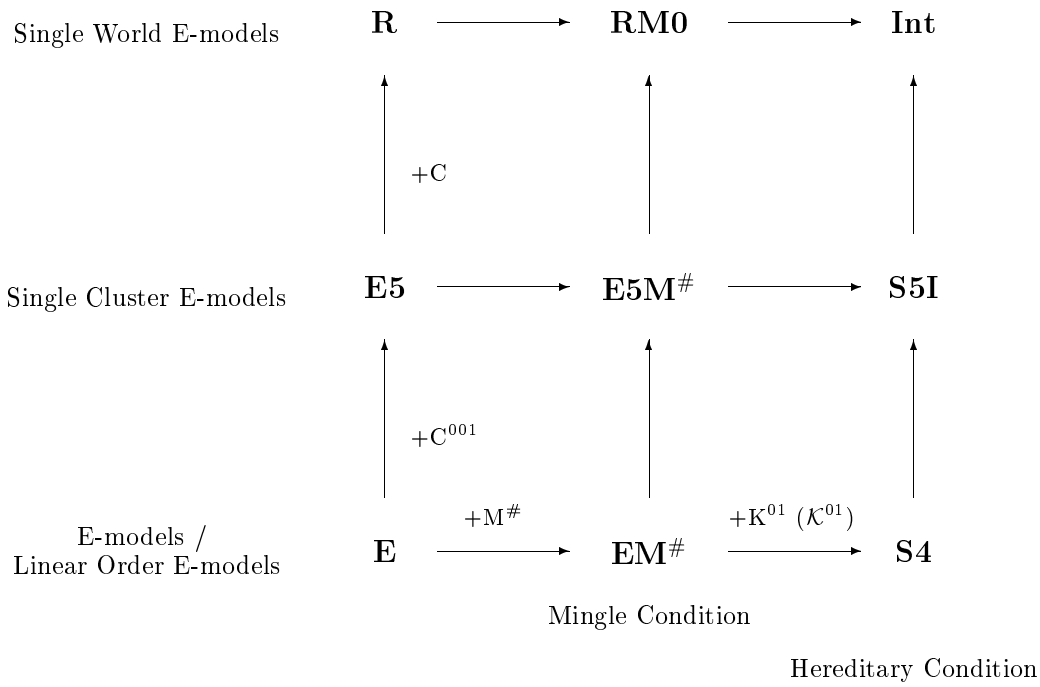
M: $A\to(A\to A)$.

This scheme is an instance of the scheme K, and therefore **RM0** is an intermediate logic between **R** and **Int**. It was shown that the logic **RM0** is complete with respect to the class of E-models satisfying both the Single World Condition and the following condition.

(Mingle Condition) If $\alpha, x \models A$ and $\beta, x \models A$, then $\alpha\cdot\beta, x \models A$.

Note that the Mingle Condition is weaker than the Hereditary Condition.

**Figure 1**




To sum up, it has been shown that the six logics **E**, **E5**, **R**, **RM0**, **Int**, and **S4**, which are located on the edges of the diagram in Figure 1, are complete with respect to the class of E-models with the additional conditions. (See [2], [5] and [10] for the proofs.) Then, bringing the diagram to completion is a natural requirement. In other words, there are two questions: (Q1) Is there an axiom scheme $\mathcal{A}$ that satisfies the following?

- The logic $\mathbf{E}{+}\mathcal{A}$ is complete with respect to the class of E-models satisfying the Mingle Condition.
- The logic $\mathbf{E5}{+}\mathcal{A}$ is complete with respect to the class of E-models satisfying both the Single Cluster Condition and the Mingle Condition.
- The logic $\mathbf{R}{+}\mathcal{A}$ is equivalent to **RM0**.

(Q2) Is the logic $\mathbf{S5I} = \mathbf{S4}{+}\mathrm{C}^{001}$ complete with respect to the class of E-models satisfying both the Single Cluster Condition and Hereditary Condition? (We will account for the name **S5I** later.)

This paper gives positive answers to these questions. (It is somewhat surprising that such a natural question has been open.) For Q1, the required scheme $\mathcal{A}$ is

$$\mathrm{M}^{\#}\colon\ (\overrightarrow{A}{\to}B){\to}(\overrightarrow{A}{\to}(B{\to}B)),$$

and the logics $\mathbf{E}{+}\mathrm{M}^{\#}$ and $\mathbf{E5}{+}\mathrm{M}^{\#}$ are denoted by $\mathbf{EM}^{\#}$ and $\mathbf{E5M}^{\#}$, respectively. (Note that $\mathrm{M}^{\#}$ is an instance of the scheme

$$\mathcal{K}^{01}\colon\ (\overrightarrow{A}{\to}B){\to}(\overrightarrow{A}{\to}(C{\to}B))$$

while this scheme and $\mathrm{K}^{01}$ are mutually inferable over **E**; see Section 2.) Moreover we show a strong version of the completeness of $\mathbf{E}/\mathbf{EM}^{\#}/\mathbf{S4}$, which claims that these logics are complete with respect to the *linear order* E-models. That is, for example, a formula $A$ is provable in **S4** if and only if $A$ is valid in any E-model in which the accessibility relation is a linear order and the Hereditary Condition holds. These complete the diagram in Figure 1, which shows a nice correspondence between simple axiom schemes and natural conditions on models.

An outline of the proof of the completeness is as follows. For each logic $X = \mathbf{E}, \mathbf{EM}^{\#}, \ldots$, we introduce two systems $\mathcal{G}X$ and $\mathcal{L}X$. The system $\mathcal{G}X$ is an ordinary "sequent calculus", and it is easy to show that a "sequent" $A_1, \ldots, A_n \mapsto B$ is provable in $\mathcal{G}X$ if and only if the formula $A_1{\to}(\cdots{\to}(A_n{\to}B)\cdots)$ is provable in $X$. The system $\mathcal{L}X$ is a "labelled sequent calculus", that is, a system to treat "labelled sequents"—sequents consisting of labelled formulas where a label reflects the structure of E-models. By a standard way, we show that $\mathcal{L}X$ is complete with respect to the class of E-models satisfying the additional conditions for $X$. In particular, if a formula $A$ is valid in any E-model of $X$, then the labelled sequent "$\Rightarrow (\emptyset, 0) : A$", which consists of $A$ and the "empty label" $(\emptyset, 0)$, is provable in $\mathcal{L}X$. Then, the following claim completes the proof of the completeness.

> If the labelled sequent $\Rightarrow (\emptyset, 0) : A$ is provable in $\mathcal{L}X$, then the unlabelled sequent $\mapsto A$ is provable in $\mathcal{G}X$.

This is proved by showing a stronger claim:

> If a labelled sequent $\mathcal{S}$ is provable in $\mathcal{L}X$, then an unlabelled sequent $\mathcal{T}$ is provable in $\mathcal{G}X$ where $\mathcal{T}$ is "extracted" from $\mathcal{S}$ in a certain way.

Since unlabelled sequents have less expressive power than labelled sequents, this "extraction" is somewhat complicated; and proving this claim is the hardest (and the most interesting) part of the proof of the completeness theorem. There have been a lot of studies on labelled sequent calculi (e.g. see [7] and [8]). Among them, the author thinks, this paper is a good example of an application of labelled sequent calculi to solve important problems on the logics, which are independent from the labelled systems.

The completion of the diagram in Figure 1 induces further questions, for which the author does not have any nontrivial answers now: (1) There are a lot of possible conditions on E-models. Is there a corresponding axiom scheme for each condition? For example, if the models lack the condition $\alpha{\cdot}\alpha = \alpha$ (idempotence), the corresponding logics might be axiomatized by deleting the scheme W: $(A{\to}(A{\to}B)){\to}(A{\to}B)$. Is this true for each logic in the diagram? (2) Introducing other connectives ($\wedge$, $\vee$, $\neg$, ...) is an important problem. It was shown ([3], [6]) that the logic $\mathbf{R}_{\to\wedge\vee}$ with an extra inference rule is complete with respect to the class of models of **R** in which $\wedge$ and $\vee$ are interpreted in a natural way. What happens for the other logics in the diagram?

If we get acceptable answers for these questions, Urquhart's semantics will be a major paradigm of "possible worlds semantics", while Routley-Meyer's "ternary relation semantics" (see, e.g., [2] or [4]) is major now. The diagram in Figure 1 is a starting point.

We make remarks about the scheme $\mathrm{M}^{\#}$ and the logic **S5I**. In literature (see, e.g., [1] or [4]), the axiom scheme M′: $\overrightarrow{A} \to (\overrightarrow{A} \to \overrightarrow{A})$ was suggested as a "Mingle for **E**". But the author thinks the scheme $\mathrm{M}^{\#}$ is the very axiom of "Mingle for **E**" because of its completeness with respect to the class of E-models satisfying the Mingle Condition. (M′ is inferable from $\mathrm{M}^{\#}$ (by $B := \overrightarrow{A}$), but the author does not know whether the converse holds.) The name **S5I** comes from [8]. In [8], a sequent style system for **S5I** was introduced, whereas the Hilbert style axiomatization **S4**+$\mathrm{C}^{001}$ did not appear. The name **S5I** reflects the fact that the models of this logic are considered to be the intuitionistic variants of the models for "S5 strict implication". We will discuss this fact in Section 3.

The plan of this paper is as follows. Let $\mathcal{X} = \{\mathbf{E}, \mathbf{EM}^{\#}, \mathbf{S4}, \mathbf{E5}, \mathbf{E5M}^{\#}, \mathbf{S5I}\}$ (the six logics in the bottom and the middle rows in Figure 1). In Section 2, we introduce implicational relevant logics by Hilbert-style axiomatizations; and then we introduce six Gentzen-style sequent calculi named $\mathcal{G}X$ for $X \in \mathcal{X}$. In Section 3, we define the E-models and prove that the Hilbert-style systems are sound with respect to the class of E-models with the additional conditions. In Section 4, we introduce labelled sequent calculi named $\mathcal{L}X$ for $X \in \mathcal{X}$, and prove their completeness with respect to the class of E-models with the additional conditions. In Section 5, we show the "extraction" from the labelled sequent calculi $\mathcal{L}X$ into the unlabelled sequent calculi $\mathcal{G}X$. Finally, in Section 6, we state the main results of this paper—completeness of the six logics in $\mathcal{X}$.

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# Semantics and Cut-free Provability of Modal Substructural Logics

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In this paper, we study the model of modal propositional Lambek logics. We give the model similar to the phase space and demonstrate its usefulness by showing the cut-free provability theorem and giving the Kripke semantics.

## 1 Preliminaries

For the definitions of Lambek logic $\mathrm{FL}_\sigma$, $\mathrm{FL}_\sigma$-algebra, $\mathrm{so}_\sigma$-monoid, modal Lambek logic $\mathrm{FL}_\sigma^m$, and modal $\mathrm{FL}_\sigma$-algebra, we refer Ono [3].

## 2 Construction by Filter

We can construct $\mathrm{so}_\sigma$-monoid from $\mathrm{FL}_\sigma$-algebra and vice versa by Ono and Komori [4]'s method.

**Definition 2.1** Let $\mathbf{A}$ be an FL-algebra. $\mathcal{SOM}(\mathbf{A})$ is a structure $\langle \mathbb{F}^-(\mathbf{A}), \cap, \cdot, \epsilon, \omega \rangle$ where $\cap$ is the set intersection, $x \cdot y \stackrel{def}{=} \mathcal{F}\{a \star b \mid a \in x,\ b \in y\}$, $\epsilon \stackrel{def}{=} 1\uparrow$, $\omega \stackrel{def}{=} A$, $\mathbb{F}^-(\mathbf{A}) \stackrel{def}{=} \{x \mid x \in \mathbb{F}(\mathbf{A}),\ x \cdot \epsilon = \epsilon \cdot x = x\}$.

**Theorem 2.2** If $\mathbf{A}$ is an $\mathrm{FL}_\sigma$-algebra, $\mathcal{SOM}(\mathbf{A})$ is an $\mathrm{so}_\sigma$-monoid.

**Definition 2.3** Let $\mathbf{D}$ be an so-monoid. $\mathcal{FLA}(\mathbf{D})$ is a structure $\langle \mathbb{F}(\mathbf{D}), \to, \cup, \cap, \star, 1, 0, \top, \bot \rangle$ where $a \to b \stackrel{def}{=} \{y \in D \mid \text{for all } x \in a,\ y \cdot x \in b\}$, $\cap$ is the set intersection, $a \cup b \stackrel{def}{=} \mathcal{F}(a \cup b)$ where $\cup$ in the righthand is the set union, $a \star b \stackrel{def}{=} \mathcal{F}\{x \cdot y \mid x \in a,\ y \in b\}$, $\top \stackrel{def}{=} D$, $\bot \stackrel{def}{=} \{\omega\}$, $1 \stackrel{def}{=} \epsilon\uparrow$, $0 \stackrel{def}{=} \bot$ (if $w \in \sigma$) or any filter (otherwise).

**Theorem 2.4** If $\mathbf{D}$ is an $\mathrm{so}_\sigma$-monoid, $\mathcal{FLA}(\mathbf{D})$ is an $\mathrm{FL}_\sigma$-algebra.

**Definition 2.5** Let $\mathbf{A} \equiv \langle A, \to, \cup, \cap, \star, 1, 0, \top, \bot \rangle$ be an FL-algebra. $\mathbf{A}^s$ is a structure $\langle A, \cup, \star, 1, \bot \rangle$.

**Theorem 2.6** If $\mathbf{A}$ is an $\mathrm{FL}_\sigma$-algebra, $\mathbf{A}^s$ is an $\mathrm{so}_\sigma$-monoid. (The order induced in $\mathbf{A}^s$ is the opposite order of the order of $\mathbf{A}$.)

## 3 Modality

**Definition 3.1** Let $\mathbf{D}$ be an $\mathrm{so}_\sigma$-monoid $\langle D, \cap, \cdot, \epsilon, \omega \rangle$. $\mathrm{Cent}(\mathbf{D})$ denotes the set $\{x \in D \mid \epsilon \leq x,\ x \cdot x \leq x,\ \text{for all } y \in D,\ x \cdot y = y \cdot x\}$. $\langle \mathcal{FLA}(\mathbf{D}), I, ¿, 0 \rangle$ is a *modal so$_\sigma$ phase structure* iff $I$ is a submonoid of $\mathrm{Cent}(\mathbf{D})$, $0 \in \mathcal{FLA}(\mathbf{D})$, and ¿ is an operation on $D$ that satisfies

1. $¿x \leq x$, $¿x \leq ¿¿x$ for $x \in D$
2. $¿(x \cdot y) \leq x \cdot ¿y$ for $x \in I$ and $y \in D$
3. if $x \in 0$, then $¿x \in 0$

A modal $\mathrm{so}_\sigma$ phase structure has operations ! and ? on $\mathbb{F}(\mathbf{D})$ which are defined as follows.

$$!a \stackrel{def}{=} \mathcal{F}(I \cap a) \qquad ?a \stackrel{def}{=} \mathcal{F}(0 \cup ¿a) \quad \text{where } ¿a \stackrel{def}{=} \{¿x \mid x \in a\}$$

**Theorem 3.2** A modal $\mathrm{so}_\sigma$ phase structure is a modal $\mathrm{FL}_\sigma$-algebra.

**Proposition 3.3** Let $\mathbf{A}$ be a modal $\mathrm{FL}_\sigma$-algebra and put $\mathbf{D} = \mathcal{SOM}(\mathbf{A})$, $I \stackrel{def}{=} \{(!u)\uparrow \mid u \in A\}$, $¿x \stackrel{def}{=} \mathcal{F}\{?u \mid u \in x\}$, and $0 \stackrel{def}{=} (0_{\mathbf{A}}\uparrow)\uparrow$ ($0_{\mathbf{A}}$ is 0 in $\mathbf{A}$). Then, $\langle \mathcal{FLA}(\mathbf{D}), I, ¿, 0\rangle$ is a modal $\mathrm{so}_\sigma$ phase structure.

**Proposition 3.4** Let $\mathbf{A}$ be a modal $\mathrm{FL}_\sigma$-algebra and put $I \stackrel{def}{=} \{!a \mid a \in A\}$ and $0 \stackrel{def}{=} 0_{\mathbf{A}^s}\uparrow$ ($0_{\mathbf{A}^s}\uparrow$ is a filter generated using the order of $\mathbf{A}^s$, therefore, $0 = \{a \mid a \leq_{\mathbf{A}} 0_{\mathbf{A}}\}$ where $\leq_{\mathbf{A}}$ is the order in $\mathbf{A}$.) Then, $\langle \mathcal{FLA}(\mathbf{A}^s), I, ?, 0\rangle$ is a modal $\mathrm{so}_\sigma$ phase structure.

**Theorem 3.5** Let $\mathbf{A}$ be a modal $\mathrm{FL}_\sigma$-algebra.
1. Let $\mathbf{B}$ be $\langle \mathcal{FLA}(\mathbf{D}), I, ¿, 0\rangle$ given in Prop. 3.3. Then, a map from $\mathbf{A}$ to $\mathbf{B}$ defined by $a \mapsto (a\uparrow)\uparrow$ is an embedding of modal $\mathrm{FL}_\sigma$-algebra.
2. Let $\mathbf{B}$ be $\langle \mathcal{FLA}(\mathbf{A}^s), I, ?, 0\rangle$ given in Prop. 3.4. Then, a map from $\mathbf{A}$ to $\mathbf{B}$ defined by $a \mapsto a\uparrow$ is an embedding of modal $\mathrm{FL}_\sigma$-algebra.

**Theorem 3.6** $\mathrm{FL}^m_\sigma$ is complete with respect to the class of all modal $\mathrm{so}_\sigma$ phase structures.

**Remark:** $\mathrm{FL}^m_{ewc}$ is equivalent to the constructive modal logic CS4 which consists of intuitionistic propositional logic, necessitation rule, and following axioms.

| | |
|---|---|
| $\Box K$: $\Box(A \to B) \to (\Box A \to \Box B)$ | $\Diamond K$: $\Box(A \to B) \to (\Diamond A \to \Diamond B)$ |
| $\Box T$: $\Box A \to A$ | $\Diamond T$: $A \to \Diamond A$ |
| $\Box 4$: $\Box A \to \Box\Box A$ | $\Diamond 4$: $\Diamond\Diamond A \to \Diamond A$ |
| | $\Diamond \bot E$: $\Diamond \bot \to A$ |

($\Box$ and $\Diamond$ are to be replaced by ! and ? respectively.) Therefore, ! and ? in $\mathrm{so}_{ewc}$-modal phase structure can be used to interpret $\Box$ and $\Diamond$ in CS4.

**Remark:** (Girard's modal op for classical phase space [1]) Let $(P, \cdot)$ be a commutative monoid, $O \subseteq P$, and $\mathcal{C}(a) \stackrel{def}{=} (a \to O) \to O$ for $a \subseteq P$. Operations ! and ? are defined on $\{a \mid \mathcal{C}(a) = a\}$.

$$!a \stackrel{def}{=} ((I \cap a) \to O) \to O \qquad ?a \stackrel{def}{=} (I \cap (a \to O)) \to O$$

# 4 Cut-Free Provability

In this section, we show the cut-free provability theorem for some $\mathrm{FL}_\sigma$ by using modal $\mathrm{so}_\sigma$ phase structure.

**Lemma 4.1** In $\mathrm{FL}^m_\sigma$, if $\Gamma_1, \alpha \vee \beta, \Gamma_2 \vdash \theta$ is cut-free provable, then $\Gamma_1, \alpha, \Gamma_2 \vdash \theta$ and $\Gamma_1, \beta, \Gamma_2 \vdash \theta$ are cut-free provable.

**Lemma 4.2** In $\mathrm{FL}^m_\sigma$ ($c \notin \sigma$ or $e \in \sigma$), if $\Gamma_1, \alpha \star \beta, \Gamma_2 \vdash \theta$ is cut-free provable, then $\Gamma_1, \alpha, \beta, \Gamma_2 \vdash \theta$ is cut-free provable.

We define the preorder $\leq$ on formulas of $\mathrm{FL}^m_\sigma$ as follows:

$\alpha \leq \beta$ iff, for every $\Gamma_1$, $\Gamma_2$, $\theta$, if $\Gamma_1, \alpha, \Gamma_2 \vdash \theta$ is cut-free provable, then $\Gamma_1, \beta, \Gamma_2 \vdash \theta$ is cut-free provable.

Let $\sim$ be the equivalence relation defined by $\alpha \sim \beta$ iff $\alpha \leq \beta$ and $\beta \leq \alpha$, $[\alpha]$ be the equivalence class to which a formula $\alpha$ belongs, and $L_\sigma$ be the quotient set. Then, a partial order $\leq$ on $L_\sigma$ is induced. We define the operations $\cap$ and $\cdot$ on $L_\sigma$ as follows: $[\alpha] \cap [\beta] \stackrel{def}{=} [\alpha \vee \beta]$, $[\alpha] \cdot [\beta] \stackrel{def}{=} [\alpha \star \beta]$. The definition of $\cap$ does not depend on the choice of the representative by Lemma 4.1 and, in the case of $\mathrm{FL}^m_\sigma$ ($c \notin \sigma$ or $e \in \sigma$), the definition of $\cdot$ does not depend on the choice of the representative by Lemma 4.2. Therefore,

if $c \notin \sigma$ or $e \in \sigma$, operations $\cap$ and $\cdot$ are well-defined and we can check that $\mathcal{L}_\sigma \stackrel{def}{=} \langle L_\sigma, \cap, \cdot, [1], [\bot] \rangle$ is an $\mathrm{so}_\sigma$-monoid. Note that the order induced from $\cap$ coincides with $\leq$, that is, $[\alpha] \cap [\beta] = [\alpha]$ iff $[\alpha] \leq [\beta]$.

Put $I \stackrel{def}{=} \{[!\alpha_1 \star \cdots \star !\alpha_n]\}$ and $¿[\alpha] \stackrel{def}{=} [?\alpha]$. Let $||\alpha|| \stackrel{def}{=} \{[\gamma] \mid \gamma \vdash \alpha \text{ is cut free provable}\}$ and put $0 \stackrel{def}{=} ||0_\sigma||$ ($0_\sigma$ is a formula 0 in $\mathrm{FL}^m_\sigma$). Then, $\mathcal{P}_\sigma \stackrel{def}{=} \langle \mathcal{FLA}(\mathcal{L}_\sigma), I, ¿, 0 \rangle$ is a modal $\mathrm{so}_\sigma$ phase structure.

Let $[\![\alpha]\!]$ be the valuation of $\alpha$ on $\mathcal{P}_\sigma$ where we put $[\![p]\!] = ||p||$ for propositional variable $p$.

**Theorem 4.3** In $\mathcal{P}_\sigma$ ($c \notin \sigma$ or $e \in \sigma$), $[\alpha] \in [\![\alpha]\!] \subseteq ||\alpha||$.

**Corollary 4.4** In $\mathrm{FL}^m_\sigma$ ($c \notin \sigma$ or $e \in \sigma$), if $\Gamma \vdash \theta$ is provable with cuts, then $\Gamma \vdash \theta$ is provable without cuts.

*Proof.* If $\Gamma \equiv \alpha_1, \ldots, \alpha_n$, we write $\Gamma^\star$ for $\alpha_1 \star \ldots \star \alpha_n$. ($\Gamma^\star \equiv 1$ if $n = 0$.) If $\Gamma \vdash \theta$ is provable, then $\Gamma^\star \vdash \theta$ is provable. By Th. 4.3, $[\Gamma^\star] \in [\![\Gamma^\star]\!] \leq [\![\theta]\!] \subseteq ||\theta||$. Therefore, $\Gamma^\star \vdash \theta$ is cut-free provable. By Lemma 4.2, $\Gamma \vdash \theta$ is cut-free provable. □

Fortunately(?), the cut-elimination theorem does not hold either in $\mathrm{FL}_c$ or $\mathrm{FL}_{cw}$.

**Remark:** The idea of using $||\alpha||$ and proving the theorem of the form Th. 4.3 comes from [2]. In [2], the cut-free provability and the cut-elimination theorems for intuitionistic and classical linear logics with modality are proved. In [5], the cut-elimination theorems for the predicate logic FLs (but not for the modal FLs) are proved.

# 5 Kripke Semantics

## 5.1 Kripke Semantics by Filter

In [3, 4], Kripke semantics of $\mathrm{FL}_\sigma$ is defined as follows.

**Definition 5.1 (Kripke model)** $\langle \mathbf{D}, 0, g \rangle$ is a *Kripke model* of $\mathrm{FL}_\sigma$ iff $\mathbf{D}$ is a $\mathrm{so}_\sigma$-monoid, $0 \in \mathbb{F}(\mathbf{D})$ where $0 = \{\omega\}$ if $w \in \sigma$, and $g$ is a map from propositional variables to $\mathbb{F}(\mathbf{D})$. The valuation relation $x \models \alpha$ for $x \in \mathbf{D}$ and a formula $\alpha$ is defined by the following clauses.

1. $x \models p$ iff $x \in g(p)$, if $p$ is a propositional variable
2. $x \models 1$ iff $\epsilon \leq x$,
3. $x \models 0$ iff $x \in 0$,
4. $x \models \top$, for every $x$,
5. $x \models \bot$ iff $x = \omega$,
6. $x \models \alpha \to \beta$ iff, for every $y$ and $z$, $y \models \alpha$ and $x \cdot y \leq z$ implies $z \models \beta$,
7. $x \models \alpha \vee \beta$ iff, there exist $y$ and $z$ such that $y \cap z \leq x$, $y \models \alpha$ and $z \models \beta$,
8. $x \models \alpha \wedge \beta$ iff $x \models \alpha$ and $x \models \beta$,
9. $x \models \alpha \star \beta$ iff, there exist $y$ and $z$ such that $y \cdot z \leq x$, $y \models \alpha$ and $z \models \beta$.

This Kripke semantics is shown to be sound by Th. 2.4, that is, by reading $x \models \alpha$ as $x \in [\![\alpha]\!]$. We will give the Kripke semantics for $\mathrm{FL}^m_\sigma$ by paraphrasing the filter generation in the definition of modal operations of Th. 3.2.

**Definition 5.2** A binary relation $R$ is a *filter relation* if it satisfies the following.

1. if $x \leq y$ and $x \mathrel{R} z$, then there exists $z'$ such that $y \mathrel{R} z'$ and $z \leq z'$
2. if $x_1 \mathrel{R} y_1$ and $x_2 \mathrel{R} y_2$, then $x_1 \cap x_2 \mathrel{R} y_1 \cap y_2$

**Definition 5.3** $\langle \mathbf{D}, R, S, 0, g \rangle$ is a *Kripke model* of $\mathrm{FL}^m_\sigma$ iff $\langle \mathbf{D}, 0, g \rangle$ is a *Kripke model* of $\mathrm{FL}_\sigma$, $R$ is a transitive relation on $D$, $S$ is a reflexive and transitive relation on $D$, and the following conditions are satisfied.

1. $R$ and $S$ are filter relations
2. if $x \mathrel{R} y$, then $x \geq y$, $y \mathrel{R} y$, and $y \in \mathrm{Cent}(\mathbf{D})$

3. if $x_1 \; R \; y_1$ and $x_2 \; R \; y_2$, then $x_1 \cdot x_2 \; R \; y_1 \cdot y_2$
4. if $x_1 \; R \; y_1$ and $x_2 \; S \; y_2$, then $x_1 \cdot x_2 \; S \; y_1 \cdot y_2$
5. if $z \in 0$ and $x \; S \; z$, then $x \in 0$

The valuation relation $x \models \alpha$ is defined by the clauses $1 - 9$ in Def. 5.1 and the following clauses.

10. $x \models !\alpha$ iff, there exists $y$ such that $x \; R \; y$ and $y \models \alpha$,
11. $x \models ?\alpha$ iff, there exists $y$ such that $x \; S \; y$ and $y \models \alpha$.

**Theorem 5.4** Let $\langle \mathbf{D}, R, S, 0, g \rangle$ be a Kripke model of $\mathrm{FL}^m_\sigma$ and ! and ? be the following operations on $\mathbb{F}(\mathbf{D})$.

$$!a \stackrel{def}{=} \{x \mid \exists y(x \; R \; y \wedge y \in a)\} \qquad ?a \stackrel{def}{=} \{x \mid \exists y(x \; S \; y \wedge y \in a)\}$$

Then, $\mathcal{FLA}(\mathbf{D})$ with ! and ? is a modal $\mathrm{FL}_\sigma$-algebra.

**Proposition 5.5** Let $\langle \mathcal{FLA}(\mathbf{D}), I, ¿, 0 \rangle$ be a modal $\mathrm{so}_\sigma$ phase structure that satisfies $(¿\omega)\uparrow \geq 0$. Put $\hat{I} \stackrel{def}{=} \{x_1 \cap \cdots \cap x_n \mid x_i \in I, \, n \geq 1\}$ and define the relations $R$ and $S$ on $D$ by $x \; R \; y \stackrel{def}{=} x \geq y \wedge y \in \hat{I}$ and $x \; S \; y \stackrel{def}{=} x \geq ¿y$. Then, $\langle \mathbf{D}, R, S, 0, g \rangle$ is a Kripke model of $\mathrm{FL}^m_\sigma$ and it holds that $x \in [\![\alpha]\!]$ in $\langle \mathcal{FLA}(\mathbf{D}), I, ¿, 0 \rangle$ (where $[\![p]\!] = g(p)$ for a propositional variable $p$) iff $x \models \alpha$ in $\langle \mathbf{D}, R, S, 0, g \rangle$.

**Theorem 5.6** $\mathrm{FL}^m_\sigma$ is complete with respect to the class of all Kripke models.

## 5.2 Traditional Kripke Semantics

Usually, $\Box$ of the intuitionistic modal logic is interpreted using universal quantifier, that is, in the simplest case, $\Box$ is interpreted like this:

($\Box$) $x \models \Box\alpha$ iff, for every $z$, $x \; R_\Box \; z$ implies $z \models \alpha$.

Here, we consider the following clause. ($dom(R) \stackrel{def}{=} \{x \mid \exists y(x \; R \; y)\}$.)

(!) $x \models !\alpha$ iff, there exists $y \leq x$ such that $y \in dom(R_!)$ and for every $y \; R_! \; z$, $z \models \alpha$.

We do not list the conditions $R_!$ should satisfy ($R_!$ is transitive, and so on), but just remark that the clause 10 in Def. 5.3 and the clause (!) have the common instance that is obtained by putting $x \; R \; y \stackrel{def}{=} x \geq y \wedge y \in R_0$ in 10 and $y \; R_! \; z \stackrel{def}{=} y \in R_0 \wedge y \leq z$ in (!). If we can take $y$ independently of $\alpha$ in (!), we write the $y$ as $r_x$ and have the clause of the form ($\Box$) by putting $x \; R_\Box \; z \stackrel{def}{=} r_x \; R_! \; z$. In the case of $\sigma = ewc$, we have an example.

**Proposition 5.7** Let $\mathbf{A}$ be a modal $\mathrm{FL}_{ewc}$-algebra and put $\mathbf{D} = \mathcal{SOM}(\mathbf{A})$, $r_x \stackrel{def}{=} \mathcal{F}\{u \mid !u \in x\}$, $¿x \stackrel{def}{=} \mathcal{F}\{?u \mid u \in x\}$, and $0 \stackrel{def}{=} (0_\mathbf{A}\uparrow)\uparrow$ ($0_\mathbf{A}$ is 0 in $\mathbf{A}$). Then, $\langle \mathcal{FLA}(\mathbf{D}), 0, g \rangle$ with $x \; R \; z \stackrel{def}{=} r_x \leq z$ and $x \; S \; z \stackrel{def}{=} x \geq ¿z$ is a model of $\mathrm{FL}^m_{ewc}$ defined by the clauses $1 - 9$ in Def. 5.1, the clause 11 in Def. 5.3, and

10'. $x \models !\alpha$ iff, for every $z$, $x \; R \; z$ implies $z \models \alpha$.

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# Similarity on a tableau method between modal propositional logic **S5** and classical propositional logic **Cl**

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**Abstract**

A.Chagrov and M.Zakharyashcev show that there exist embeddings between propositional modal logic **S5** and classical propositional logic **Cl** in terms of frame semantics ([CZ97]). So, it is natural to consider that there is also something similar on tableau method between **S5** and **Cl**. Combining a few propositions in [CZ97], we can quite easily find a new tableau method for **S5**, which resembles for **Cl**.

## 1 Notation

Through this report, except treating classical tableau, we will use propositional modal formulas, only. They are introduced in usual way; Let $\mathcal{P}$ be the set of *propositional variables*. Letters $p, q, r, \ldots, p_0, p_1, \ldots$ will denote propositional variables. The set of *propositional modal formulas* is constructed from $\mathcal{P}$, $\wedge$, $\vee$, $\supset$, $\Box$, $\Diamond$ and $\bot$ in usual way. Greek letters $\varphi, \psi, \chi, \ldots$ will denote formulas. We consider $\top$ and the negation $\neg\varphi$ of formula $\varphi$ as abbreviations of $\bot \supset \bot$ and $\varphi \supset \bot$, respectively. We call a formula which outermost connective is $\Box$ *box formula*. In box formulas, we separate formulas which second outermost connective are disjunction, and call such a formula *box disjunctive formula*.

As for model, we suppose frame semantics. The *truth relation* and *validity* are introduced similar to [CZ97], as follows; Let $W$ and $R$ be a denumerable set of *possible worlds* and *equationl relation* on $W$, respectively. A *frame* $\mathcal{F}$ is a pair $\langle W, R\rangle$, a *valuation* $\mathcal{V}$ is a function from $\mathcal{P}$ to $2^W$, and a model $\mathcal{M}$ is a pair $\langle \mathcal{F}, \mathcal{V}\rangle$. Suppose $x$ and $y$ are arbitrary possible worlds, and $\mathcal{C}$ is a class of frames, then;

$$
\begin{aligned}
(\mathcal{M}, x) &\not\models \bot, \\
(\mathcal{M}, x) &\models p \quad \text{iff} \quad x \in \mathcal{V}(p), \\
(\mathcal{M}, x) &\models \varphi \wedge \psi \quad \text{iff} \quad (\mathcal{M}, x) \models \varphi \text{ and } (\mathcal{M}, x) \models \psi, \\
(\mathcal{M}, x) &\models \varphi \vee \psi \quad \text{iff} \quad (\mathcal{M}, x) \models \varphi \text{ or } (\mathcal{M}, x) \models \psi, \\
(\mathcal{M}, x) &\models \varphi \supset \psi \quad \text{iff} \quad (\mathcal{M}, x) \not\models \varphi \text{ or } (\mathcal{M}, x) \models \psi, \\
(\mathcal{M}, x) &\models \Box\varphi \quad \text{iff} \quad \text{for all } y(xRy \Rightarrow (\mathcal{M}, y) \models \varphi), \\
(\mathcal{M}, x) &\models \Diamond\varphi \quad \text{iff} \quad \text{for some } y(xRy \text{ and } (\mathcal{M}, y) \models \varphi), \\
\mathcal{M} &\models \varphi \quad \text{iff} \quad \text{for all } x((\mathcal{M}, x) \models \varphi), \\
\mathcal{F} &\models \varphi \quad \text{iff} \quad \text{for all } \mathcal{V}(\mathcal{M} \models \varphi), \\
(\mathcal{F}, x) &\models \varphi \quad \text{iff} \quad \text{for all } \mathcal{V}((\mathcal{M}, x) \models \varphi), \\
\mathcal{C} &\models \varphi \quad \text{iff} \quad \text{for all } \mathcal{F}(\mathcal{F} \in \mathcal{C} \Rightarrow \mathcal{F} \models \varphi).
\end{aligned}
$$

$\varphi$ is *true* at $x$ in $\mathcal{M}$ if $(\mathcal{M}, x) \models \varphi$. $\varphi$ is *false* at $x$ in $\mathcal{M}$ if $\varphi$ is not true. $\varphi$ is *true* in $\mathcal{M}$ if $\mathcal{M} \models \varphi$. $\varphi$ is *valid* in $\mathcal{F}$ if $\mathcal{F} \models \varphi$. A set $\Gamma$ of formulas is *characterized* by a class $\mathcal{C}$ of frames if $\mathcal{C} \models \varphi$ for any $\varphi(\in \Gamma)$, and is denoted by $\mathcal{C} \models \Gamma$. A *generated subframe* of a given frame $\langle W, R\rangle$ is a pair $\langle V, S\rangle$ such that $V$ and $S$ is a closure of $R$ for some subset of $W$ and a restriction of $R$ by $V$, respectively. A *generated submodel*

of a given model $\langle \mathcal{F}, \mathcal{V} \rangle$ is a pair $\langle \mathcal{G}, \mathcal{U} \rangle$ where $\mathcal{G}$ is a generated subframe of $\mathcal{F}$ and $\mathcal{U}$ is a restriction of $\mathcal{V}$ by the base set of $\mathcal{G}$.

# 2 Frame which is constructed from one cluster

In our report, a *cluster* has a main role. For any possible world $x$, a cluster $C(x)$ is defined as follows;

$$C(x) = \{y : xRy \text{ and } yRx\}.$$

Under an equivalence relation $R$, $C(x)$ means an equivalence class of $W/R$. A.Chagrov and M.Zakharyascev introduced the following proposition for a frame with a transitive relation.

**Proposition 2.1 (3.7 of [CZ97])** *Suppose $x$ is a possible worlds in a model $\mathcal{M}$ built on a transitive frames and $\varphi$ an arbitrary fromula. Then for every $y \in C(x)$,*

$$\begin{array}{lll} (\mathcal{M}, x) \models \Box\varphi & \textit{iff} & (\mathcal{M}, y) \models \Box\varphi, \\ (\mathcal{M}, x) \models \Diamond\varphi & \textit{iff} & (\mathcal{M}, y) \models \Diamond\varphi. \end{array}$$

This proposition 2.1 asserts that any box formula has same truth value in each clusters. Next,

**Proposition 2.2 (3.11 of [CZ97])** *If $\mathcal{N}$ is a generated submodel of $\mathcal{M}$ then, for every point $x$ in $\mathcal{N}$ and every modal formula $\varphi$,*

$$(\mathcal{N}, x) \models \varphi \textit{ iff } (\mathcal{M}, x) \models \varphi.$$

**Proposition 2.3 (3.13 of [CZ97])** *Let* **K** *be the set of theorems of propositional modal logic* **K**. *Then,*

$$\mathbf{K} = \{\varphi : \mathcal{F} \models \varphi \textit{ for all rooted frames } \mathcal{F}\}.$$

The set of theorems of **S5** is the set of formulas which are valid in all frames with equivalence relation. As for the case of **S5**, a rooted frame means that one cluster frames. By proposition 2.2 and 2.3, we can easily show that **S5** is characterized by the class of frames which is constructed from one cluster. That is,

**Proposition 2.4** *Let* **S5** *be the set of theorems of* **S5**. *Then,*

$$\mathbf{S5} = \{\varphi : \mathcal{F} \models \varphi \textit{ for any frame which is constructed from one cluster } \mathcal{F}\}.$$

By proposition 2.4, it is sufficient to know cardinal of $W$ when expressing frame for **S5**. Combining propositions 2.1and 2.4, it is natural to dispose box formulas from another types of formulas if we consider tableau proof for **S5**.

# 3 Tableau proof of S5

We will introduce a tableau proof method of **S5**. The essence of this method was introduced first in [Suz97], but the method in [Suz97] is difficult to read because it is intended to implement in a computer system. To simplify it, a similar point between our method and tableau proof of **Cl** becomes clear.

Basically, tableau method makes a tree structure which is called *tableau tree*. *Beth tableau* is a pair $(\Gamma, \Delta)$ of two sets $\Gamma$, $\Delta$ of formulas. Roughly speaking, $\Gamma$ of Beth tableau indicates a set of formulas which truth value are true, and $\Delta$ indicates a set of formulas which truth value are false. For normal tableau system, each node of a tableau tree is Beth tableau, and each Beth tableau corresponds to a possible world of a frame. In our system, each node of tableau tree corresponds to a model (that is, a set of Beth tableaus).

In this report, **S5** *tableau* is a set of Beth tableau. To dispose box formulas and box disjunctive formulas, we prepare special Beth tableaus in **S5** tableau. Each **S5** tableau has two special Beth tableaus.

One is called $\Box$-*tableau* (denoted by $(\Gamma_\Box, \Delta_\Box)$), and another $\Box$-$\vee$-*tableau* (denoted by $(\Gamma_{\Box\vee}, \Delta_{\Box\vee})$), respectively. Thus, **S5** tableau itself is denoted by $[(\Gamma_\Box, \Delta_\Box) : (\Gamma_{\Box\vee}, \Delta_{\Box\vee}) : (\Gamma_1, \Delta_1), \ldots, (\Gamma_n, \Delta_n)]$. **S5** *tabelau tree* is a tree structure which is constructed from **S5** tableaus.

In our tableau method, two types of tableau extension rules are introduced. Each rule is applied to a formula of Beth tableau on leaf **S5** tableaus. Formula which is applied rule is marked with check mark. *Checked formulas* do not become target of applying rules. Checked formula ($\overline{\varphi}$) and non-checked formula ($\varphi$) are treated the identical element (checked-formula) in any set if both $\overline{\varphi}$ and $\varphi$ are elements of the set. *Tableau extension rule* is applied to non-checked formulas of all Beth tableaus except $\Box$-$\vee$-tableau, and *tableau box disjunctive extension rule* is applied to non-checked formulas of $\Box$-$\vee$-tableau.

Tableau extension rule is defined as follows;

- $\bot \in \Gamma(\Gamma_\Box)$: Stop applying extension rule to this **S5** tableau.
- $\neg\varphi \in \Gamma(\Gamma_\Box)$: Check $\neg\varphi$, and add $\varphi$ into $\Delta$ ($\Delta_\Box$).
- $\varphi \wedge \psi \in \Gamma(\Gamma_\Box)$: Check $\varphi \wedge \psi$, and add $\varphi$, $\psi$ into $\Gamma$ ($\Gamma_\Box$).
- $\varphi \vee \psi \in \Gamma$: Make two copies of **S5** tableau, and add them as children of this node. Then, check $\varphi \vee \psi$ in $\Gamma$ in each child, and add $\varphi$ into $\Gamma$ of one child and $\psi$ into $\Gamma$ of another child.
- $\varphi \supset \psi \in \Gamma$: Make two copies of **S5** tableau, and add them as children of this node. Then, check $\varphi \supset \psi$ from $\Gamma$ in each child, and add $\varphi$ into $\Delta$ of one child and $\psi$ into $\Gamma$ of another child.
- $\varphi \vee \psi \in \Gamma_\Box$: Check $\varphi \vee \psi$ in $\Gamma_\Box$, and add $\varphi \vee \psi$ to $\Gamma_{\Box\vee}$.
- $\varphi \supset \psi \in \Gamma_\Box$: Check $\varphi \supset \psi$ in $\Gamma_\Box$, and add $\neg\varphi \vee \psi$ to $\Gamma_{\Box\vee}$.
- $\Box\varphi \in \Gamma(\Gamma_\Box)$: Check $\Box\varphi$, and add $\varphi$ into $\Gamma_\Box$ ($\Gamma_\Box$).
- $\Diamond\varphi \in \Gamma(\Gamma_\Box)$: Suppose $\{\varphi_1, \ldots, \varphi_h\}$ and $\{\varphi_{h+1}, \ldots, \varphi_k\}$ are the set of checked formulas of $\Gamma_{\Box\vee}$ and the set of checked formulas of $\Delta_{\Box\vee}$, respectively. For each i, $\varphi_i$ is denoted by $\alpha_i^0 \vee \alpha_i^1$ when $1 \leq i \leq h$ and by $\alpha_i^0 \wedge \alpha_i^1$ when $h+1 \leq i \leq k$. Make $2^k$ copies of **S5** tableau and indexes each copies like $\mathcal{T}_1, \ldots, \mathcal{T}_{2^k}$, and add them as children of this **S5** tableau. For each **S5** tableau children $\mathcal{T}_s$, add a Beth tableau $(\{\varphi, \alpha_1^{j_1}, \ldots, \alpha_h^{j_h}\}, \{\alpha_{h+1}^{j_{h+1}}, \ldots, \alpha_k^{j_k}\})$ such that $s = j_1 2^0 + j_2 2^1 + \cdots + j_k 2^{k-1}$. Check $\Diamond\varphi$ of all $\Gamma$-nodes of Beth tableaus including $\Box$-tableau at last.
- $\bot \in \Delta(\Delta_\Box)$: Check $\bot$.
- $\varphi \wedge \psi \in \Delta$: Make two copies of **S5** tableau, and add them as children of this node. Then, check $\varphi \wedge \psi$ from $\Delta$ in each child, and add $\varphi$ into $\Delta$ of one child and $\psi$ into $\Delta$ of another child.
- $\varphi \wedge \psi \in \Delta_\Box$: Check $\varphi \wedge \psi$, and add $\varphi \wedge \psi$ to $\Delta_{\Box\vee}$.
- $\varphi \vee \psi \in \Delta(\Delta_\Box)$: Check $\varphi \vee \psi$, and add $\varphi$, $\psi$ into $\Delta$ ($\Delta_\Box$).
- $\varphi \supset \psi \in \Delta(\Delta_\Box)$: Check $\varphi \supset \psi$, and add $\varphi$ into $\Gamma$ ($\Gamma_\Box$) and $\psi$ into $\Delta$ ($\Delta_\Box$).
- $\Box\varphi \in \Delta(\Delta_\Box)$: Suppose $\{\varphi_1, \ldots, \varphi_h\}$ and $\{\varphi_{h+1}, \ldots, \varphi_k\}$ are the set of checked formulas of $\Gamma_{\Box\vee}$ and the set of checked formulas of $\Delta_{\Box\vee}$, respectively. For each i, $\varphi_i$ is denoted by $\alpha_i^0 \vee \alpha_i^1$ when $1 \leq i \leq h$ and by $\alpha_i^0 \wedge \alpha_i^1$ when $h+1 \leq i \leq k$. Make $2^k$ copies of **S5** tableau and indexes each copies like $\mathcal{T}_1, \ldots, \mathcal{T}_{2^k}$, and add them as children of this **S5** tableau. For each **S5** tableau children $\mathcal{T}_s$, add a Beth tableau $(\{\alpha_1^{j_1}, \ldots, \alpha_h^{j_h}\}, \{\varphi, \alpha_{h+1}^{j_{h+1}}, \ldots, \alpha_k^{j_k}\})$ such that $s = j_1 2^0 + j_2 2^1 + \cdots + j_k 2^{k-1}$. Check $\Box\varphi$ of all $\Delta$-nodes of Beth tableaus including $\Box$-tableau at last.
- $\Diamond\varphi \in \Delta(\Delta_\Box)$: Check $\Diamond\varphi$, and add $\varphi$ into $\Delta_\Box$ ($\Delta_\Box$).

Tableau box disjunctive extension rule is defined as follows;

- $\varphi \vee \psi \in \Gamma_{\Box\vee}$: Check $\varphi \vee \psi$. Denote $\varphi$ and $\psi$ by $\alpha^0$ and $\alpha^1$, respectively. Suppose $n$ is the number of Beth Tableaus of **S5** tableau $[(\Gamma_\Box, \Delta_\Box) : (\Gamma_{\Box\vee}, \Delta_{\Box\vee}) : (\Gamma_1, \Delta_1), \ldots, (\Gamma_n, \Delta_n)]$. Make $2^n$ copies of **S5** tableau and indexes each copies like $\mathcal{T}_1, \ldots, \mathcal{T}_{2^n}$, and add them as children of this **S5** tableau. For each Beth tableau $(\Gamma_i, \Delta_i)$ of $\mathcal{T}_j$, add $\alpha^{j_i}$ to $\Gamma_i$ such that $j = j_1 2^0 + j_2 2^1 + \cdots + j_n 2^{n-1}$.

- $\varphi \wedge \psi \in \Delta_{\Box\vee}$: Check $\varphi \wedge \psi$. Denote $\varphi$ and $\psi$ by $\alpha^0$ and $\alpha^1$, respectively. Suppose $n$ is the number of Beth Tableaus of **S5** tableau $[(\Gamma_\Box, \Delta_\Box) : (\Gamma_{\Box\vee}, \Delta_{\Box\vee}) : (\Gamma_1, \Delta_1), \ldots, (\Gamma_n, \Delta_n)]$. Make $2^n$ copies of **S5** tableau and indexes each copies like $\mathcal{T}_1, \ldots, \mathcal{T}_{2^n}$, and add them as children of this **S5** tableau. For each Beth tableau $(\Gamma_i, \Delta_i)$ of $\mathcal{T}_j$, add $\alpha^{j_i}$ to $\Delta_i$ such that $j = j_1 2^0 + j_2 2^1 + \cdots + j_n 2^{n-1}$.

We will show two examples of applying to **S5** tableau.

**Example 3.1** *Let* **S5** *tableau* $[(\emptyset,\emptyset) : (\emptyset,\emptyset) : (\emptyset, \{\Diamond p \supset \Box\Diamond p\})]$ *be given. Then,*

$$
\begin{aligned}
[(\emptyset,\emptyset) : (\emptyset,\emptyset) : (\emptyset, \{\Diamond p \supset \Box\Diamond p\})] &\Rightarrow [(\emptyset,\emptyset) : (\emptyset,\emptyset) : (\{\Diamond p\}, \{\Box\Diamond p, \overline{\Diamond p \supset \Box\Diamond p}\})] \\
&\Rightarrow [(\emptyset,\emptyset) : (\emptyset,\emptyset) : (\{\overline{\Diamond p}\}, \{\Box\Diamond p, \overline{\Diamond p \supset \Box\Diamond p}\}), (\{p\}, \emptyset)] \\
&\Rightarrow [(\emptyset,\emptyset) : (\emptyset,\emptyset) : (\{\overline{\Diamond p}\}, \{\Box\Diamond \overline{p}, \overline{\Diamond p \supset \Box\Diamond p}\}), (\{p\}, \emptyset), (\emptyset, \{\Diamond p\})] \\
&\Rightarrow [(\emptyset,\{p\}) : (\emptyset,\emptyset) : (\{\overline{\Diamond p}\}, \{\Box\Diamond \overline{p}, \overline{\Diamond p \supset \Box\Diamond p}\}), (\{p\}, \emptyset), (\emptyset, \{\overline{\Diamond p}\})]
\end{aligned}
$$

**Example 3.2** *Let* **S5** *tableau* $[(\emptyset,\emptyset) : (\emptyset,\emptyset) : (\{\Box(p \vee \Diamond q)\}, \emptyset)]$ *be given. Then,*

$$
\begin{aligned}
[(\emptyset,\emptyset) : (\emptyset,\emptyset) : (\{\Box(p \vee \Diamond q)\}, \emptyset)] &\Rightarrow [(\{p \vee \Diamond q\}, \emptyset) : (\emptyset,\emptyset) : (\{\overline{\Box(p \vee \Diamond q)}\}, \emptyset)] \\
&\Rightarrow [(\{\overline{p \vee \Diamond q}\}, \emptyset) : (\{p \vee \Diamond q\}, \emptyset) : (\{\overline{\Box(p \vee \Diamond q)}\}, \emptyset)] \\
&\Rightarrow \begin{cases} [(\{\overline{p \vee \Diamond q}\}, \emptyset) : (\{\overline{p \vee \Diamond q}\}, \emptyset) : (\{p, \overline{\Box(p \vee \Diamond q)}\}, \emptyset)] \\ [(\{\overline{p \vee \Diamond q}\}, \emptyset) : (\{\overline{p \vee \Diamond q}\}, \emptyset) : (\{\Diamond q, \overline{\Box(p \vee \Diamond q)}\}, \emptyset)] \end{cases} \\
&\Rightarrow \begin{cases} [(\{\overline{p \vee \Diamond q}\}, \emptyset) : (\{\overline{p \vee \Diamond q}\}, \emptyset) : (\{p, \overline{\Box(p \vee \Diamond q)}\}, \emptyset)] \\ \begin{cases} [(\{\overline{p \vee \Diamond q}\}, \emptyset) : (\{\overline{p \vee \Diamond q}\}, \emptyset) : (\{\overline{\Diamond q}, \overline{\Box(p \vee \Diamond q)}\}, \emptyset), (\{p, q\}, \emptyset)] \\ [(\{\overline{p \vee \Diamond q}\}, \emptyset) : (\{\overline{p \vee \Diamond q}\}, \emptyset) : (\{\overline{\Diamond q}, \overline{\Box(p \vee \Diamond q)}\}, \emptyset), (\{\overline{\Diamond q}, q\}, \emptyset)] \end{cases} \end{cases}
\end{aligned}
$$

It is natural condsidered from the above examples, *closeness* on **S5** tableau is defined as follows.

**Definition 3.3** **S5** *tableau* $[(\Gamma_\Box, \Delta_\Box) : (\Gamma_{\Box\vee}, \Delta_{\Box\vee}) : (\Gamma_1, \Delta_1), \ldots, (\Gamma_n, \Delta_n)]$ *is closed if* $(\Gamma_\Box \cup \Gamma_i) \cap (\Delta_\Box \cup \Delta_i) \neq \emptyset$ *holds for some* $i$.

*Closeness* of **S5** tabelau tree and $\vdash$ are defined in usual way. Then, the following completeness result holds.

**Proposition 3.4** *For any formula* $\varphi$, $\vdash \varphi$ *holds, if and only if,* $\models \varphi$.

# 4 Conclusion

Modal logic **S5** is characterized by the class of one cluster frames. Meanwhile propositional classical logic **Cl** is characterized by the singleton of the one point frame. Combining with proposition 2.1 and 2.4, we could devide modal formulas into three layers in our tableau method. We tried to simplify extension rules. If box disjunctive formula in $\Gamma$ (and its dual in $\Delta$) does not appear in our tableau tree, our method quite resembles to **Cl** tableau. Box disjunctive formulas are the reason why increasing nodes of **S5** tableau. We conjecture that box disjunctive formula becomes the cause of deviding **S5** from **Cl**.

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# Products of Modal Logics by General Frames

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## 1 Products of Modal Logics by Kripke Frames

Let $\mathcal{F} = (W, R)$ and $\mathcal{G} = (V, S)$ be Kripke frames. As Shehtman introduced in [3], we define the *products of Kripke frames* as $\mathcal{F} \times_s \mathcal{G} := (W \times V, R^\vee, S^\vee)$, where $R^\vee = \{\big((x,y),(z,y)\big) \mid xRz\}$, $S^\vee = \{\big((x,y),(x,z)\big) \mid ySz\}$. Moreover we will define as follows.

For any modal logic $L$, $\boldsymbol{F}(L) := \{\mathcal{F} \mid \mathcal{F} \models L\}$.

For any class $\mathcal{C}$ of Kripke frames, $\boldsymbol{L}(\mathcal{C}) := \{\varphi \mid \mathcal{F} \models \varphi \text{ for any } \mathcal{F} \in \mathcal{C}\}$.

For any classes $\mathcal{C}_1, \mathcal{C}_2$ of Kripke frames, $\mathcal{C}_1 \times_s \mathcal{C}_2 := \{\mathcal{F} \times_s \mathcal{G} \mid \mathcal{F} \in \mathcal{C}_1, \mathcal{G} \in \mathcal{C}_2\}$.

For modal logics $L_1, L_2$, we define the *products of logics* as $L_1 \times_s L_2 := \boldsymbol{L}(\boldsymbol{F}(L_1) \times_s \boldsymbol{F}(L_2))$.

The properties of $L_1 \times_s L_2$ is, for example, investigated as follows (see [1]).

- The logic $L_1 \times_s L_2$ is Kripke complete. (Trivial.)
- $L_1 \times_s L_2 \supseteq [L_1, L_2]$, where $[L_1, L_2] := L_1 * L_2 \oplus \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p \oplus \Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p$.
- If $L_1, L_2$ are PTC-logics (i.e. the logics axiomatized by a set of pseudotransitive formulas $\Diamond^m\Box p \rightarrow \Box^n p$ and/or closed formulas), then $L_1 \times_s L_2$ is product-matching (i.e. $L_1 \times_s L_2 = [L_1, L_2]$).
- If $L$ is axiomatized by a set of $\Box p \rightarrow \Box^n p$, $\Diamond\Box p \rightarrow p$ and/or closed formulas, then $L \times_s \mathbf{S5}$ has f.m.p..

But the products $(\times_s)$ of logics have a bad property. We do NOT always have

$$\boldsymbol{L}(\mathcal{C}_1) \times_s \boldsymbol{L}(\mathcal{C}_2) = \boldsymbol{L}(\mathcal{C}_1 \times_s \mathcal{C}_2), \tag{1}$$

for any classes $\mathcal{C}_1, \mathcal{C}_2$ of Kripke frames. The logic $\mathbf{LIN} \times_s \mathbf{S5}$ give us a counterexample [2].

In the following (Sections 4), we will define another products $(\times)$ of logics on which the property (1) holds.

## 2 Products of Modal Logics by General Frames

Let $\mathcal{F} = (W, R, \mathcal{P})$ and $\mathcal{G} = (V, S, \mathcal{Q})$ be general frames. That is, $\mathcal{P}$ is a modal algebra of sets on Kripke frame $\mathcal{F}$, which contains $\emptyset$ and is closed under $\cap, \cup, -, \Box$, where $\Box X = \{x \in W \mid xRy \Rightarrow y \in X\}$. Also we set $\kappa\mathcal{F}$ as $(W, R, 2^W)$. We define the *products of general frames* as follows.
$\mathcal{F} \divideontimes \mathcal{G} := (W \times V, R \times S, \mathcal{P} \divideontimes \mathcal{Q})$, where

$$R \times S := \{\big((x,y),(t,u)\big) \mid xRt, ySu\},$$

$$\mathcal{P} \divideontimes \mathcal{Q} := ma(\{X \times Y \mid X \in \mathcal{P}, Y \in \mathcal{Q}\}),$$

i.e. the smallest modal algebra containing $\{X \times Y \mid X \in \mathcal{P}, Y \in \mathcal{Q}\}$.

Note that $(W, R) \times_s (V, S) = (W \times V, R \times \Delta, \Delta \times S)$, where $\Delta$ denotes a diagonal relation.

**Theorem 2.1** $\quad \mathcal{P} \divideontimes \mathcal{Q} = \{\bigcup_{i \in I} X_i \times Y_i \mid X_i \in \mathcal{P}, Y_i \in \mathcal{Q}, I : finite\}$.

**Proof.** It is implied by following (2)–(7).

- $$\emptyset = \emptyset \times \emptyset (= \emptyset \times V = W \times \emptyset). \tag{2}$$
- $$\bigcup_{i\in I} X_i \times Y_i \cap \bigcup_{j\in J} T_j \times U_j = \bigcup_{(i,j)\in I\times J} (X_i \cap T_j) \times (Y_i \cap U_j). \tag{3}$$
- $$\bigcup_{i\in I} X_i \times Y_i \cup \bigcup_{j\in J} T_j \times U_j = \bigcup_{k\in I\sqcup J} A_k \times B_k, \tag{4}$$
  where $I \sqcup J = \{(i,0) \mid i \in I\} \cup \{(j,1) \mid j \in J\}, A_{(i,0)} = X_i, A_{(j,1)} = T_j, B_{(i,0)} = Y_j, B_{(j,1)} = U_j$.
- $$-\bigcup_{i\in I} X_i \times Y_i = \bigcup_{J\in 2^I} (-\bigcup_{i\in J} X_i) \times (-\bigcup_{i\in -J} Y_i). \tag{5}$$
- $$\Diamond \bigcup_{i\in I} X_i \times Y_i = \bigcup_{i\in I} \Diamond X_i \times \Diamond Y_i. \tag{6}$$
- $$\Box \bigcup_{i\in I} X_i \times Y_i = \bigcup_{\mathcal{A}\subseteq 2^I} (\Box \bigcap_{J\in\mathcal{A}} \bigcup_{i\in J} X_i) \times (\Box \bigcup_{J\in\mathcal{A}} \bigcap_{i\in J} Y_i). \tag{7}$$

∎

Moreover, the following holds.

- $$\bigcup_{i\in I} X_i \times Y_i = \bigcup_{j\in J} T_j \times U_j$$
  $$\iff \quad \forall i \in I, \exists \mathcal{A} \subseteq 2^J \text{ such that } X_i \subseteq \bigcap_{J'\in\mathcal{A}} \bigcup_{j\in J'} T_j, Y_i \subseteq \bigcup_{J'\in\mathcal{A}} \bigcap_{j\in J'} U_j, \tag{8}$$
  $$\text{and} \quad \forall j \in J, \exists \mathcal{A} \subseteq 2^I \text{ such that } T_j \subseteq \bigcap_{I'\in\mathcal{A}} \bigcup_{i\in I'} X_i, U_j \subseteq \bigcup_{I'\in\mathcal{A}} \bigcap_{i\in I'} Y_i.$$

**Proposition 2.2** *$\kappa\mathcal{F} \divideontimes \kappa\mathcal{G} = \kappa(\mathcal{F} \divideontimes \mathcal{G}) \iff$ either $\mathcal{F}$ or $\mathcal{G}$ is finite.*

**Proposition 2.3** *Let $\alpha$ be $\Diamond^{m_1}\Box^{n_1}p_1 \wedge \cdots \wedge \Diamond^{m_k}\Box^{n_k}p_k \to \varphi(p_1,\ldots,p_k)$,*
*where $\varphi$ is constructed by $\wedge, \to, \Box, \Diamond$ from $p_1,\ldots,p_k$.*
*Then $\mathcal{F} \models \alpha, \mathcal{G} \models \alpha \Rightarrow \mathcal{F} \divideontimes \mathcal{G} \models \alpha$,*
*and $\kappa\mathcal{F} \models \alpha, \kappa\mathcal{G} \models \alpha \Rightarrow \kappa(\mathcal{F} \divideontimes \mathcal{G}) \models \alpha$.*

# 3 The Dual Algebras

For any general frame $\mathcal{F} = (W, R, \mathcal{P})$, we denote the *dual algebra* of $\mathcal{F}$ by $\mathcal{F}^+ = (\mathcal{P}, \cap, \cup, -, \emptyset, \Box)$.

For any (normal) modal algebra $\boldsymbol{A} = (A, \wedge, \vee, \neg, 0, \Box)$, we denote the *dual frame* of $\boldsymbol{A}$ by $\boldsymbol{A}_+ = (PF(\boldsymbol{A}), R_{\boldsymbol{A}}, \mathcal{P}_{\boldsymbol{A}})$, where

1. $PF(\boldsymbol{A})$ is the set of all prime filters of $\boldsymbol{A}$
2. $xR_{\boldsymbol{A}}\ y \overset{def}{\iff} \forall a \in A(\Box a \in x \Rightarrow a \in y)$,
3. $\mathcal{P}_{\boldsymbol{A}} = \{\{x \in PF(\boldsymbol{A}) \mid a \in x\} \mid a \in A\}$.

We define the *products of modal algebras* as follows.
For modal algebras $\boldsymbol{A}, \boldsymbol{B}$, $\boldsymbol{A} \divideontimes \boldsymbol{B} := (\boldsymbol{A}_+ \divideontimes \boldsymbol{B}_+)^+$

By (2)–(8), we can easily get the following proposition.

**Proposition 3.1** *$\mathcal{F}_1^+ \simeq \mathcal{F}_2^+, \mathcal{G}_1^+ \simeq \mathcal{G}_2^+ \Rightarrow (\mathcal{F}_1 \divideontimes \mathcal{G}_1)^+ \simeq (\mathcal{F}_2 \divideontimes \mathcal{G}_2)^+$.*
*Especially, $\boldsymbol{A} \simeq \mathcal{F}^+, \boldsymbol{B} \simeq \mathcal{G}^+ \Rightarrow \boldsymbol{A} \divideontimes \boldsymbol{B} \simeq (\mathcal{F} \divideontimes \mathcal{G})^+$.*

Hence for general frames $\mathcal{F}, \mathcal{G}$, we have

$$(\mathcal{F} \ast\!\!\ast \mathcal{G})^+ \simeq \mathcal{F}^+ \ast\!\!\ast \mathcal{G}^+. \tag{9}$$

Recall that
a general frame $\mathcal{F} = (W, R, \mathcal{P})$ is descriptive (i.e. $\exists \boldsymbol{A}$: modal algebra, $\mathcal{F} \simeq \boldsymbol{A}_+$ )
$\Longleftrightarrow$

- differentiated $\quad \forall x, y \in W(x \neq y \Rightarrow \exists X \in \mathcal{P}$ such that $x \in X, y \notin X)$,
- tight $\quad \forall x, y \in W(x \not R y \Rightarrow \exists X \in \mathcal{P}$ such that $x \in X, y \notin X)$,
- compact $\quad \forall \mathcal{A} \subseteq \mathcal{P}\big((\bigcap \mathcal{A}_0 \neq \emptyset$ for finite $\mathcal{A}_0 \subseteq \mathcal{A}) \Rightarrow \bigcap \mathcal{A} \neq \emptyset\big)$.

**Proposition 3.2** *If general frames $\mathcal{F}, \mathcal{G}$ are descriptive, then the product $\mathcal{F} \ast\!\!\ast \mathcal{G}$ is also descriptive.*

Hence for modal algebras $\boldsymbol{A}, \boldsymbol{B}$, we have

$$(\boldsymbol{A} \ast\!\!\ast \boldsymbol{B})_+ \simeq \boldsymbol{A}_+ \ast\!\!\ast \boldsymbol{B}_+. \tag{10}$$

# 4 Products of Modal Logics by General Frames

For any modal logic $L$, we denote by $\boldsymbol{F}(L)$ the set of all general frames which validate $L$, and by $\boldsymbol{V}(L)$ the set of all modal algebras which validate $L$. For modal logics $L_1, L_2$, we define

$$L_1 \ast\!\!\ast L_2 := \boldsymbol{L}(\boldsymbol{F}(L_1) \ast\!\!\ast \boldsymbol{F}(L_2)).$$

By (9) and (10), we have

$$L_1 \ast\!\!\ast L_2 = \boldsymbol{L}(\boldsymbol{V}(L_1) \ast\!\!\ast \boldsymbol{V}(L_2)).$$

Let $\mathcal{K}$ be a class of modal algebras.
$\boldsymbol{V}(\mathcal{K})$ denotes the smallest variety containing $\mathcal{K}$.
$\boldsymbol{H}(\mathcal{K})$ denotes the smallest class containing $\mathcal{K}$ closed under homomorphic images.
$\boldsymbol{S}(\mathcal{K})$ denotes the smallest class containing $\mathcal{K}$ closed under subalgebras.
$\boldsymbol{P}(\mathcal{K})$ denotes the smallest class containing $\mathcal{K}$ closed under direct product.
For any modal algebra $\boldsymbol{A}$, we denote by $f_{\boldsymbol{A}}$ a isomorphism from $\boldsymbol{A}$ to $(\boldsymbol{A}_+)^+$.

**Theorem 4.1**

1. *If $h : \boldsymbol{A}_1 \to \boldsymbol{A}_2$ and $g : \boldsymbol{B}_1 \to \boldsymbol{B}_2$ are homomorphic onto maps, then $h \ast\!\!\ast g : \boldsymbol{A}_1 \ast\!\!\ast \boldsymbol{B}_1 \to \boldsymbol{A}_2 \ast\!\!\ast \boldsymbol{B}_2$ defined by*

$$h \ast\!\!\ast g(\bigcup_{i \in I} X_i \times Y_i) = \bigcup_{i \in I} f_{\boldsymbol{A}_2} \circ h \circ f^{-1}_{\boldsymbol{A}_1}(X_i) \times f_{\boldsymbol{B}_2} \circ g \circ f^{-1}_{\boldsymbol{B}_1}(Y_i)$$

   *is also homomorphic onto map.*

2. *If $\boldsymbol{A}_1$ is a subalgebra of $\boldsymbol{A}_2$ and $\boldsymbol{B}_1$ is a subalgebra of $\boldsymbol{B}_2$, then $\boldsymbol{A}_1 \ast\!\!\ast \boldsymbol{B}_1$ is also a subalgebra of $\boldsymbol{A}_2 \ast\!\!\ast \boldsymbol{B}_2$.*

3. $\prod_{\lambda \in \Lambda} \boldsymbol{A}_\lambda \ast\!\!\ast \prod_{\mu \in M} \boldsymbol{B}_\mu$ *is a subalgebra of* $\prod_{(\lambda, \mu) \in \Lambda \times M} \boldsymbol{A}_\lambda \ast\!\!\ast \boldsymbol{B}_\mu$.

Hence for classes $\mathcal{K}_1, \mathcal{K}_2$ of modal algebras,

1. $\boldsymbol{H}(\mathcal{K}_1) \ast\!\!\ast \boldsymbol{H}(\mathcal{K}_2) \subseteq \boldsymbol{H}(\mathcal{K}_1 \ast\!\!\ast \mathcal{K}_2)$,
2. $\boldsymbol{S}(\mathcal{K}_1) \ast\!\!\ast \boldsymbol{S}(\mathcal{K}_2) \subseteq \boldsymbol{S}(\mathcal{K}_1 \ast\!\!\ast \mathcal{K}_2)$,
3. $\boldsymbol{P}(\mathcal{K}_1) \ast\!\!\ast \boldsymbol{P}(\mathcal{K}_2) \subseteq \boldsymbol{SP}(\mathcal{K}_1 \ast\!\!\ast \mathcal{K}_2)$.

Therefore we have the following theorem.

**Theorem 4.2** *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be classes of modal algebras, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of general frames. Then*

1. $\boldsymbol{V}\big(\boldsymbol{V}(\mathcal{K}_1) \ast\!\!\!\ast \boldsymbol{V}(\mathcal{K}_2)\big) = \boldsymbol{V}(\mathcal{K}_1 \ast\!\!\!\ast \mathcal{K}_2)$,
2. $\boldsymbol{L}(\mathcal{K}_1) \ast\!\!\!\ast \boldsymbol{L}(\mathcal{K}_2) = \boldsymbol{L}(\mathcal{K}_1 \ast\!\!\!\ast \mathcal{K}_2)$,
3. $\boldsymbol{L}(\mathcal{C}_1) \ast\!\!\!\ast \boldsymbol{L}(\mathcal{C}_2) = \boldsymbol{L}(\mathcal{C}_1 \ast\!\!\!\ast \mathcal{C}_2)$.

Now we define another products ($\times$) of logics instead of products ($\times_s$) of logics.

Let $\boldsymbol{A} = (A, \wedge, \vee, \neg, 0, \Box_{\boldsymbol{A}})$ and $\boldsymbol{B} = (B, \wedge, \vee, \neg, 0, \Box_{\boldsymbol{B}})$ be modal algebras, and let $\mathcal{F} = (W, R, \mathcal{P})$ and $\mathcal{G} = (V, S, \mathcal{Q})$ be general frames. Then we define as follows.
$\boldsymbol{A}^\circ := (A, \wedge, \vee, \neg, 0, \Box_{\boldsymbol{A}}, \Box_{\boldsymbol{B}})$ with $\Box_{\boldsymbol{B}} b = b$,
$^\circ\boldsymbol{B} := (B, \wedge, \vee, \neg, 0, \Box_{\boldsymbol{A}}, \Box_{\boldsymbol{B}})$ with $\Box_{\boldsymbol{A}} a = a$.
$\mathcal{F}^\circ := (W, R, \Delta, \mathcal{P})$,
$^\circ\mathcal{G} := (V, \Delta, S, \mathcal{Q})$.
$\mathcal{F} \times \mathcal{G} := \mathcal{F}^\circ \ast\!\!\!\ast {}^\circ\mathcal{G}$.
For modal logics $L_1, L_2$,
$L_1 \times L_2 := \boldsymbol{L}\big(\boldsymbol{F}(L_1) \times \boldsymbol{F}(L_2)\big)$.

Since $(\boldsymbol{A}_+)^\circ = (\boldsymbol{A}^\circ)_+$ and $(\mathcal{F}^+)^\circ = (\mathcal{F}^\circ)^+$ hold, we have the following theorem.

**Theorem 4.3** *Let $\mathcal{K}_1, \mathcal{K}_2$ be classes of modal algebras, and let $\mathcal{C}_1, \mathcal{C}_2$ be classes of general frames. Then*

1. $\boldsymbol{V}(\mathcal{K}^\circ) = \boldsymbol{V}(\mathcal{K})^\circ$, $\boldsymbol{V}({}^\circ\mathcal{K}) = {}^\circ\boldsymbol{V}(\mathcal{K})$,
2. $\boldsymbol{L}(\mathcal{K}_1) \times \boldsymbol{L}(\mathcal{K}_2) = \boldsymbol{L}(\mathcal{K}_1^\circ \ast\!\!\!\ast {}^\circ\mathcal{K}_2)$,
3. $\boldsymbol{L}(\mathcal{C}_1) \times \boldsymbol{L}(\mathcal{C}_2) = \boldsymbol{L}(\mathcal{C}_1 \times \mathcal{C}_2)$.

Note that $L_1 \times L_2 = \big(L_1 * (\mathbf{K} \oplus \Box p \leftrightarrow p)\big) \ast\!\!\!\ast \big((\mathbf{K} \oplus \Box p \leftrightarrow p) * L_2\big)$.
The following proposition about properties of $L_1 \times L_2$ holds.

**Proposition 4.4** *Let $L_1, L_2$ be modal logics. Then*

1. *If $L_1$ and $L_2$ are consistent mono-modal logics, then $L_1 \times L_2$ is a conservative extension of $L_1$ and $L_2$.*
2. $L_1 \times L_2 \supseteq [L_1, L_2]$,
3. *If $L_1$ and$L_2$ have f.m.p., then $L_1 \times L_2$ has product f.m.p.(i.e. $L_1 \times L_2 = \boldsymbol{L}\big(\boldsymbol{F}_f(L_1) \times \boldsymbol{F}_f(L_2)\big)$), hence $L_1 \times L_2$ also has f.m.p.. ($\boldsymbol{F}_f(L)$ denotes the set of all general frames which validate $L$.)*
4. *If one of $L_1$ and $L_2$ has f.m.p. and the other is Kripke complete, then $L_1 \times L_2$ is also Kripke complete.*

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# On some modal predicate logics with Glivenko type characterization.

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## 1 Problem

Ghilardi [3] showed that **QS4.1**, the least predicate extension of $\mathbf{S4.1} = \mathbf{S4} + \Box\Diamond p \supset \Diamond\Box p$, is not Kripke complete. Hughes-Cresswell [4] showed that **QS4.1** + BF is Kripke incomplete, where BF is the Barcan formula $\forall x \Box A(x) \supset \Box \forall x\ A(x)$.

So, we consider the following problem.

**Problem 1**

(1). *Find the Kripke completion of* **QS4.1**.

*That is, find the logic characterized by the modal predicate Kripke frames in which* $\Box\Diamond p \supset \Diamond\Box p$ *is valid.*

(2). *Find the Kripke completion of* **QS4.1** + BF.

*That is, find the logic characterized by the modal predicate Kripke frames of constantdomains in which* $\Box\Diamond p \supset \Diamond\Box p$ *is valid.*

Note that $\Box\Diamond p \supset \Diamond\Box p$ is valid in a Kripke frame $(\mathbf{M}, U)$ if and only if $\mathbf{M}$ has the following property:

> For every $u \in M$, there is a maximal trivial cluster above $w$, that is, there is an element $v$ such that $uRv$ and $vRw$ implies $w = v$ for all $w$.

In this talk, I show a syntactical characterization of the Kripke completion of **QS4.1** can be obtained by the method described in Shimura [6], and that of **QS4.1**+BF by modifying the method in Shimura [7].

## 2 Glivenko type characterization

It is well-known that the cut-free system **GS4** for the modal propositional logic **S4** gives a useful machinery to prove some syntactical properties and the completeness of **S4**. Shimura [6] gave some kinds of cut-free sequent style systems and showed that a similar machinery works for the logics defined by them.

The propositional logic **S4.1** is one of the logics defined by the sequent style systems. That is, we can define it as the logic defined by the the sequent style system obtained from **GS4** by adding the sequents of the form $\Box\Gamma \ \rightarrow\ $ provable in **GTriv**, where $\mathbf{Triv} = \mathbf{S4} + p \supset \Box p$.

In other words, **S4.1** can be characterized as the least logic with the following propety:

$\mathbf{Triv} \ \vdash\ A \Leftrightarrow \mathbf{L} \ \vdash\ \Diamond\Box A$.

The completeness proof of **S4.1** by means of this sequent style system tells us that the characterization of the frames validating **S4.1** is nothing but the following:

For every world $w$, there exits a world $w'$ above $w$ in which **Triv** is valid.

By transferring the above fact to predicate case, we can obtain the answer to Problem 1 (1).

**Theorem 1** *The Kripke completion is the logic defined by the sequent style system obtained from* **GQS4** *by adding all the sequents of the form* $\Box\Gamma \rightarrow$ *provable in* **GQTriv**.

# 3 Axiomatization.

We should find a formula $C$ with the folloing property to give an axiomatizatiion of the Kripke completion of **QS4.1**.

$$\mathbf{QTriv} \vdash A \Rightarrow \mathbf{QS4.1} + C \vdash \Diamond\Box A.$$

If we try to prove the above assertion by induction on the complexity of the proof in **GQTriv**, the crucial case is that the last inference is $(\rightarrow \forall)$ or $(\exists \rightarrow)$. So, we should take $C$ as $\Box\forall x\Diamond\Box A(x) \supset \Diamond\Box\forall x\ A(x)$, which is the formula given by Corsi-Ghilardi [1].

Note that this formula is equivalent to the McKinsey-Tarski translation $T(K)$ of Kroda axiom $K = \forall x\neg\neg A(x) \supset \neg\neg\forall x\ A(x)$.

The logic axiomatized by $T(K)$ over **QS4** has the following characterization.

**Theorem 2** $\mathbf{QS4} + T(K)$ *is characteized as the least logic* $\mathbf{L}$ *with the folloing Glivenko type property.*

$$\mathbf{QS5} \vdash A \Leftrightarrow \mathbf{L} \vdash \Diamond\Box A.$$

We can summarize the above facts as follows:

**Fact 1**

- $\mathbf{QS4.1} + T(K)$ *is Glivenko type characterization by* **QTriv**
- $\mathbf{QS4.1} + T(K)$ *is the Kripke completion of* **QS4.1**,
- $\mathbf{QS4.1} + T(K)$ *has modal disjunction property,*
- $\mathbf{QS4.1} + T(K)$ *has modal existence property,*
- $\mathbf{QS4.1} + T(K)$ *has interpolation property.*

**Fact 2**

- $\mathbf{QS4} + T(K)$ *has the Glivenko type characterization by* **QS5**
- $\mathbf{QS4} + T(K)$ *has modal disjunction property,*
- $\mathbf{QS4} + T(K)$ *has modal existence property,*

Note that $\mathbf{QS4} + T(K)$ does not have the interpolation property since **QS5** does not have it.

We have few examples of modal predicate logics with the interpolation property. Moreover, we have no general method to determine whether a given logic has the interpolation property or not. So the following question may be interesting.

**Question 1**

(1). *Does* **QS4.1** *have interpolation property?*

(2). *Does* $\mathbf{QS4.1} + T(K) + \mathrm{BF}$ *have interpolation property?*

# 4 Completeness for the logics with the Barcan axiom.

First note that $T(K)$ must be valid in the Kripke frames in which $\Box\Diamond p \supset \Diamond\Box p$ is valid. This implies the Kripke completion of **QS4.1** + BF must contains $T(K)$.

So we have to investigate the structure of the Kripke models with constant domains in which all the instances of $T(K)$ are true.

**Lemma 3 (cf. Shimura [7, Lemma 3.8])** *Let $(\mathbf{M}, D, \models)$ be a canonical model with constant domains and $(\Gamma, \Delta) \in M$ the root of $\mathbf{M}$. Then, if all the instances of $T(K)$ belong to $\Gamma$, there exits a final cluster above any element of $\mathbf{M}$.*

**Lemma 4** *Let $(\mathbf{M}, D)$ be a Kripke frame with constant domains in which there exits a final cluster above any element of $\mathbf{M}$. Then, $\Box\Diamond p \supset \Diamond\Box p$ is not valid in $(\mathbf{M}, D)$ if and only if there is a non-trivial final cluster in $\mathbf{M}$.*

Since **QS4** + BF is strongly Kripke complete with respect to Kripke frames with constant domains, we can apply the method of canonical models by the aid of the above lemmas.

**Theorem 5**

(1). **QS4** + $T(K)$ + BF *is Kripke complete.*

(2). **QS4.1** + $T(K)$ + BF *is Kripke complete.*

This implies the following:

**Fact 3** **QS4** + $T(K)$ + BF *and* **QS4.1** + $T(K)$ + BF *have the following properties:*

- *modal disjunction property,*
- *modal existence property.*

We can generalize the above completeness result as follows:

**Theorem 6 (cf. Shimura [7])** *Let $\mathbf{M}$ be the frame in the following figure and $\{\mathbf{M}_i\}$ a set of finite Kripke frames. If there is no cofinal subreduction from $\mathbf{M}_i$ to $\mathbf{M}$ for every $\mathbf{M}_i$, **QS4** + $\{\alpha(\mathbf{M}_i, \bot)\}$ + $T(K)$ + BF is Kripke complete.*

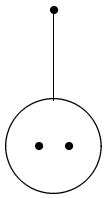

Figure 1:

It is natural to ask whether $\mathbf{L} + T(K) + \mathrm{BF}$ is complete or not for every cofinal subframe logic $\mathbf{L}$. The answer was implicitely given in Isoda and Shimura [5]. In fact, the incompleteness proof of $\mathbf{QS4.1.4} + \mathrm{BF}$ actually proves that of $\mathbf{QS4.1.4} + T(K) + \mathrm{BF}$.

**Theorem 7 (Isoda-Shimura [5])** *Then, $\mathbf{QS4.1.4} + T(K) + \mathrm{BF} = \mathbf{QS4} + \alpha(\mathbf{M}, \bot) + T(K) + \mathrm{BF}$ is Kripke incomplete.*

**Added in proof:** The author noticed after the talk that Theorem 6 is wrong. It seems that we can correct the statement by adding two more conditions on non-exsistence of cofinal subreductions to the frames $\mathbf{M}_i$, but the author does not check whether this correction works yet.

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# Quantum Set Theory

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## 1 Introduction

Quantum physics is interpreted in a Hilbert space model : Physical states are represented as vectors in a Hilbert space, and observables are represented as self-adjoint operators on the Hilbert space. Let $\mathcal{H}$ be a Hilbert space realizing a physical system and let $\mathcal{Q}(\mathcal{H})$ be the complete orthomodular lattice consisting of closed linear subspaces of $\mathcal{H}$, where

$$\begin{aligned} \bigvee_\alpha p_\alpha &= \overline{\bigcup_\alpha p_\alpha}, \\ \bigwedge_\alpha p_\alpha &= \bigcap_\alpha p_\alpha, \\ p^\perp &= \text{the orthocomplement of } p. \end{aligned}$$

$\mathcal{Q}(\mathcal{H})$ is the algebraic expression of quantum logic on the physical system.

We deal with a theory on $\mathcal{Q}(\mathcal{H})$-valued sets, in classical set theory. Let $V$ be the universe of classical set theory as our standpoint, and $\mathcal{Q} = \mathcal{Q}(\mathcal{H})$. $\mathcal{Q}$-valued sets and $\mathcal{Q}$-valued universe $V^{\mathcal{Q}}$ consisting of all $\mathcal{Q}$-valued sets are constructed in $V$. We call $V$ the *external universe for* $V^{\mathcal{Q}}$. Set theory on the $\mathcal{Q}$-valued universe $V^{\mathcal{Q}}$ is called *quantum set theory*.

In [1] and [2], G.Takeuti developed the quantum set theory and showed that each real number in $V^{\mathcal{Q}}$, which is defined as a Dedekind cut of rational numbers, represents a self-adjoint operator in the external universe $V$. We call these reals in $V^{\mathcal{Q}}$ *q-reals*. Thus, observables are represented by q-reals. G.Takeuti defined addition $+$ and multiplication $\cdot$ for commutative q-reals and showed that they represent the corresponding operations on the set of self-adjoint operators. Even if self-adjoint operators $u, v$ are not commutative, $u + v$ and $u \cdot v$ are defined as linear operators.

We develop Takeuti's quantum set theory by adding a strong implication to the logic, and extend the concept of q-reals and q-complexes.

## 2 Quantum Logic

Let $C$ be a property of the states of a physical system $S$. $C$ is considered as an observable as follows: Each measurement which distinguished between the presence or absence of $C$ is considered as a measurement of this observable, such that its value is 1 if $C$ is verified, and zero in the opposite case. The observable that is identified with property $C$ is represented by a projection $[\![C]\!]$ on $\mathcal{H}$. Namely, $[\![C]\!]$ is the projection on the closed subspace consisting of all states $\varphi$ which possess the property $C$ with certainty. The closed subspace which is the range of $[\![C]\!]$ will also denoted by $[\![C]\!]$ without any confusion. $[\![C]\!]$ is an element of $\mathcal{Q}$, and is also a q-real in $V^{\mathcal{Q}}$ as a projection.

Now we consider the property $C$ as a proposition with the truth value $[\![C]\!]$, and logical operators $\wedge$, $\vee$ are interpreted as lattice operators $\wedge$, $\vee$ on $\mathcal{Q}$. Complete orthomodular lattice $\mathcal{Q} = \mathcal{Q}(\mathcal{H})$ consisting of closed subspaces of a complex Hilbert space $\mathcal{H}$ is called quantum logic. $\mathcal{Q}$ is irreducible, that is, the center $Z(\mathcal{Q})$ of $\mathcal{Q}$ consists of two elements 0 and 1. We define implication $\to$ on $\mathcal{Q}$ by

$$(p \to q) \stackrel{\text{def}}{=} \bigvee \{x \in Z(\mathcal{Q}) \mid p \wedge x \leq q\} = \begin{cases} 1 & \text{if } p \leq q \\ 0 & \text{otherwise.} \end{cases}$$

This implication represents the lattice order $\leq$ on $\mathcal{Q}$, and is called the *basic implication*.

Generally, lattice valued universe $V^{\mathcal{L}}$ is constructed for any complete lattice $\mathcal{L}$. In [3], we showed that a lattice valued set theory on the lattice valued universe $V^{\mathcal{L}}$ can be developed by introducing the basic implication $\rightarrow$.

Our quantum logic has, as primitive logical symbols, $\wedge$, $\vee$, $\forall$, $\exists$, orthogonal complementation $^{\perp}$, and basic implication $\rightarrow$.

# 3 Quantum set theory

Quantum valued universe $V^{\mathcal{Q}}$, consisting of all $\mathcal{Q}$-valued sets, is constructed in the same way as Boolean valued universe $V^B$ in the external universe $V$ of classical set theory. G.Takeuti developed a quantum set theory in [1] and [2] using the notation $p \rightarrow q$ by which he denotes $p^{\perp} \vee (p \wedge q)$. We call $p^{\perp} \vee (p \wedge q)$ quantum implication, and denote it by $\rightarrow_Q$ in order to distinguish from the basic implication $\rightarrow$. That is,

$$p \rightarrow_Q q \stackrel{\text{def}}{\Longleftrightarrow} p^{\perp} \vee (p \wedge q).$$

$\rightarrow_Q$ is weaker than $\rightarrow$ in the sense that $(p \rightarrow q) \leq (p \rightarrow_Q q)$. Equality and membership relation $=_Q$ and $\in_Q$ corresponding to the quantum implication are interpreted on the universe $V^{\mathcal{Q}}$ in [1] and [2] as follows :

$$\begin{aligned} [\![u =_Q v]\!] &= \bigwedge_{x \in \mathcal{D}u} (u(x) \rightarrow_Q [\![x \in_Q v]\!]) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \rightarrow_Q [\![x \in_Q u]\!]) \\ [\![u \in_Q v]\!] &= \bigvee_{x \in \mathcal{D}v} [\![u =_Q x]\!] \wedge v(x) \end{aligned}$$

For the quantum equality $=_Q$, the axiom of equality is not valid, because of non-distributivity of $\mathcal{Q}$. Distributive law

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

holds only in the case that $p$ commutative with both of $q$ and $r$, where "$p$ commutative with $q$" means that

$$p = (p \wedge q) \vee (p \wedge q^{\perp}).$$

Thus, the axiom of equality for $=_Q$ is restricted. This means that the quantum equality $=_Q$ is not strong enough to be qualified for an equality.

So, as primitive predicate symbles, we introduce $=$ and $\in$ corresponding to the basic implication $\rightarrow$. They are interpreted on $V^{\mathcal{Q}}$ as

$$\begin{aligned} [\![u{=}v]\!] &= \bigwedge_{x \in \mathcal{D}u} (u(x) \rightarrow [\![x{\in}v]\!]) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \rightarrow [\![x{\in}u]\!]) \\ [\![u{\in}v]\!] &= \bigvee_{x \in \mathcal{D}v} [\![u{=}x]\!] \wedge v(x). \end{aligned}$$

These $=$ and $\in$ are called the basic equality and the basic membership relation. The quantum equality $=_Q$ and the quantum membership relation $\in_Q$ will be defined in our quantum set theory.

We use the following abbreviations.

$$\begin{aligned} \varphi \rightarrow_Q \psi &\stackrel{\text{def}}{\Longleftrightarrow} \varphi^{\perp} \vee (\psi \wedge \varphi) \\ \varphi \downarrow_{\circ} \psi &\stackrel{\text{def}}{\Longleftrightarrow} \varphi \rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \psi^{\perp}) \\ \downarrow_{\circ}(\Gamma) &\stackrel{\text{def}}{\Longleftrightarrow} \bigwedge_{\varphi, \psi \in \Gamma} \varphi \downarrow_{\circ} \psi \\ \Box\varphi &\stackrel{\text{def}}{\Longleftrightarrow} (\varphi \rightarrow \varphi) \rightarrow \varphi \end{aligned}$$

Then the following nonlogical axioms GA1–GA11 of lattice valued set theory LZFZ in [3] are valid on $V^{\mathcal{Q}}$.

**GA1. Equality** $\forall u \forall v\,(u = v \wedge \varphi(u) \to \varphi(v))\,.$

**GA2. Extensionality** $\forall u, v\,(\forall x(x \in u \leftrightarrow x \in v) \to u = v).$

**GA3. Pairing** $\forall u, v \exists z\,(\forall x(x \in z \leftrightarrow (x = u \vee x = v))).$

The set $z$ satisfying $\forall x(x \in z \leftrightarrow (x = u \vee x = v))$ is denoted by $\{u, v\}$.

**GA4. Union** $\forall u \exists z\,(\forall x(x \in z \leftrightarrow \exists y \in u(x \in y)))\,.$

The set $z$ satisfying $\forall x(x \in z \leftrightarrow \exists y \in u(x \in y))$ is denoted by $\bigcup u$.

**GA5. Power set** $\forall u \exists z\,(\forall x(x \in z \leftrightarrow x \subset u))$, where

$$x \subset u \overset{\text{def}}{\Longleftrightarrow} \forall y(y \in x \to y \in u).$$

The set $z$ satisfying $\forall x(x \in z \leftrightarrow x \subset u)$ is denoted by $\mathcal{P}(u)$.

**GA6. Infinity** $\exists u\,(\exists x(x \in u) \wedge \forall x(x \in u \to \exists y \in u(x \in y)))\,.$

**GA7. Separation** $\forall u \exists v\,(\forall x(x \in v \leftrightarrow x \in u \wedge \varphi(x))).$

The set $v$ satisfying $\forall x(x \in v \leftrightarrow x \in u \wedge \varphi(x))$ is denoted by $\{x \in u \mid \varphi(x)\}$.

**GA8. Collection**

$$\forall u \exists v \left(\forall x(x \in u \to \exists y \varphi(x, y)) \to \forall x(x \in u \to \exists y \overset{\scriptscriptstyle\square}{\in} v \varphi(x, y))\right).$$

**GA9. $\in$-induction** $\forall x\,(\forall y(y \in x \to \varphi(y)) \to \varphi(x)) \to \forall x \varphi(x).$

**GA10. Zorn** $\mathrm{Gl}(u) \wedge \forall v\,(\mathrm{Chain}(v, u) \to \bigcup v \in u) \to \exists z \mathrm{Max}(z, u)$, where

$$\begin{aligned} \mathrm{Gl}(u) &\overset{\text{def}}{\Longleftrightarrow} \forall x(x \in u \to x \overset{\scriptscriptstyle\square}{\in} u), \\ \mathrm{Chain}(v, u) &\overset{\text{def}}{\Longleftrightarrow} v \subset u \wedge \forall x, y(x, y \in v \to x \subset y \vee y \subset x), \\ \mathrm{Max}(z, u) &\overset{\text{def}}{\Longleftrightarrow} z \in u \wedge \forall x(x \in u \wedge z \subset x \to z = x). \end{aligned}$$

**GA11. Axiom of $\Diamond$** $\forall u \exists z \forall t(t \in z \leftrightarrow \Diamond(t \in u)).$

The set $z$ satisfying $\forall t(t \in z \leftrightarrow \Diamond(t \in u))$ is denoted by $\Diamond u$.

We take GA1–GA11 as nonlogical axioms of quantum set theory. Then the induction principle is provable in the quantum set theory, and enables us to define the quantum equality $=_Q$ and quantum membership relation $\in_Q$, inductively.

# 4 Quantum numbers

The external universe $V$, in which $V^{\mathcal{Q}}$ is constructed, can be embedded in $V^{\mathcal{Q}}$ :

$$u \mapsto \check{u} \in V^{\mathcal{Q}} \qquad \text{for } u \in V$$

The set of rational numbers in $V^{\mathcal{Q}}$ is $\check{\mathbf{Q}}$ corresponding to the set of rational numbers $\mathbf{Q}$ in $V$ by the embedding. Real numbers are defined as Dedekind cuts of $\check{\mathbf{Q}}$. (cf.[1]). Since $V^{\mathcal{Q}}$ has more members than $\check{V}$, Dedekind cuts of $\check{\mathbf{Q}}$ in $V^{\mathcal{Q}}$ are more rich than those in $V$. The set of q-reals is denoted by $R^{\mathcal{Q}}$. Q-reals represent self-adjoint operators on the Hilbert space $\mathcal{H}$, hence q-reals represent observables in the physical system. If $u, v$ are commutative q-reals, then $u + v$ and $u \cdot v$ are defined in $R^{\mathcal{Q}}$, as shown in [1]. Q-complexes are defined as $u + iv$ where $u, v$ are commutative q-reals. Each q-complex represents normal operator on $\mathcal{H}$. The set of all q-complexes is denoted by $C^{\mathcal{Q}}$, and the set of all bounded q-complexes is denoted by $C_b^{\mathcal{Q}}$.

Let $L(\mathcal{H})$ be the set of closed and densely defined linear operators on $\mathcal{H}$. Self-adjoint operators and normal operators are in $L(\mathcal{H})$, that is, $R^{\mathcal{Q}}$ and $C^{\mathcal{Q}}$ are subsets of $L(\mathcal{H})$. We extend the concept of numbers such as q-reals and q-complexes to closed and densely defined linear operators. For that, we represent $\mathcal{H}$ as a Hilbert space $H$ in $V^{\mathcal{Q}}$.

For a q-real $A$, that is, a self-adjoint operator $A$ on $\mathcal{H}$, there exists a unique resolution $E$ of the identity such that $A = \int \lambda dE(\lambda)$. Let $\mathcal{E}$ be the set of all resolutions of the identity. We can show that for each resolution of the identity, $E \in \mathcal{E}$, there exists an isomorphism of $\mathcal{H}$ into $C^{\mathcal{Q}}$ :

$$\sigma_E : \mathcal{H} \to C^{\mathcal{Q}}$$

$\sigma_E(\mathcal{H})$ is a Hilbert space over $\check{\mathbf{C}}$. It follows that each vector $x$ in $\mathcal{H}$ is considered as a mapping of $\mathcal{E}$ into $C^{\mathcal{Q}}$ :

$$\hat{x} : \mathcal{E} \to C^{\mathcal{Q}} \quad \text{by } \hat{x}(E) = \sigma_E(x)$$

For each $E \in \mathcal{E}$, let

$$H(E) = \{\hat{x}(E) \mid x \in \mathcal{H}\}$$

$$H_b(E) = H(E) \cap C_b^{\mathcal{Q}}$$

$H(E)$ is isomorphic to $\mathcal{H}$ as Hilbert space. $H_b(E)$ is dense in $H(E)$ and forms an abelian von Neumann algebra. Each $u \in H(E)$ defines a multiplication operator $\hat{u}$ on $H_b(E)$ :

$$\hat{u} : \hat{x}(E) \mapsto (ux)\hat{}(E) = u \cdot \hat{x}(E)$$

We represent $\mathcal{H}$ as the set of $\hat{x} \in \Pi_{E \in \mathcal{E}} H(E) \in V^{\mathcal{Q}}$, and denote it by $H$.

$$H = \{\hat{x} : \mathcal{E} \to C^{\mathcal{Q}} \mid x \in \mathcal{H}\} \subset \Pi_{E \in \mathcal{E}} H(E) \in V^{\mathcal{Q}}.$$

Thus, q-real (or q-complex) $\int \lambda dE(\lambda)$ is in $H(E)$, and represents a self-adjoint operator (or normal operator) on $H$.

$H$ can be defined in the language of quantum set theory, and q-reals (or q-complexes) are considered as closed and densely defined linear operators on $H$. Let $L(H)$ be the set of all closed and densely defined linear operators on $H$. Members of $L(H)$ we call *q-numbers*.

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# A real vector space which has a base but which has no well-orderable one

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*This paper is not included in this on-line proceedings.*

# Epistemic Models of Shallow Depths and Decision Making in Games

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# A new system of common knowledge predicate logic

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**Abstract**

We propose a Kripke complete sequent calculus for common knowledge predicate logic[1]. Though the system is infinitary, it is finitary except for one inference rule.

Kaneko and Nagashima [KN96, KN97] proposed a system $\mathrm{GL}_\omega$ for predicate common knowledge logic. The system $\mathrm{GL}_\omega$ is infinitary in the following sense: (1) $\mathrm{GL}_\omega$ allows conjunctions and disjunctions of countably many formulae. Common knowledge is defined as an infinite conjunction. (2) $\mathrm{GL}_\omega$ has inference rules with countably many premises.

Halpern and Moses [HM92] proposed a finitary system for propositional common knowledge logic. In contrast to $\mathrm{GL}_\omega$, Halpern and Moses' system has a symbol for common knowledge. The system has a set of an axiom and an inference rule which defines common knowledge as a fixed point.

A seemingly natural system for predicate common knowledge logic would be obtained by adding quantification rules to Halpern and Moses' system. However, this cannot be Kripke complete. Indeed, any Kripke complete system for predicate common knowledge logic cannot be finitary because of the following:

FACT. (Wolter [W99]) Any Kripke complete system for predicate common knowledge logic is not recursively enumerable.

Here we propose a Kripke complete sequent calculus CK for common knowledge predicate logic. Though the system CK is infinitary, it is finitary except for one inference rule. The system CK has a symbol for common knowledge, and the meaning of this symbol is determined with two inference rules.

The language $\mathcal{L}$ of CK consists of the following symbols. A positive integer $n$ is fixed throughout this article.
Free variables $a_0$, $a_1$, ... (countably many).
Bound variables $x_0$, $x_1$, ... (countably many).
Individual constants (at most finite).
Predicate letters.
Logical symbols $\neg$, $\supset$, $\wedge$, $\vee$, $\forall$, $\exists$.
Epistemic symbols $K_1$, ..., $K_n$, C.

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[1]We consider common knowledge logics based upon KD4, but similar arguments may be carried out for some other modal logics such as K, S4 and so on.

$\mathcal{N} = \{1, \ldots, n\}$ is the set of agents or players. $K_i$ means that *agent $i$ knows* (or *believes*) a proposition. C means that *it is common knowledge among the agents in $\mathcal{N}$.*

Formulae are constructed as in the classical predicate logic with the additional rules: If $A$ is a formula then $K_iA$ is a formula ($i = 1, \ldots, n$); If $A$ is a formula then $\mathrm{C}A$ is a formula.

A sequent is an expression of the form $\Gamma \longrightarrow \Theta$ where $\Gamma$ and $\Theta$ are finite *sets* of formulae.

Let $\mathcal{N}^*$ be the set of all words $i_1 \ldots i_m$ ($i_1 \in \mathcal{N}, \ldots, i_m \in \mathcal{N}$, $m \geq 0$) on the alphabet $\mathcal{N} = \{1, \ldots, n\}$. The concatenation of words $u \in \mathcal{N}^*$ and $v \in \mathcal{N}^*$ is denoted as $uv$. The length of a word $w \in \mathcal{N}^*$ is denoted as $|w|$. Let $\mathcal{N}^{\langle\omega\rangle} = \{i_1 \ldots i_m \mid m \geq 0,\ i_1 \neq i_2 \neq \ldots \neq i_m\} \subset \mathcal{N}^*$. For a word $e = i_1 \ldots i_m \in \mathcal{N}^{\langle\omega\rangle}$, $K_{i_1} \cdots K_{i_m}$ is abbreviated as $K_e$. For $e = i_1 i_2 \ldots i_m \in \mathcal{N}^{\langle\omega\rangle}$ and $e' = j_1 j_2 \ldots j_l \in \mathcal{N}^{\langle\omega\rangle}$, $e * e'$ is defined as

$$e * e' = \begin{cases} i_1 i_2 \ldots i_m j_2 \ldots j_l & \text{if } e \neq \varepsilon,\ e' \neq \varepsilon \text{ and } i_m = j_1, \\ i_1 i_2 \ldots i_m j_1 j_2 \ldots j_l = ee' & \text{otherwise.} \end{cases}$$

$\mathcal{N}^{\langle\omega\rangle}$ is closed under this operation. Now we introduce a notation $K[u; \Delta; A]$ where $A$ is a formula, $u \in \mathcal{N}^*$, and $\Delta$ is a sequence of formulae such that $|u| = |\Delta|$. $K[i_1 i_2 \ldots i_k; D_1, D_2, \ldots, D_k; A]$ means

$$K_{i_1}(D_1 \supset K_{i_2}(D_2 \supset \ldots \supset K_{i_k}(D_k \supset A)\ldots)).$$

The notation $K[u; \Delta; A]$ is meaningful only when $|u| = |\Delta|$.

The system CK has the following schemas for initial sequents and inference rules. The only infinitary rule is ($\to$ C), which has countably many premises.

Initial Sequent:

$$A \longrightarrow A$$

Inference rules:

$$\frac{\Gamma \longrightarrow \Theta}{\Gamma, \Delta \longrightarrow \Theta, \Lambda}\ (\text{set})$$

$$\frac{\Gamma \longrightarrow \Theta, A \quad A, \Delta \longrightarrow \Lambda}{\Gamma, \Delta \longrightarrow \Theta, \Lambda}\ (\text{cut})$$

$$\frac{\Gamma \longrightarrow \Theta, A}{\neg A, \Gamma \longrightarrow \Theta}\ (\neg \to) \qquad \frac{A, \Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta, \neg A}\ (\to \neg)$$

$$\frac{\Gamma \longrightarrow \Theta, A \quad B, \Delta \longrightarrow \Lambda}{A \supset B, \Gamma, \Delta \longrightarrow \Theta, \Lambda}\ (\supset \to) \qquad \frac{A, \Gamma \longrightarrow \Theta, B}{\Gamma \longrightarrow \Theta, A \supset B}\ (\to \supset)$$

$$\frac{B, \Gamma \longrightarrow \Theta}{A \wedge B, \Gamma \longrightarrow \Theta}\ (\wedge \to) \qquad \frac{\Gamma \longrightarrow \Theta, A \quad \Gamma \longrightarrow \Theta, B}{\Gamma \longrightarrow \Theta, A \wedge B}\ (\to \wedge)$$

$$\frac{A, \Gamma \longrightarrow \Theta}{A \wedge B, \Gamma \longrightarrow \Theta}\ (\wedge \to)$$

$$\frac{A, \Gamma \longrightarrow \Theta \quad B, \Gamma \longrightarrow \Theta}{A \vee B, \Gamma \longrightarrow \Theta}\ (\vee \to) \qquad \frac{\Gamma \longrightarrow \Theta, A}{\Gamma \longrightarrow \Theta, A \vee B}\ (\to \vee)$$

$$\frac{\Gamma \longrightarrow \Theta, B}{\Gamma \longrightarrow \Theta, A \vee B}\ (\to \vee)$$

$$\frac{F(t), \Gamma \longrightarrow \Theta}{\forall x\, F(x), \Gamma \longrightarrow \Theta}\ (\forall \to) \qquad \frac{\Gamma \longrightarrow \Theta, K[u; \Delta; F(a)]}{\Gamma \longrightarrow \Theta, K[u; \Delta; \forall x\, F(x)]}\ (\to \forall) \quad (u \in \mathcal{N}^*)$$

$$\frac{F(a), \Gamma \longrightarrow \Theta}{\exists x\, F(x), \Gamma \longrightarrow \Theta}\ (\exists \to) \qquad \frac{\Gamma \longrightarrow \Theta, F(t)}{\Gamma \longrightarrow \Theta, \exists x\, F(x)}\ (\to \exists)$$

$$\frac{\Gamma, K_i\Delta \longrightarrow \Theta}{K_i\Gamma, K_i\Delta \longrightarrow K_i\Theta}\ (K_i) \quad (i \in \mathcal{N},\ |\Theta| \leq 1)$$

$$\frac{K_e A, \Gamma \longrightarrow \Theta}{\mathrm{C}A, \Gamma \longrightarrow \Theta}\ (\mathrm{C} \to) \quad (e \in \mathcal{N}^{\langle\omega\rangle}) \qquad \frac{\{\Gamma \longrightarrow \Theta, K[u; \Delta; K_e A] \mid e \in \mathcal{N}^{\langle\omega\rangle}\}}{\Gamma \longrightarrow \Theta, K[u; \Delta; \mathrm{C}A]}\ (\to \mathrm{C}) \quad (u \in \mathcal{N}^*)$$

Condition on variables: In the rules ($\to \forall$) and ($\exists \to$), the free variable $a$ should not occur in the lower sequent.

We define the fixed-point systems HM[2], QHM and QHM + BF as follows. The language $\mathcal{L}'$ of HM is the propositional fragment of $\mathcal{L}$. The system HM is Hilbert style calculus for propositional multimodal

[2] Slightly different from the original systems in [HM92] but the difference is not essential.

$\mathrm{KD4}_n$ with an additional axiom

$$\mathrm{C}A \supset A \wedge K_1\mathrm{C}A \wedge \ldots \wedge K_n\mathrm{C}A$$

and an additional inference rule

$$\frac{B \supset A \wedge K_1 B \wedge \ldots \wedge K_n B}{B \supset \mathrm{C}A} \quad .$$

The quantificational extension QHM is a system over $\mathcal{L}$, and it has additional quantification rules same as in classical predicate logic. The system QHM + BF is QHM with additional Barcan axiom BF:

$$\forall x\, K_i F(x) \supset K_i \forall x\, F(x).$$

THEOREM 1. CK is Kripke complete. □

Proof will be reported at *proof 2000* meeting[3].

LEMMA 2. $\vdash_{\mathrm{CK}} K_u A \longrightarrow K[u;\Delta;A]$ and $\vdash_{\mathrm{CK}} K[u;\top,\ldots,\top;A] \longrightarrow K_u A$, where $\top$ is $A_0 \supset A_0$ and $A_0$ is any closed atomic formula. □

THEOREM 3. [Barcan rule] $\vdash_{\mathrm{CK}} \forall x\, K_i F(x) \longrightarrow K_i \forall x\, F(x)$. □

Proof. By Lemma 2,

$$\dfrac{\dfrac{\dfrac{K_i F(a) \longrightarrow K_i(\top \supset F(a))}{\forall x\, K_i F(x) \longrightarrow K_i(\top \supset F(a))}\,(\forall\to)}{\forall x\, K_i F(x) \longrightarrow K_i(\top \supset \forall x\, F(x))}\,(\to\forall) \qquad K_i(\top \supset \forall x\, F(x)) \longrightarrow K_i \forall x\, F(x)}{\forall x\, K_i F(x) \longrightarrow K_i \forall x\, F(x)}\,(\mathrm{cut})$$

□

LEMMA 4. If $\vdash_{\mathrm{CK}} K_{e_1}A_1, \ldots, K_{e_m}A_m \longrightarrow \Theta$ and if $|\Theta| \leq 1$ then $\vdash_{\mathrm{CK}} K_{e*e_1}A_1, \ldots, K_{e*e_m}A_m \longrightarrow K_e\Theta$. □

LEMMA 5. [Barcan rule for $K_e$] $\vdash_{\mathrm{CK}} \forall x\, K_e F(x) \longrightarrow K_e \forall x\, F(x)$. □

THEOREM 6. [Barcan rule for C] $\vdash_{\mathrm{CK}} \forall x\, \mathrm{C}F(x) \longrightarrow \mathrm{C}\forall x\, F(x)$. □

THEOREM 7. [Conservativeness] If $A$ contains none of $K_1$, ..., $K_n$, C then $\vdash_{\mathrm{CK}} A \iff \vdash_{\mathrm{LK}} A$. □

Proof. Similarly as in proof of Proposition 4.1 of [KN96], the result of eliminating all occurrences of $K_1$, ..., $K_n$, C from a CK-proof can be easily transformed into an LK-proof. □

THEOREM 8. [Conservativeness] If $A$ contains no C then $\vdash_{\mathrm{CK}} A$ if and only if $A$ is provable in $\mathrm{QKD4}_n$ + BF, where $\mathrm{QKD4}_n$ is the quantificational extension of multimodal $\mathrm{KD4}_n$. □

CONJECTURE. If no C occurs positively in a CK-provable formula $A$ then there exists a CK-proof of $A$ without $(\to \mathrm{C})$. □

LEMMA 9. (1) $\vdash_{\mathrm{CK}} \mathrm{C}A \longrightarrow A$. (2) $\vdash_{\mathrm{CK}} \mathrm{C}A \longrightarrow K_i\mathrm{C}A$. (3) $\vdash_{\mathrm{CK}} \mathrm{C}A \longrightarrow \mathrm{CC}A$.
(4) If $\vdash_{\mathrm{CK}} \Gamma \longrightarrow A$ then $\vdash_{\mathrm{CK}} \mathrm{C}\Gamma \longrightarrow \mathrm{C}A$. □

THEOREM 10. A CK-provable formula containing none of $K_1$, ..., $K_n$ is provable in S4 if we regard C as the box symbol $\Box$. Conversely, a S4-provable formula is provable in CK if we regard $\Box$ as C. □

Proof. First we replace all occurrences of $K_1$, ..., $K_n$, C by $\Box$ in a given CK-proof. The result of this replacement can be easily converted into an S4-proof. The converse follows from Lemma 9. □

THEOREM 11. [Conservativeness] If a quantifier-free sequent $\Gamma \longrightarrow \Theta$ is provable in CK then there exists a CK-proof of $\Gamma \longrightarrow \Theta$ without quantifier rules. □

COROLLARY 12. If a quantifier-free formula is provable in CK, then it is provable in HM.[4] □

THEOREM 13. [Fixed point] $\vdash_{\mathrm{CK}} \mathrm{C}A \longrightarrow A \wedge K_1\mathrm{C}A \wedge \ldots \wedge K_n\mathrm{C}A$ and $\vdash_{\mathrm{CK}} A, K_1\mathrm{C}A, \ldots, K_n\mathrm{C}A \longrightarrow \mathrm{C}A$. □

[3] "Shōmeiron kenkyūshūkai", January 18–20, Tokyo University, Komaba, Tokyo

[4] Cf. Kaneko's Theorem 3.4 in [K99]

THEOREM 14. If $\vdash_{\mathrm{CK}} B \longrightarrow A \wedge K_1 B \wedge \ldots K_n B$ then $\vdash_{\mathrm{CK}} B \longrightarrow \mathrm{C}A$. □

DEFINITION. [K99] A formula in $\mathrm{GL}_\omega$ is a *cc-formula* if it satisfies the following conditions.
(1) It contains no infinitary disjunction.
(2) If it contains an infinitary conjunction then the conjunction has the form $\wedge\{K_e B \mid e \in \mathcal{N}^{\langle\omega\rangle}\}$ for some $B$. □

DEFINITION. [K99] We define a mapping $\psi$ which transforms a $\mathcal{L}$-formula into a cc-formula. Definition is by induction on the structure of formula.
(1) If no C occurs in $A$, then $\psi(A)$ is $A$.
(2) $\psi(A \supset B)$ is $\psi(A) \supset \psi(B)$ and similarly for the other logical symbols.
(3) $\psi(K_i A)$ is $K_i(\psi(A))$.
(4) $\psi(\mathrm{C}A)$ is $\wedge\{K_e(\psi(A)) \mid e \in \mathcal{N}^{\langle\omega\rangle}\}$. □

Kaneko [K99] proved that $\psi$ is a bijection from the set of $\mathcal{L}$-formulae to the set of cc-formulae.

THEOREM 15. If a formula $A$ is provable in CK, then $\psi(A)$ is provable in $\mathrm{GL}_\omega$. □

THEOREM 16. If a cc-formula $A$ is provable in $\mathrm{GL}_\omega$, then $\psi^{-1}(A)$ is provable in CK. □

For a certain propositional fragment of $\mathrm{GL}_\omega$, a corresponding theorem is proved by Kaneko [K99].

Proof. If a cc-formula $A$ is provable in $\mathrm{GL}_\omega$, then $\psi^{-1}(A)$ is valid in every Kripke model of CK, hence provable. □

Any formula provable in QHM + BF is provable in CK, but not conversely.

Open problems. (1) Find a formula that is CK-provable but not provable in QHM + BF. (2) [Hiroakira ONO] What $X$ makes QHM + BF + $X$ equivalent to CK?

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# Ultraproduct of Kripke sheaves and Halldén-completeness in super-intuitionistic predicate logics

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**Abstract**

The Kripke sheaf semantics is an extended Kripke-type semantics, and was introduced in Shehtman-Skvortsov[1]. We can introduce *ultraproduct* of Kripke sheaves and present an application to variants of Halldén-completeness in super-intuitionistic predicate logics.

## 1 Kripke sheaves and their ultraproducts

We fix a pure first-order language $\mathcal{L}$. Note that $\mathcal{L}$ contains neither individual constants nor function symbols. For each non-empty set $U$, we denote by $\mathcal{L}[U]$ the language obtained from $\mathcal{L}$ by adding the name $\overline{u}$ of each $u \in U$. In what follows, we will sometimes use the same letter $u$ for the name of $u$.

**Definition 1.1** A partially ordered set $\mathbf{M} = \langle M, \leq \rangle$ with the least element $0_{\mathbf{M}}$ is said to be a *Kripke base* (for super-intuitionistic logics). We can regard a Kripke base $\mathbf{M}$ as a category in the usual way. Let $\mathcal{S}$ denote the category of all non-empty sets. A covariant functor $D$ from a Kripke base $\mathbf{M}$ to $\mathcal{S}$ is called a *domain-sheaf* over $\mathbf{M}$. That is,

DS1) $D(a)$ is a non-empty set for every $a \in M$,
DS2) for every $a, b \in M$ with $a \leq b$, there exists a mapping $D_{ab} : D(a) \to D(b)$,
DS3) $D_{aa}$ is the identity mapping $id_{D(a)}$ of $D(a)$ for every $a \in M$,
DS4) $D_{ac} = D_{bc} \circ D_{ab}$ for every $a, b, c \in M$ with $a \leq b \leq c$.

A pair $\mathcal{K} = \langle \mathbf{M}, D \rangle$ of a Kripke base $\mathbf{M}$ and a domain-sheaf $D$ over $\mathbf{M}$ is called a *Kripke sheaf* (for super-intuitionistic predicate logics).

If every $D_{ab}$ $(a \leq b)$ is the set-theoretic inclusion, $\langle \mathbf{M}, D \rangle$ is said to be a *Kripke frame* (for super-intuitionistic predicate logics). Intuitively, each $D(a)$ is the individual domain of the world $a \in M$. For each $d \in D(a)$ and each $b \in M$ with $a \leq b$, $D_{ab}(d)$ is said to be the *inheritor* of $d$ at $b$. For each formula $A$ of $\mathcal{L}[D(a)]$ and each $b \in M$ with $a \leq b$, the *inheritor* $A_{a,b}$ of $A$ at $b$ is a formula of $\mathcal{L}[D(b)]$ obtained from $A$ by replacing occurrences of $\overline{u}$ $(u \in D(a))$ by the name $\overline{v}$ of the inheritor $v$ of $u$ at $b$. That is, $A_{ab}$ is $A^{D_{ab}}$.

A binary relation $\models$ between each $a \in M$ and each atomic sentence of $\mathcal{L}[D(a)]$ is said to be a *valuation* on $\langle \mathbf{M}, D \rangle$ if for every $a, b \in M$ and every atomic sentence $A$ of $\mathcal{L}[D(a)]$, $a \models A$ and $a \leq b$ imply $b \models A_{a,b}$. We extend $\models$ to a relation between each $a \in M$ and each sentence of $\mathcal{L}[D(a)]$ inductively as follows:

- $a \models A \wedge B$ if and only if $a \models A$ and $a \models B$,
- $a \models A \vee B$ if and only if $a \models A$ or $a \models B$,
- $a \models A \supset B$ if and only if for every $b \in M$ with $a \leq b$, either $b \not\models A_{a,b}$ or $b \models B_{a,b}$,
- $a \models \neg A$ if and only if for every $b \in M$ with $a \leq b$, $b \not\models A_{a,b}$,
- $a \models \forall x A(x)$ if and only if for every $b \in M$ with $a \leq b$ and every $u \in D(b)$, $b \models A_{a,b}(\overline{u})$,
- $a \models \exists x A(x)$ if and only if there exists $u \in D(a)$ such that $a \models A(\overline{u})$.

A pair $(\mathcal{K}, \models)$ of a Kripke sheaf $\mathcal{K}$ and a valuation $\models$ on it is said to be a *Kripke-sheaf model*. A formula $A$ of $\mathcal{L}$ is said to be *true* in a Kripke-sheaf model $(\mathcal{K}, \models)$ if $0_{\mathbf{M}} \models \overline{A}$, where $\overline{A}$ is the universal closure of $A$. A formula $A$ of $\mathcal{L}$ is said to be *valid* in a Kripke sheaf $\mathcal{K}$ if for every valuation $\models$ on $\mathcal{K}$, $A$ is true in $(\mathcal{K}, \models)$. The set of formulas of $\mathcal{L}$ valid in $\mathcal{K} = \langle \mathbf{M}, D \rangle$ is denoted by $L(\mathcal{K})$ or $L\langle \mathbf{M}, D \rangle$. The following propositions are fundamental properties of Kripke-sheaf semantics.

**Proposition 1.2** *For every Kripke-sheaf model $(\langle \mathbf{M}, D \rangle, \models)$, every $a, b \in M$, and every sentence $A \in \mathcal{L}[D(a)]$, if $a \models A$ and $a \leq b$, then $b \models A_{a,b}$.*

**Proposition 1.3** *For each Kripke-sheaf $\mathcal{K}$, the set $L(\mathcal{K})$ contains all formulas provable in $\mathbf{H}_*$, and is closed under the modus ponens, the rule of generalization and the rule of substitution. Namely, $L(\mathcal{K})$ is a super-intuitionistic predicate logic.*

By virtue of this Proposition, we can apply the Kripke sheaf semantics to the study of super-intuitionistic predicate logics. Now we introduce ultraproduct of Kripke sheaves. First, we state a first-order representation of Kripke sheaves.

**Definition 1.4** A triple $\langle W, R, \rho \rangle$ of a non-empty set $W$, a quasi-order $R$ on $W$ and a partial order $\rho$ on $W$ is said to be a *double-ordered structure*, if

WS0) $W$ has an $R$-minimum element $0_R$.
WS1) for every $a, b \in W$, $a \rho b$ implies $aRb$.
WS2) for every $a, b \in W$, $aRb$ implies that there exists exactly one $c$ such that $a \rho c$ and $c \sim_R b$.

where $\sim_R$ is the equivalence relation induced by $R$. The $c$ determined by $a$ and $b$ is denoted by $[a]_b$

Double-ordered structures are first-order representation of Kripke sheaves. Let $\langle W, R, \rho \rangle$ be a double-ordered structure. We denote by $W^*$ the partially ordered set induced by the quasi-ordered set $(W, R)$. Define $D^*(a/\sim_R) = a/\sim_R$ for each $a/\sim_R \in W^*$, and for every $a/\sim_R$, $b/\sim_R \in W^*$ with $aRb$,

$$D^*_{a/\sim_R \, b/\sim_R}(d) = [d]_b$$

for each $d \in a/\sim_R$. Then $\langle W^*, D^* \rangle$ is a Kripke sheaf with the least element $0_R/\sim_R$ of $W^*$.

**Theorem 1.5** *For every Kripke sheaf $\langle M, D \rangle$, there exits a double-ordered structure $\langle W, R, \rho \rangle$ such that $\langle W^*, D^* \rangle$ is isomorphic to $\langle M, D \rangle$.*

Conversely, we can show that every double-ordered structure is obtained from a Kripke sheaf. Since double-ordered structures are first-order structures, we can introduce ultraproducts of them very easily as first-order structures. Let $\{\langle \mathbf{M}_i, D_i, \models_i \rangle; i \in I\}$ be a family of Kripke sheaf models and $\mathcal{U}$ an ultrafilter over $I$. We denote the untraproduct by $\langle \Pi_{\mathcal{U}} \mathbf{M}_i, \Pi_{\mathcal{U}} D_i, \Pi_{\mathcal{U}} \models_i \rangle$. Note that the Kripke base of the ultraproduct product $\langle \Pi_{\mathcal{U}} \mathbf{M}_i, \Pi_{\mathcal{U}} D_i, \Pi_{\mathcal{U}} \models_i \rangle$ is (isomorphic to the partially ordered set) obtained from $\{\mathbf{M}_i; i \in I\}$ by taking ultraproduct. Therefore we write it by $\Pi_{\mathcal{U}} \mathbf{M}_i$.

**Theorem 1.6 (à la Łoś)** *Let $A(x_1, \ldots, x_n)$ be a formula of $\mathcal{L}$ having no free variables other than $x_1, \ldots, x_n$. Then for every $a/\mathcal{U} \in \Pi_{\mathcal{U}} \mathbf{M}_i$ and every $d_1/\mathcal{U}, \ldots, d_n/\mathcal{U} \in \Pi_{\mathcal{U}} D_i(a/U)$,*
*$a/\mathcal{U} \Pi_{\mathcal{U}} \models_i A(d_1/\mathcal{U}, \ldots, d_n/\mathcal{U})$ if and only if*
*$\{i \in I; d_1(i) \in D_i(a(i)), \ldots, d_n(i) \in D_i(a(i))$ and $a(i) \models_i A(d_1(i), \ldots, d_n(i))\} \in \mathcal{U}$.*

# 2 Halldén-completeness

**Definition 2.1 (Cf. [2] and [4])** A super-intuitionistic predicate logic $\mathbf{L}$ is said to be *Halldén-complete* (H-complete), if for every formulas $A$ and $B$ containing no predicate variable in common, $A \vee B \in \mathbf{L}$ implies either $A \in \mathbf{L}$ or $B \in \mathbf{L}$. A super-intuitionistic predicate logic $\mathbf{L}$ is said to be *sentential Halldén-complete* (SH-complete), if for every sentences $S$ and $T$ containing no predicate variable in common, $S \vee T \in \mathbf{L}$ implies either $S \in \mathbf{L}$ or $T \in \mathbf{L}$.

For super-intuitionistic propositional logics, H-completeness and SH-completeness are the same concept by the definition. We have the following facts for super-intuitionistic propositional logics.

**Fact 2.2** *Let* **L** *be a super-intuitionistic propositional logic.*
*(1)* **L** *is H-complete if and only if* **L** *is meet-irreducible.*
*(2) (Wroński [4])* **L** *is H-complete if and only if* **L** *is characterized by a single subdirectly irreducible Heyting algebra.*

Since every logic is a (possibly infinite) meet of meet-irreducible elements, every super-intuitionistic propositional logic turns out to be a (possibly infinite) meet of H-complete logics.

We will try to deal with these concepts in super-intuitionistic *predicate* logics. For super-intuitionistic predicate logics, Semantical tools do not work well as they do in the study of propositional logics. We try to use Kripke sheaves here. The following can be shown quite easily.

**Proposition 2.3** *Let* **L** *be a super-intuitionistic predicate logic characterized by a single Kripke sheaf. Then* **L** *is SH-complete.*

We can show a partial converse of Proposition 2.3 by making use of the ultraproduct construction.

**Theorem 2.4** *Let* **L** *be a super-intuitionistic predicate logic characterized by a class $\mathcal{C}$ of Kripke sheaves such that $\mathcal{C}$ is closed under ultraproducts. Then the following conditions are equivalent.*
*(1)* **L** *is SH-complete.*
*(2)* **L** *is characterized by a single Kripke sheaf.*

Clearly, H-completeness implies SH-completeness. But the converse does not hold. To show this, we introduce the following condition on a class $\mathcal{C}$ of Kripke sheaves called the *joint p-morphism property* :

(JPP) for every $\mathcal{K}_1$ and $\mathcal{K}_2$ in $\mathcal{C}$, there exist a member $\mathcal{K} = \langle \mathbf{M}, D\rangle$ in $\mathcal{C}$, elements $a_1, a_2 \in \mathbf{M}$, and $p$-morphisms $f_1$ of $\mathcal{K}^{a_1}$ to $\mathcal{K}_1$ and $f_2$ of $\mathcal{K}^{a_2}$ to $\mathcal{K}_2$.

where $\mathcal{K}^{a_1}$ and $\mathcal{K}^{a_2}$ are generated subsheaves determined by $a_1$ and $a_2$, respectively.

**Lemma 2.5** *Suppose that a class $\mathcal{C}$ of Kripke sheaves satisfies (JPP). Then a super-intuitionistic predicate logic characterized by $\mathcal{C}$ is SH-complete.*

Consider the following condition (#) on a Kripke sheaf $\langle \mathbf{M}, D\rangle$:

(#) $\forall x \in \mathbf{M}$ : $D(x)$ has exactly two elements.
i.e. $\exists d_1\, \exists d_2 \in D(x)$ $d_1 \neq d_2$ and $\forall e \in D(x)(e = d_1$ or $e = d_2)$.

Let $\mathcal{D}$ be the class of Kripke sheaves satisfying (#). Clearly, $\mathcal{D}$ satisfies the condition (JPP). Let $\mathbf{L}_0$ be a super-intuitionistic predicate logic characterized by $\mathcal{D}$. Then,

(1) $\mathcal{D}$ is closed under ultraproducts,
(2) $\mathbf{L}_0$ is SH-complete,
(3) Hence, $\mathbf{L}_0$ is characterized by a single Kripke sheaf.

However $\mathbf{L}_0$ is not H-complete. Let $A$ and $B$ be $p(x) \equiv p(y)$ and $(q(y) \equiv q(z)) \vee (q(z) \equiv q(x))$, respectively, where $p$ and $q$ are distinct unary predicate variables and $x, y, z$ are mutually distinct individual variables. Then $A$ and $B$ contain no predicate variable in common. We have $A \vee B \in \mathbf{L}_0$, $A \notin \mathbf{L}_0$, and $B \notin \mathbf{L}_0$. Therefore,

**Theorem 2.6** *SH-completeness does not imply H-completeness.*

## 3 One more variant of Halldén-completeness

We can introduce another variant of Halldén-completeness as follows.

**Definition 3.1** A super-intuitionistic predicate logic $\mathbf{L}$ is said to be *weakly Halldén-complete* (WH-complete), if for every formulas $A$ and $B$ containing no predicate variable nor free individual variable in common, $A \vee B \in \mathbf{L}$ implies either $A \in \mathbf{L}$ or $B \in \mathbf{L}$.

It is obvious that WH-completeness implies SH-completeness. Under a condition, WH-completeness is equivalent to SH-completeness.

**Proposition 3.2** *Let $Dis$ be the sentence $\forall x(p(x) \vee r) \supset (\forall x p(x) \vee r)$, where $p$ and $r$ are a unary predicate variable and a propositional variable, respectively. Suppose that a super-intuitionistic predicate logic $\mathbf{L}$ contains $Dis$. Then, $\mathbf{L}$ is SH-complete if and only if $\mathbf{L}$ is WH-complete.*

Note that the example $\mathbf{L}_0$ introduced in the previous section contains $Dis$. Therefore, even if we assume $\mathbf{L} \ni Dis$, SH-completeness and WH-completeness do not imply H-completeness. Now we show

**Theorem 3.3** *SH-completeness does not imply WH-completeness.*

*Proof.* Let $\mathbf{2}$ be $\{0,1\}$ with the canonical order, $U$ a mapping of $\mathbf{2}$ defined by: $U(0) = \{0\}$ and $U(1) = \omega = \{i : i < \omega\}$. Then $\langle \mathbf{2}, U\rangle$ is a Kripke sheaf. Let $\mathbf{L}_1$ be a super-intuitionistic predicate logic characterized by $\langle \mathbf{2}, U\rangle$. Clearly $\mathbf{L}_1$ is SH-complete. Let $C$ and $D$ be formulas $p(x) \equiv p(y)$ and $r \vee \neg r$, respectively, where $p$ and $r$ are a unary predicate variable and a propositional variable, respectively. Then $C$ and $D$ contain no predicate variable nor free individual variable in common. Verify that

(0) $C \vee D$ is valid in $\langle \mathbf{2}, U\rangle$,
(1) neither $C$ nor $D$ is valid in $\langle \mathbf{2}, U\rangle$.

Namely, $C \vee D \in \mathbf{L}_1$, $C \notin \mathbf{L}_1$, and $D \notin \mathbf{L}_1$. Therefore, $\mathbf{L}_1$ is not WH-complete. *q.e.d.*

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# Propositional Dynamic Logic has interpolation

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## 1 Prelude

Propositional Dynamic Logic (**PDL**) is the logic corresponding in a natural way to the variety $\mathcal{D}$ of *dynamic algebras*. Consult e.g., [5] for details on dynamic algebras. Precisely speaking, the correspondence holds between $\mathcal{D}$ and the so-called *test-free* **PDL**, but since it is proved in [2] that *test-free* **PDL** has interpolation iff *full* **PDL** does, we will systematically disregard the distinction between them. It is also conjectured in [2] that **PDL** indeed has interpolation.

## 2 Dynamic algebras

A dynamic algebra is an algebra $\mathbf{A} = \langle A; \vee, \wedge, \neg, 0, 1, \pi(\pi \in \Pi)\rangle$, such that:

1. $\langle A; \vee, \wedge, \neg, 0, 1\rangle$ is a boolean algebra;
2. $\Pi$ is the universe of the absolutely free algebra $\mathbf{\Pi} = \langle \Pi; \circ, \cup, *\rangle$, of the type $\langle 2, 2, 1\rangle$, countably generated by the set $\Xi = \{\xi_n : n \in \omega\}$.
3. $\mathbf{A}$ satisfies:
   (i) $\pi 0 = 0$,
   (ii) $\pi(x \vee y) = \pi x \vee \pi y$,
   (iii) $(\pi \cup \sigma)x = \pi x \vee \sigma x$,
   (iv) $(\pi\sigma)x = \pi(\sigma x)$,
   (v) $x \vee \pi\pi^* x \leq \pi^* x \leq x \vee \pi^*(\neg x \wedge \pi x)$.

An algebra $\mathbf{A}$, of the type of dynamic algebras, is called $*$-*continuous* iff $\pi^* x = \bigvee\{\pi^n : n \in \omega\}$, for all $\pi \in \Pi$, with $\pi^n x$ defined inductively by: $\pi^0 x = x$, $\pi^{n+1} x = \pi\pi^n x$. All $*$-continuous algebras are dynamic algebras. A class $\mathcal{K}$ of dynamic algebras is $*$-continuous iff if all its members are.

**Fact 1** *All finite dynamic algebras are $*$-continuous. If a class $\mathcal{K}$ is $*$-continuous, then so is $SP(\mathcal{K})$.*

Recall, that an algebra is called *residually finite* iff it is a subdirect product of finite algebras.

**Fact 2** ([5] Theorem 5.4.) *Free dynamic algebras are residually finite.*

**Lemma 1** *Free dynamic algebras are $*$-continuous.*

# 3 Amalgamation and interpolation

A triple $\langle \mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2 \rangle$ of similar algebras, is called a V-formation iff there are embeddings $i_1 : \mathbf{A}_0 \longrightarrow \mathbf{A}_1$ and $i_2 : \mathbf{A}_0 \longrightarrow \mathbf{A}_2$. In other words, in a V-formation, $\mathbf{A}_1$ and $\mathbf{A}_2$ share an isomorphic copy of $\mathbf{A}_0$ as a subalgebra. Thus, dealing with V-formations we will always tacitly assume $i_1$ and $i_2$ to be identity maps.

A V-formation $\langle \mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2 \rangle$, with $\mathbf{A}_i \in \mathcal{K}$, for $i = 0, 1, 2$ and $\mathcal{K}$ a class of similar algebras, can be *amalgamated* in $\mathcal{K}$ iff there is an algebra $\mathbf{B} \in \mathcal{K}$ and embeddings $e_1 : \mathbf{A}_1 \longrightarrow \mathbf{B}$, $e_2 : \mathbf{A}_2 \longrightarrow \mathbf{B}$, such that $e_1 \circ i_1 = e_2 \circ i_2$ (or, by our convention, $e_1|_{A_0} = e_2|_{A_0}$).

A class $\mathcal{K}$ of similar algebras has the *amalgamation property* (AP) iff every V-formation from $\mathcal{K}$ can be amalgamated in $\mathcal{K}$.

Assume that the universes of algebras in $\mathcal{K}$ are partially ordered. Then $\mathcal{K}$ is said to have the *superamalgamation property* (SupAP) iff for any V-formation $\langle \mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2 \rangle$, there is a $\mathbf{B}$ that amalgamates it, with the embeddings $f_1 : \mathbf{A}_1 \longrightarrow \mathbf{B}$ and $f_2 : \mathbf{A}_2 \longrightarrow \mathbf{B}$, such that we have: for any $x \in A_k$ and $y \in A_j$, if $f_k(x) \leq f_j(y)$, then there is a $z \in A_0$ with $x \leq i_k(z)$, $i_j(z) \leq y$, where $\{k, j\} = \{1, 2\}$ (or, again, by our convention, simply $x \leq z$, $z \leq y$).

Clearly, SupAP is stronger than AP; thus, by obvious extensions of the definitions above we will speak of V-formations that superamalgamate, or of superamalgams of certain V-formations.

A (propositional) logic $\Lambda$, with a certain reasonable implication $\rightarrow$, possesses *Craig interpolation property* iff for any formulas $\phi$, $\psi$ in the appropriate language, the following holds:

(†) if $\vdash_\Lambda \phi \rightarrow \psi$, then there is a formula $\chi$, whose variables are a subset of those that $\phi$ and $\psi$ have in common, and such that $\vdash_\Lambda \phi \rightarrow \chi$ and $\vdash_\Lambda \chi \rightarrow \psi$.

The next, crucial, proposition follows easily from Madarász's generalisation of Maximova's celebrated theorem (see [3] and [4]), and the folklore about dynamic algebras recalled above. Let $\mathbf{F}_\mathcal{K}(Z)$ stand for the $\mathcal{K}$-free algebra generated by $Z$. We say that free algebras from $\mathcal{K}$ superamalgamate iff every V-formation $\langle \mathbf{F}_\mathcal{K}(X \cap Y), \mathbf{F}_\mathcal{K}(X), \mathbf{F}_\mathcal{K}(Y) \rangle$ (which implicitly assumes the existence of all the algebras involved) can be superamalgamated in the variety $\mathcal{V}(\mathcal{K})$. If the above holds for finite $X$ and $Y$, we say that free finitely generated algebras from $\mathcal{K}$ superamalgamate.

**Proposition 1** *For the variety $\mathcal{D}$ of dynamic algebras the following are equivalent:*

(i) *$\mathcal{D}$ has SupAP,*

(ii) *free finitely generated algebras from $\mathcal{D}$ superamalgamate,*

(iii) *$\Lambda(\mathcal{D})$ has Craig interpolation.*

# 4 Boolean interlude

We will need two constructions well-known in the theory of boolean algebras (BAs). For lack of space we will not recall any definitions, instead, we refer the reader to [1]. The first construction is that of *minimal completion.*

**Fact 3** *For any BA $\mathbf{A}$ there is a unique complete BA $\mathbf{C}$, such that $\mathbf{A} \subseteq \mathbf{C}$ and $\mathbf{C}$ preserves all joins that exist in $\mathbf{A}$.*

The unique complete algebra $\mathbf{C}$ above is called the minimal completion of $\mathbf{A}$. Let $\mathbf{A} \subseteq \mathbf{B}$ be BAs, and $\mathbf{C}$, $\mathbf{D}$ their respective minimal completions. We have:

**Lemma 2** *If $\mathbf{B}$ preserves all joins that exist in $\mathbf{A}$, then the embedding $i : \mathbf{A} \longrightarrow \mathbf{B}$ extends in a natural way to a complete embedding $e : \mathbf{C} \longrightarrow \mathbf{D}$.*

Let $\mathbf{F}(X)$, $\mathbf{F}(Y)$ be BAs freely generated by $X$ and $Y$, with $Y \subseteq X$ and $|Y| \leq |X| = \omega$.

**Lemma 3** *The identity embedding $i : \mathbf{F}(Y) \longrightarrow \mathbf{F}(X)$ preserves all joins that exist in $\mathbf{F}(Y)$.*

Let $\langle \mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2 \rangle$ be a V-formation of BAs. The *amalgamated free product* $\mathbf{B}$ of $\mathbf{B}_1$, $\mathbf{B}_2$ over $\mathbf{B}_0$ satisfies the following characterisation lemma (dual of [2] Corollary 11.22).

**Fact 4** $\mathbf{B}$ *is the amalgamated free product of* $\mathbf{B}_1$, $\mathbf{B}_2$ *over* $\mathbf{B}_0$, *with the embeddings* $e_1$, $e_2$ *iff* $e_1(B_1) \cup e_2(B_2)$ *generates* $\mathbf{B}$ *and for any two elements* $b_1 \in B_1$, $b_2 \in B_2$ *with* $e_1(b_1) \vee e_2(b_2) = 1$, *there is a* $c \in B_0$ *such that* $b_1 \geq \neg c$ *and* $b_2 \geq c$.

Under certain completeness assumptions amalgamated free products behave extremely well (cf.[1] Lemma 12.16).

**Fact 5** *If the algebras* $\mathbf{B}_0$, $\mathbf{B}_1$, $\mathbf{B}_2$, *and the identity embeddings are all complete, then the amalgamated free product* $\mathbf{B}$ *of* $\langle \mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2 \rangle$ *preserves all joins from* $\mathbf{B}_i$ *(*$i = 1, 2$*).*

# 5 Finale

First we extend the notion of minimal completion to dynamic algebras. Recall that, by Lemma 1, free dynamic algebras are $*$-continuous.

**Lemma 4** *Fact 3 and Lemma 2 remain valid for* $*$*-continuous algebras. Lemma 3 holds also for free dynamic algebras.*

Then, take the V-formation $\langle \mathbf{F}(X \cap Y), \mathbf{F}(X), \mathbf{F}(Y) \rangle$. As $\mathcal{D}$ has constants, the free algebras involved always exist. Clearly, the boolean reducts of the algebras above also constitute a V-formation.

Since $\mathbf{F}(X \cap Y)$, $\mathbf{F}(X)$, and $\mathbf{F}(Y)$ are all $*$-continuous, it follows by the lemma above that there are unique complete $*$-continuous algebras $\mathbf{G}_{X \cap Y}$, $\mathbf{G}_X$, and $\mathbf{G}_Y$ with $\mathbf{F}(Z) \subseteq \mathbf{G}_Z$, for $Z \in \{X, Y, X \cap Y\}$ such that $\mathbf{G}_Z$ preserves all joins existing in $\mathbf{F}(Z)$. It also follows that for $Z \in \{X, Y\}$, $\mathbf{G}_Z$ preserves all joins existing in $\mathbf{F}(X \cap Y)$. Consider the V-formation $\langle \mathbf{G}_{X \cap Y}, \mathbf{G}_X, \mathbf{G}_Y \rangle$.

**Fact 6** *The identity embeddings in* $\langle \mathbf{G}_{X \cap Y}, \mathbf{G}_X, \mathbf{G}_Y \rangle$ *are complete.*

Let $\mathbf{G}$ be the *amalgamated free product* of the boolean reducts of $\mathbf{G}_X$ and $\mathbf{G}_Y$ over $\mathbf{G}_{X \cap Y}$. Let $\mathbf{C}$ be the minimal completion of $\mathbf{G}$ (as a BA).

**Fact 7** $\mathbf{C}$ *can be expanded to a* $*$*-continuous algebra* $\mathbf{D}$, *with* $D = C$. *Moreover,* $\mathbf{D}$ *is a superamalgam of* $\langle \mathbf{F}(X \cap Y), \mathbf{F}(X), \mathbf{F}(Y) \rangle$.

**Corollary 1** *Free finitely generated algebras from* $\mathcal{D}$ *superamalgamate.*

By Proposition 1, we obtain:

**Theorem 1** *The variety* $\mathcal{D}$ *of dynamic algebras has SupAP. Full propositional dynamic logic has Craig interpolation property.*

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# On interpretability logics

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The idea of interpretability logics arose in Visser [Vis90]. He introduced the logics as extensions of the provability logic **GL** with a binary modality $\triangleright$. The arithmetic realization of $A \triangleright B$ in a theory $T$ will be that $T$ plus the realization of $B$ is interpretable in $T$ plus the realization of $A$. ($T + A$ *interprets* $T + B$). More precisely, there exists a function $f$ (the relative interpretation) on the formulas of the language of $T$ such that $T + B \vdash C$ implies $T + A \vdash f(C)$.

The basic interpretability logic **IL** is the smallest set of formulas containing **GL** and axioms

$(J1) : \Box(p \supset q) \supset (p \triangleright q)$,
$(J2) : (p \triangleright q) \wedge (q \triangleright r) \supset (p \triangleright r)$,
$(J3) : (p \triangleright r) \wedge (q \triangleright r) \supset (p \vee q \triangleright r)$,
$(J4) : (p \triangleright q) \supset (\Diamond p \supset \Diamond q)$,
$(J5) : \Diamond p \triangleright p$,

and closed under modus ponens, substitution and necessitation. The principles of **IL** are arithmetically sound for a wide class of theories and for various interpretations of its main connective $\triangleright$. The theory is not arithmetically complete for any known interpretation. The motivation for studying this specific set of formulas comes form its modal simplicity and elegance.

The modality $\triangleright$ has more than one interpretation. Another most salient interpretation is $\Pi_1$-*conservativity*. More precisely, the arithmetic realization of $A \triangleright B$ in a theory $T$, containing $\mathbf{I\Sigma_1}$, will be that $T$ plus the realization of $B$ is $\Pi_1$-conservative over $T$ plus the interpretation of $A$. In Berarducci [Ber90] and Shavrukov [Sha88], it was proved that the interpretability logic (**ILM**) obtained by adding Montagna's axiom

$$M : (p \triangleright q) \supset (p \wedge \Box r) \triangleright (q \wedge \Box r)$$

to **IL** is complete for this arithmetic interpretation. The definition of the arithmetic interpretation $*$ given for the proof of Solovay's theorem was extended to the language of interpretability logics by stipulating that

$(A \triangleright B)^* = Conserv(\langle A^* \rangle, \langle B^* \rangle)$,

where $Conserv(\langle A^* \rangle, \langle B^* \rangle)$ is an intensional formalization of "$T + B^*$ is $\Pi_1$-conservative over $T + A^*$", and it was proved

$$A \in \mathbf{ILM} \text{ if and only if } T \vdash A^* \text{ for any } *.$$

The interpretability logic **ILP**, an extension of **IL** by adding the axiom

$$P : (p \triangleright q) \supset \Box(p \triangleright q),$$

is also complete for another arithmetic interpretation. $P$ is valid for interpretations in finitely axiomatized arithmetical theories extending, say, $I\Delta_0 + \Omega_1$, and an arithmetic completeness for this interpretation was proved in [Vis90].

The completeness with respect to Kripke semantics due to Veltman was, for **IL**, **ILM** and **ILP**, proved in de Jongh and Veltman [JV90]. The fixed point theorem of **GL** can be extended to **IL** and hence **ILM** and **ILP**(de Jongh and Visser [JV91]). The unary pendant "$T$ interprets $T + A$" is much less expressive and was studied in de Rijke [Rij92]. For an overview of interpretability logic, see Visser [Vis97], and Japaridze and de Jongh [JJ98].

Here we give a cut-free sequent system for the basic interpretability logic **IL**. First, we give a cut-free system for a sublogic **IK4**, whose $\triangleright$-free fragment is the modal logic **K4**. In other words, **IK4** is the logic obtained from **K4** by adding the axioms $(J1), (J2), (J3), (J4)$ and $(J5)$. Using the system and a property of Löb's axiom presented in [Sas99c] we obtain a cut-free system for **IL**.

Let us define a sequent system for **IK4**. We use Greek letters, possibly with suffixes, for finite sets of formulas, in particular, we use $\Sigma$, possibly with suffices, for finite set of $\triangleright$-formulas[1]. The expression $\Gamma_A$ denotes $\Gamma - \{A\}$. For each prefix $\bigcirc \in \{\Box, \neg\}$, the expression $\bigcirc\Gamma$ denotes the set $\{\bigcirc X | X \in \Gamma\}$. By a sequent, we mean the expression

$$\Gamma \to \Delta.$$

For brevity's sake, we write

$$A_1, \cdots, A_n, \Gamma_1, \cdots, \Gamma_m \to B_1, \cdots, B_\ell, \Delta_1, \cdots, \Delta_k$$

instead of

$$\{A_1, \cdots, A_n\} \cup \Gamma_1 \cup \cdots \cup \Gamma_m \to \{B_1, \cdots, B_\ell\} \cup \Delta_1 \cup \cdots \cup \Delta_k.$$

Let $S_1, \cdots, S_n$ be a sequence of sequents. The depth $d_{S_1,\cdots,S_n}(o_i)$ of the occurrence $o_i$ of $S_i$ denotes $n + 1 - i$. If there is no confusion, we omit the subscript of $d$.

Let $\Sigma$ be a finite set of $\triangleright$-formulas. A sequent is said to be a sequent on a sequent $\Sigma, \Box\Gamma \to A \triangleright B$ if it is of the form

$$C, \Box\neg(B^*, \Delta), \Gamma, \Box\Gamma \to B^*, \Delta,$$

where $C \in \{A\} \cup \{Y | X \triangleright Y \in \Sigma\}$, $\Delta \subseteq \{X | X \triangleright Y \in \Sigma\}$ and

$$B^* = \begin{cases} \emptyset & \text{if } B = \bot \\ B & \text{otherwise.} \end{cases}$$

We put

$$cl(C, \Box\neg(B^*, \Delta), \Gamma, \Box\Gamma \to B^*, \Delta) = C,$$
$$cr(C, \Box\neg(B^*, \Delta), \Gamma, \Box\Gamma \to B^*, \Delta) = \Delta.$$

Immediately, a sequent $S$ on $\Sigma, \Box\Gamma \to A \triangleright B$ is of the form

$$cl(S), \Box\neg(B^*, cr(S)), \Gamma, \Box\Gamma \to B^*, cr(S).$$

For an occurrence $o$ of a sequent $S$ in **SEQ**, we also use the expressions $cl(o)$ and $cr(o)$ for $cl(S)$ and $cr(S)$, respectively. We say that a sequence **SEQ** of sequents is on a sequent $\Sigma, \Box\Gamma \to A \triangleright B$ if it is finite sequence of sequents on $\Sigma, \Box\Gamma \to A \triangleright B$ and $cl(o) = A$ for some occurrence $o$ of a sequent in **SEQ**.

Let $\Sigma'$ be a subset of $\Sigma$. A mapping

$$f : \{(o, X) | o \text{ is an occurrence of a sequent in } \mathbf{SEQ}, X \in cr(o)\}$$

$$\to \{o | o \text{ is an occurrence of a sequent in } \mathbf{SEQ}, cl(o) \neq A\}$$

is called a mapping on **SEQ**$/\Sigma'$ if $X \triangleright cl(f(o, X)) \in \Sigma'$ and $d(o) > d(f(o, X))$ for any occurrence $o$ and an formula $X \in cr(o)$.

We say that a sequence **SEQ** of sequents is *essential* on a sequent $\Sigma, \Box\Gamma \to A \triangleright B$ if it is on $\Sigma, \Box\Gamma \to A \triangleright B$ and the following two conditions hold:

(I) there exists an onto mapping on **SEQ**$/\Sigma$,

(II) $cl(o_1) \neq cl(o_2)$ for any different occurrences $o_1$ and $o_2$ of sequents in **SEQ**.

Let **SEQ** be an essential sequence on $\Sigma, \Box\Gamma \to A \triangleright B$ and let $f$ be an onto mapping on **SEQ**$/\Sigma$. Since **SEQ** is finite, we can see a sequent

$$C, \Box\neg(B^*, \Delta), \Gamma, \Box\Gamma \to B^*, \Delta$$

[1] formulas of the form $A \triangleright B$

at the end occurrence $o$ of **SEQ**. If $X \in \Delta$, then $0 = d(o) > d(f(o, X))$. This is a contradiction. So, we have $\Delta = \emptyset$, i.e., we can see a sequent

$$Y, \Box\neg B^*, \Gamma, \Box\Gamma \to B^*$$

at the end of the sequence **SEQ**.

Here we show some examples of essential sequences. A sequence of a sequent

$$p, \Box\neg q, r, \Box r \to q$$

is only one essential sequence on

$$\Box r \to p \triangleright q.$$

Two sequences of sequents

$$p, \Box\neg q, \Box\neg p_1 \to q, p_1 \qquad q_1, \Box\neg q \to q$$

and

$$p, \Box\neg q \to q$$

are essential on

$$p_1 \triangleright q_1 \to p \triangleright q.$$

Also two sequences

$$p, \Box\neg q, \Box\neg p_1 \to q, p_1 \quad q_1, \Box\neg q, \Box\neg p_2, \to q, p_2 \quad q_2, \Box\neg q \to q$$

and

$$p, \Box\neg q, \Box\neg p_1, \Box\neg p_2 \to q, p_1, p_2 \quad q_1, \Box\neg q \to q \quad q_2, \Box\neg q \to q$$

are essential on

$$p_1 \triangleright q_1, p_2 \triangleright q_2 \to p \triangleright q.$$

Our system **GIK4** is defined from the following axioms and inference rules in the usual way.

**Axioms of GIK4**

$$A \to A$$

$$\bot \to$$

**Inference rules of GIK4**

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}(T \to) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}(\to T)$$

$$\frac{\Gamma \to \Delta, A \qquad A, \Pi \to \Lambda}{\Gamma, \Pi_A \to \Delta_A, \Lambda}(\text{cut})$$

$$\frac{A_i, \Gamma \to \Delta}{A_1 \wedge A_2, \Gamma \to \Delta}(\wedge \to_i) \qquad \frac{\Gamma \to \Delta, A \qquad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}(\to \wedge)$$

$$\frac{A, \Gamma \to \Delta \qquad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}(\vee \to) \qquad \frac{\Gamma \to \Delta, A_i}{\Gamma \to \Delta, A_1 \vee A_2}(\to \vee_i)$$

$$\frac{\Gamma \to \Delta, A \qquad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}(\supset \to) \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}(\to \supset)$$

$$\frac{\textbf{SEQ}}{X_1 \triangleright Y_1, \cdots, X_n \triangleright Y_n, \Box\Gamma \to A \triangleright B}(\triangleright_{IK4})$$

where **SEQ** is essential on the lower sequent,

$$\frac{\textbf{SEQ}}{X_1 \triangleright Y_1, \cdots, X_n \triangleright Y_n, \Box\Gamma \to \Box A}(\Box_{IK4})$$

where **SEQ** is essential on

$$X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Box\Gamma \to \neg A \rhd \bot$$

A proof figure for a sequent $\Gamma \to \Delta$ is defined as usual. We say that $\Gamma \to \Delta$ is provable in **GIK4**, and write $\Gamma \to \Delta \in$ **GIK4**, if there exists a proof figure for $\Gamma \to \Delta$.

Then we have the following two theorems.

**Theorem 1.** *$A \in$ **IK4** if and only if $\to A \in$ **GIK4**.*

**Theorem 2.** *If $\Gamma \to \Delta \in$ **GIK4**, then there exists a cut-free proof figure for $\Gamma \to \Delta$ in **GIK4**.*

Next, we define a sequent system **GIL** for **IL**. The system **GIL** is obtained from **GIK4** by replacing $(\Box_{IK4})$ and $(\rhd_{IK4})$ by the following two inference rules, respectively:

$$\frac{\textbf{SEQ}}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Box\Gamma \to A \rhd B}(\rhd_{IL})$$

where **SEQ** is essential on the sequent $X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Box\Gamma \to (A \wedge \Box\neg A) \rhd B$,

$$\frac{\textbf{SEQ}}{X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Box\Gamma \to \Box A}(\Box_{IL})$$

where **SEQ** is essential on the sequent $X_1 \rhd Y_1, \cdots, X_n \rhd Y_n, \Box\Gamma \to (\neg A \wedge \Box A) \rhd \bot$.

**Theorem 3.** *$A \in$ **IL** if and only if $\to A \in$ **GIL**.*

For cut-elimination theorem, we use the method in [Sas99c]. By sketching his proof, we obtain

**Theorem 4.** *If $\Gamma \to \Delta \in$ **GIL**, then there exists a cut-free proof figure for $\Gamma \to \Delta$.*

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# Cut-free Sequent Calculi for Visser's Propositional Logics

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## 1 Introduction

Visser [5] introduced Basic Propositional Logic (BPL) and Formal Propositional Logic (FPL) in the form of natural deduction systems which are complete with respect to transitive models and finite irreflexive transitive models, respectively. A sequent style formulation of BPL can be found in [1], but it is not satisfactory because it does not enjoy the subformula property. Here we introduce another sequent calculus for BPL as well as one for FPL, and show the admissibility of the cut rule through semantical completeness. These are extensions of systems in [2].

## 2 A sequent calculus for BPL

**Definition 2.1** *Formulas* are constructed from $\{\rightarrow, \wedge, \vee, \bot\}$ and the set $\mathcal{V}$ of propositional variables. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$ where $\Gamma$ and $\Delta$ are finite sets of formulas. As far as sequents are concerned, we write $A_1, ..., A_n$ for $\{A_1, ..., A_n\}$, and $A, \Gamma$ for $\{A\} \cup \Gamma$, etc.

**Definition 2.2** A (*transitive*) *model* is a triple $\langle W, R, V\rangle$ where $W$ is a non-empty set, $R$ is a transitive relation on $W$, and $V$ is a mapping from $\mathcal{V}$ to $2^W$ such that $x \in V(p)$ and $xRy$ imply $y \in V(p)$. If $W$ is a finite set, we say that a model $\langle W, R, V\rangle$ is *finite*. Given a model $M = \langle W, R, V\rangle$, the truth-relation $\models$ is defined as follows:

$$\begin{array}{lll}
(M,x) \models p & \text{iff} & x \in V(p) \quad \text{for each propositional variable } p,\\
(M,x) \not\models \bot, & & \\
(M,x) \models A \wedge B & \text{iff} & (M,x) \models A \text{ and } (M,x) \models B,\\
(M,x) \models A \vee B & \text{iff} & (M,x) \models A \text{ or } (M,x) \models B,\\
(M,x) \models A \rightarrow B & \text{iff} & \forall y \in W[xRy \text{ and } (M,y) \models A \text{ imply } (M,y) \models B].
\end{array}$$

We say that a formula $A$ is *true* in $M$ if $(M,x) \models A$ for every $x \in W$. The truth-relation for sequents is defined by

$$(M,x) \models \Gamma \Rightarrow \Delta \quad \text{iff} \quad \forall A \in \Gamma[(M,x) \models A] \text{ implies } \exists A \in \Delta[(M,x) \models A].$$

We say that a sequent $\Gamma \Rightarrow \Delta$ is *true* in $M$ if $(M,x) \models \Gamma \Rightarrow \Delta$ for every $x \in W$.

**Definition 2.3** The system **LBP** consists of the following initial sequents

$$A \Rightarrow A,$$
$$\bot \Rightarrow$$

and the following rules of inference:

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (\text{weakening left}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ (\text{weakening right})$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\wedge\text{right})$$

$$\frac{A,\Gamma \Rightarrow \Delta \quad B,\Gamma \Rightarrow \Delta}{A \vee B,\Gamma \Rightarrow \Delta}\ (\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ (\vee\text{right})$$

$$\frac{\Delta_1,\Sigma,A \Rightarrow B,\Gamma_1 \quad \Delta_2,\Sigma,A \Rightarrow B,\Gamma_2 \quad \cdots \quad \Delta_{2^n},\Sigma,A \Rightarrow B,\Gamma_{2^n}}{\Sigma, C_1{\to}D_1,...,C_n{\to}D_n \Rightarrow A{\to}B}\ (\to)$$

where $n \geq 0$, $\Gamma_i = \{C_j \mid j \in \gamma(i)\}$, $\Delta_i = \{D_j \mid j \in \delta(i)\}$, and the sets $\gamma(i)$ and $\delta(i)$ of natural numbers are defined as follows. Enumerate all the "partitions" of $\{1,...,n\}$:

$$\langle\{\},\{1,...,n\}\rangle, \langle\{1\},\{2,...,n\}\rangle, \langle\{2\},\{1,3,...,n\}\rangle, ...,$$
$$\langle\{1,2\},\{3,...,n\}\rangle, \langle\{2,3\},\{1,4,...,n\}\rangle, ..., \langle\{1,...,n\},\{\}\rangle.$$

The number of the partitions is $2^n$, and $\langle\delta(i),\gamma(i)\rangle$ is defined to be the $i$-th partition.
For example, when $n = 0, 1, 2$, the rule $(\to)$ is of the forms

$$\frac{\Sigma, A \Rightarrow B}{\Sigma \Rightarrow A{\to}B}\ (\to), \qquad \frac{\Sigma, A \Rightarrow B, C_1 \quad D_1,\Sigma,A \Rightarrow B}{\Sigma, C_1{\to}D_1 \Rightarrow A{\to}B}\ (\to),$$

$$\frac{\Sigma, A \Rightarrow B, C_1, C_2 \quad D_1,\Sigma,A \Rightarrow B, C_2 \quad D_2,\Sigma,A \Rightarrow B, C_1 \quad D_1, D_2,\Sigma,A \Rightarrow B}{\Sigma, C_1{\to}D_1, C_2{\to}D_2 \Rightarrow A{\to}B}\ (\to).$$

**Theorem 2.4 (Soundness)** *If $\Gamma \Rightarrow \Delta$ is provable in* **LBP**, *then $\Gamma \Rightarrow \Delta$ is true in any model.*

To prove the completeness theorem, we need the following notion.

**Definition 2.5** A sequent $\Gamma \Rightarrow \Delta$ is said to be *saturated* if, for all formulas $A, B$,

$$\begin{array}{lllll} A \wedge B \in \Gamma & \text{implies} & A \in \Gamma & \text{and} & B \in \Gamma, \\ A \wedge B \in \Delta & \text{implies} & A \in \Delta & \text{or} & B \in \Delta, \\ A \vee B \in \Gamma & \text{implies} & A \in \Gamma & \text{or} & B \in \Gamma, \\ A \vee B \in \Delta & \text{implies} & A \in \Delta & \text{and} & B \in \Delta. \end{array}$$

Given a finite set $\Lambda$ of formulas, $\mathcal{S}(\Lambda)$ is defined to be the set of all sequents $\Gamma \Rightarrow \Delta$ satisfying the following conditions:

- $\Gamma \Rightarrow \Delta$ is saturated,
- $\Gamma \Rightarrow \Delta$ is not provable in **LBP**,
- $\Gamma \cup \Delta \subseteq \mathrm{Sub}(\Lambda)$, where $\mathrm{Sub}(\Lambda)$ is the set of all the subformulas in $\Lambda$.

**Lemma 2.6** *If a sequent $\Gamma \Rightarrow \Delta$ is not provable in* **LBP**, *then there is a sequent $\Gamma' \Rightarrow \Delta'$ in $\mathcal{S}(\Gamma \cup \Delta)$ such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.*

**Theorem 2.7 (Completeness)** *$\Gamma \Rightarrow \Delta$ is provable in* **LBP** *if and only if $\Gamma \Rightarrow \Delta$ is true in any model.*

(Sketch of proof) Suppose that $\Gamma \Rightarrow \Delta$ is not provable in **LBP**. Then by Lemma 2.6, there is a sequent $\Gamma' \Rightarrow \Delta'$ in $\mathcal{S}(\Gamma \cup \Delta)$ such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. Now define a model $M = \langle W, R, V\rangle$ as follows:

- $W = \mathcal{S}(\Gamma \cup \Delta)$,
- $(\Sigma \Rightarrow \Theta)R(\Phi \Rightarrow \Psi)$ iff $\Sigma \subseteq \Phi$ and $\forall(A{\to}B) \in \Sigma[A \in \Psi$ or $B \in \Phi]$,
- $V(p) = \{\Sigma \Rightarrow \Theta \in W \mid p \in \Sigma\}$.

We can verify that $R$ is transitive and that $\Sigma \Rightarrow \Theta \in V(p)$ and $(\Sigma \Rightarrow \Theta)R(\Phi \Rightarrow \Psi)$ imply $\Phi \Rightarrow \Psi \in V(p)$. To see that $\Gamma \Rightarrow \Delta$ is not true in $M$ because of $(M, \Gamma' \Rightarrow \Delta') \not\models \Gamma \Rightarrow \Delta$, it suffices to show that, for any sequent $\Sigma \Rightarrow \Theta \in W$, $\forall A \in \Sigma[(M, \Sigma \Rightarrow \Theta) \models A]$ and $\forall A \in \Theta[(M, \Sigma \Rightarrow \Theta) \not\models A]$. □

The above proof provides the model $M$ based on the finite set $W = \mathcal{S}(\Gamma \cup \Delta)$. So we have

**Theorem 2.8 (Finite model property)** *If* $\Gamma \Rightarrow \Delta$ *is not provable in* **LBP**, *then there is a finite model in which* $\Gamma \Rightarrow \Delta$ *is not true.*

As a consequence of the completeness theorem, we derive

**Corollary 2.9** *The rule*

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ (cut)}$$

*is admissible in* **LBP**.

# 3 A sequent calculus for FPL

**Definition 3.1** An *irreflexive model* is a model $\langle W, R, V \rangle$ in which $R$ is irreflexive, i.e., there is no $x \in W$ such that $xRx$.

**Definition 3.2** The system **LFP** is obtained from **LBP** by providing the left hand side of each upper sequent in the rule $(\rightarrow)$ with the formula $A \rightarrow B$. Then, the rule $(\rightarrow)$ of **LFP** is of the form

$$\frac{\Delta_1, \Sigma, A \rightarrow B, A \Rightarrow B, \Gamma_1 \quad \Delta_2, \Sigma, A \rightarrow B, A \Rightarrow B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A \rightarrow B, A \Rightarrow B, \Gamma_{2^n}}{\Sigma, C_1 \rightarrow D_1, ..., C_n \rightarrow D_n \Rightarrow A \rightarrow B} \ (\rightarrow)$$

where $n \geq 0$, and $\Gamma_i$ and $\Delta_i$ are as in the rule $(\rightarrow)$ of **LBP**.

**Theorem 3.3 (Soundness)** *If* $\Gamma \Rightarrow \Delta$ *is provable in* **LFP**, *then* $\Gamma \Rightarrow \Delta$ *is true in any finite irreflexive model.*

**Theorem 3.4 (Completeness)** $\Gamma \Rightarrow \Delta$ *is provable in* **LFP** *if and only if* $\Gamma \Rightarrow \Delta$ *is true in any finite irreflexive model.*

(Sketch of proof) We construct a model $M = \langle W, R, V \rangle$ similar to that in the proof of Theorem 2.7 except that $R$ is defined by

- $(\Sigma \Rightarrow \Theta) R (\Phi \Rightarrow \Psi)$ iff $\Sigma \subsetneq \Phi$ and $\forall (A \rightarrow B) \in \Sigma [A \in \Psi \text{ or } B \in \Phi]$.

We can verify that $M$ defines a finite irreflexive model. The rest of the proof is similar to that of Theorem 2.7. □

**Corollary 3.5** *The rule*

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ (cut)}$$

*is admissible in* **LFP**.

# 4 Relationships with other systems

**Definition 4.1** We define $\bigwedge \Gamma$ ($\bigvee \Gamma$) as the conjunction (the disjunction) of all formulas in $\Gamma$ if $\Gamma$ is non-empty, as $\bot \rightarrow \bot$ ($\bot$, respectively) if $\Gamma$ empty.

**Definition 4.2** The Hilbert-style system **HB** [4] consists of the following axiom schemes and inference rule:

| | |
|---|---|
| $A \rightarrow A$ | $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$ |
| $A \wedge B \rightarrow A$ | $A \wedge B \rightarrow B$ |
| $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$ | $A \rightarrow A \vee B$ |
| $B \rightarrow A \vee B$ | $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$ |
| $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$ | $\bot \rightarrow A$ |
| $A \rightarrow (B \rightarrow A)$ | $A \rightarrow (B \rightarrow A \wedge B)$ |

$$\frac{A \quad A \rightarrow B}{B} \text{ (modus ponens)}.$$

**Lemma 4.3** *If $A$ is provable in* **HB**, *then $\Rightarrow A$ is provable in* **LBP**.

**Lemma 4.4** *If $\Gamma \Rightarrow \Delta$ is provable in* **LBP**, *then $\bigwedge \Gamma \to \bigvee \Delta$ is provable in* **HB**.

Combining Lemmas 4.3 and 4.4 with Theorem 2.7, we derive

**Theorem 4.5 (Completeness of HB)** *$A$ is provable in* **HB** *if and only if $A$ is true in any model.*

**Definition 4.6** Sequents of the system $\mathbf{GVPL}^+$ [3] consist of finite sequences of formulas which are possibly of the form $(A \to B)^+$ where $A$ and $B$ are ordinary ones. The system $\mathbf{GVPL}^+$ consists of the following initial sequents

$$A \Rightarrow A,$$

$$\bot \Rightarrow$$

and the following rules of inference:

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ (\Rightarrow \wedge) \qquad \frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow_1) \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow_2)$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ (\Rightarrow \vee_1) \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ (\Rightarrow \vee_2) \qquad \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow)$$

$$\frac{A, \Gamma^+ \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (\Rightarrow \to) \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow (A \to B)^+}\ (\Rightarrow \to^+) \qquad \frac{\Gamma \Rightarrow A \quad B, \Pi \Rightarrow \Delta}{(A \to B)^+, \Gamma, \Pi \Rightarrow \Delta}\ (\to^+ \Rightarrow)$$

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathcal{D}}\ (\Rightarrow T) \qquad \frac{\Gamma \Rightarrow \Delta}{\mathcal{D}, \Gamma \Rightarrow \Delta}\ (T \Rightarrow) \qquad \frac{\mathcal{D}, \mathcal{D}, \Gamma \Rightarrow \Delta}{\mathcal{D}, \Gamma \Rightarrow \Delta}\ (C \Rightarrow)$$

$$\frac{\Gamma, \mathcal{C}, \mathcal{D}, \Pi \Rightarrow \Delta}{\Gamma, \mathcal{D}, \mathcal{C}, \Pi \Rightarrow \Delta}\ (I \Rightarrow) \qquad \frac{\Gamma \Rightarrow \mathcal{D} \quad \mathcal{D}, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ (\text{cut})$$

where $\Delta$ is the empty sequence or a formula, and $\Gamma^+$ is obtained from $\Gamma$ by replacing every $C \to D$ with $(C \to D)^+$.

In the following, we let $\Gamma$ denote both a finite set of ordinary formulas and a finite sequence of them, and let $A$ denote an ordinary formula.

**Lemma 4.7** *If $\Gamma \Rightarrow \Delta$ is provable in* **LBP**, *then $\Gamma \Rightarrow \bigvee \Delta$ is provable in* $\mathbf{GVPL}^+$.

**Lemma 4.8** *If $\Gamma, \Sigma \Rightarrow A$ is provable in* $\mathbf{GVPL}^+$, *then $\Sigma^- \Rightarrow \bigwedge \Gamma \to A$ is provable in* **LBP**, *where $\Sigma^-$ is obtained from $\Sigma$ by replacing every $(C \to D)^+$ with $C \to D$.*

**Theorem 4.9** *$\Gamma \Rightarrow A$ is provable in* **LBP** *if and only if $\Gamma \Rightarrow A$ is provable in* $\mathbf{GVPL}^+$.

**Definition 4.10** The system $\mathbf{GFPL}^+$ [3] is obtained from $\mathbf{GVPL}^+$ by replacing $(\Rightarrow \to)$ with

$$\frac{A, A \to B, \Gamma^+ \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (\Rightarrow \to_f).$$

Lemmas 4.7, 4.8, and Theorem 4.9 hold also between **LFP** and $\mathbf{GFPL}^+$.

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# Logics of finite width for Some Intuitionistic Modal Logic

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**Abstract**

Fine [1] showed the completeness of logics of finite width for **K4**. We treat the logic containing the **S4**-type intuitionistic modal logic $\mathbf{L}_0$.

For the logic $\mathbf{L}_0$, we show the completeness of the logics of finite width with respect to one relation.

## 1 Preliminaries.

In [2], Ono introduced the intuitionistic modal logic $\mathbf{L}_0$ and it is shown the logic is complete for the **I**-frame.

We repeat the axioms of the intuitionistic modal logic $\mathbf{L}_0$ and **I**-frame. $\mathbf{L}_0$ is the logic obtained from the intuitionistic logic **H** by adding the following axioms, $\Box p \supset p$, $\Box p \supset \Box\Box p$, $\Box(p \supset q) \supset (\Box p \supset \Box q)$, and closed under the rules of substitution, necessitation and modus ponens. A triple $\mathbf{F} = (M, \leq, R)$ is called an **I**-frame if the following conditions are satisfied.

- $M$ is a non empty set.
- $\leq$ is a partial order on $M$.
- $R$ is a pre-order on $M$, that is, $x \leq y$ implies $xRy$ for each $x, y \in M$.

A relation $\models$ between $M$ and the set of propositional variables is said to be a valuation on **F** if

$$a \models p \text{ and } a \leq b \Rightarrow b \models p$$

holds for every $a, b \in M$ and every variable $p$. We can extend a valuation on $M$ and the set of formulas in the following way.

1. $a \models A \wedge B \Leftrightarrow a \models A$ and $a \models B$,
2. $a \models A \vee B \Leftrightarrow a \models A$ or $a \models B$,
3. $a \models A \supset B \Leftrightarrow \forall b(a \leq b) b \not\models A$ or $b \models B$,

4. $a \models \neg A \Leftrightarrow \forall b(a \leq b$ implies $b \not\models A)$,

5. $a \models \Box A \Leftrightarrow \forall b(aRb$ implies $b \models A)$.

The pair $(\mathbf{F}, \models)$ of **I**-frame $\mathbf{F}$ and a valuation $\models$ on $\mathbf{F}$ is said to be an **I**-model.

A formula $A$ is said to be true in $(\mathbf{F}, \models)$ if $a \models A$ holds for any $a \in M$. A formula $A$ is said to be valid in $\mathbf{F}$ if $a$ is true in any model $(\mathbf{F}, \models)$, and any $a \in M$.

We write $\mathbf{F_L}$ instead of the canonical frame for a logic $\mathbf{L}$ and $\mathbf{M_L}$ instead of the canonical model for a logic $\mathbf{L}$. If a logic $\mathbf{L}'$ is the restriction of $\mathbf{L}$ to finite subset of propositional valiables then $\mathbf{L}'$ is said to be weak.

**Lemma 1.1** *Suppose* $\mathbf{M} = (M, \leq, R, \models)$ *is an* **I***-model,* $A'$ *is a formula substituted* $B_i$ *for* $p_i$ *in* $A$ *and* $\mathbf{M}' = (M, \leq, R, \models')$ *is an* **I***-model such that*

$$\forall x \in M(\mathbf{M}', x) \models' p_i \Leftrightarrow (\mathbf{M}, x) \models B_i, i = 0, 1, \cdots, n.$$

*Then*

$$(\mathbf{M}', x) \models' A \Leftrightarrow (\mathbf{M}, x) \models A'.$$

**Lemma 1.2** *Suppose* $\Delta$ *is closed under substitution and true in* $\mathbf{M} = (M, \leq, R, \models_M)$. *Then* $\Delta$ *is true in each definable variant* $\mathbf{N} = (M, \leq, R, \models_N)$.

**Lemma 1.3** *Suppose* $\mathbf{L}'$ *is the weak restriction of a logic* $\mathbf{L}$. *Then* $\mathbf{L}$ *is true in* $\mathbf{M_{L'}}$.

# 2 The logics of finite $\leq$-width.

In this section, we define the formula of $\leq$-width $n$.

For $n \geq 1$, let $BW_n$ be the formula

$$\bigvee_{i=0}^{n} \Box(p_i \supset \bigvee_{j \neq i} p_j).$$

$w$ and $v$ are said to be $\leq$-incomparable if the elements are incomparable with respect to the relation $\leq$. An **I**-frame $(M, \leq, R)$ is of $\leq$-width $n$ if it does not contain $(n+1)$ mutually $\leq$-incomparable elements. A frame $\mathbf{F}$ is of finite $\leq$-width if it is of $\leq$-width $n$ for some $n$. A logic $\mathbf{L}$ is of finite $\leq$-width if $\mathbf{L} \supseteq \mathbf{L}_0 + BW_n$ for some $n > 0$.

**Theorem 2.1** *Suppose* $\mathbf{L}$ *is a logic of finite* $\leq$*-width and a frame* $\mathbf{F} = (M, \leq, R)$ *is generated from* $\mathbf{F_L}$ *by* $w$. *Then* $\mathbf{F}$ *is of finite* $\leq$*-width.*

**Lemma 2.2** *Suppose an* **I***-frame* $(M, \leq, R)$ *is of finite* $\leq$*-width. If there is a infinite sequence* $w_0, w_1, \cdots$ *in* $M$ *such that* $w_j \not R^{\leftarrow} w_i for j > i$. *Then the sequence* $w_0, w_1, \cdots$ *consists of infinite subsequence* $v_0, v_1, \cdots$ *such that* $v_i \leq v_{i+1}$ *and* $v_i \neq v_{i+1}$ *for* $i = 0, 1, \cdots$.

Let $(M, \leq, R)$ be an **I**-model. $w$ is $\leq$-deep if there is a elements $w_0$, $w_1$, $\cdots$ such that $w_i \in M$, $w_0 = w$, $w_i \leq w_{i+1}$ and $w_i \neq w_{i+1}$ for $i = 0, 1, \cdots$. Let $U_w^{\leq} = \{u \in M : w \leq u \text{ and } u \text{ is not } \leq\text{-deep}\}$ and $U_w^R = \{u \in M : wRu \text{ and } u \text{ is not } \leq\text{-deep}\}$. A $\leq$-deep $w$ is called $\leq$-static if $U_w^{\leq} = U_v^{\leq}$ for any $\leq$-deep $v \in M(w \leq v)$. A $\leq$-deep $w$ is called $R$-static if $U_w^R = U_v^R$ for any $\leq$-deep $v \in M(w \leq v)$.

For $u \in M$, let $u_p = \{p : u \models p(p \text{ is a propositional variable})\}$ and let $|u|_p = \{v_p : v \in M, u \leq v \text{ and } v \text{ is } \leq\text{-deep}\}$. A $\leq$-deep $w$ is called $\leq$-stationary if $|w|_p = |v|_p$ for any $\leq$-deep $v \in M(w \leq v)$.

**Lemma 2.3** *Let $(M, \leq, R, \models)$ be an weak* **I***-frame of finite $\leq$-width. For any $\leq$-deep $w \in M$, there is $v(v \in M, w \leq v)$ such that $\leq$-deep and $\leq$-static.*

**Lemma 2.4** *Let $(M, \leq, R, \models)$ be an weak* **I***-frame of finite $\leq$-width. For any $\leq$-deep $w \in M$, there is $v(v \in M, w \leq v)$ such that $\leq$-deep and $R$-static.*

**Lemma 2.5** *Let $(M, \leq, R, \models)$ be an weak* **I***-frame of finite $\leq$-width. For any $\leq$-deep $w \in M$, there is $v(v \in M, w \leq v)$ such that $\leq$-deep and $\leq$-stationary.*

By Lemma 2.3, Lemma 2.4 and Lemma 2.5 For any $\leq$-deep $w \in M$, there is $v(v \in M, w \leq v)$ such that $\leq$-deep, $\leq$-static, $R$-static and $\leq$-stationary. So, we can get the next theorem.

**Theorem 2.6** *Suppose* **L** *is a logic of finite $\leq$-width. Then $\mathbf{M_L}$ contains no infinite $\leq$-ascending chain.*

**Theorem 2.7** *Suppose* **L** *is a logic of finite $\leq$-width. Then $\mathbf{M_L}$ contains no infinite $R$-ascending chain.*

Let $\mathbf{F} = (M, \leq, R)$ is an **I**-frame. A set $V$ is said to be a $\leq$-cover for $U \subset M$ if $V \subseteq U$ and $(\forall u \in U)(\exists v \in V)(u = v \text{ or } u < v)$. $\mathbf{F}$ has finite $\leq$-cover property if each $U \subset M$ has a finite $\leq$-cover.

**Theorem 2.8** *Let $\mathbf{L}'$ be a weak logic of finite $\leq$-width and $\mathbf{F} = (M, \leq, R)$ be generated from $\mathbf{F_L}$ by $w$. Then $f$ has finite $\leq$-cover property.*

# 3 Completeness results.

Let $\mathbf{M} = (M, \leq_M, R_M, \models_M)$ and $\mathbf{N} = (N, \leq_N, R_N, \models_N)$ be **I**-models and $w \in M$, $v \in N$. We defind the relations $(\mathbf{M}, w) \leq_A (\mathbf{N}, v)$, $(\mathbf{M}, w) \equiv_A (\mathbf{N}, v)$ and $(\mathbf{M}, w) \sim_A (\mathbf{N}, v)$ by induction on the construction of a formula $A$:

$$
\begin{aligned}
(\mathbf{M}, w) \sim_0 (\mathbf{N}, v) &\Leftrightarrow (w \models_M p \leftrightarrow v \models_N p) \text{ for any propositional variable p;} \\
(\mathbf{M}, w) \leq_{B \supset C} (\mathbf{N}, v) &\Leftrightarrow \forall u \in M \exists t \in N[w \neq_M u \text{ and } w \leq_M u \Rightarrow \\
&\quad v \leq_N t, (\mathbf{M}, u) \sim_B (\mathbf{N}, t) \text{ and } (\mathbf{M}, u) \sim_C (\mathbf{N}, t)]; \\
(\mathbf{M}, w) \leq_{\neg B} (\mathbf{N}, v) &\Leftrightarrow \forall u \in M \exists t \in N[w \neq_M u \text{ and } w \leq_M u \Rightarrow
\end{aligned}
$$

$$
\begin{array}{rcl}
 & & v \leq_N t \text{ and } (\mathbf{M}, u) \sim_B (\mathbf{N}, t)]; \\
(\mathbf{M}, w) \leq_{\Box B} (\mathbf{N}, v) & \Leftrightarrow & \forall u \in M \exists t \in N[w \neq_M u \text{ and } wR_M u \Rightarrow \\
 & & vR_N t \text{ and } (\mathbf{M}, u) \sim_B (\mathbf{N}, t)]; \\
(\mathbf{M}, w) \equiv_A (\mathbf{N}, v) & \Leftrightarrow & (\mathbf{M}, w) \leq_A (\mathbf{N}, v) \text{ and } (\mathbf{N}, v) \leq_A (\mathbf{M}, w); \\
(\mathbf{M}, w) \sim_A (\mathbf{N}, v) & \Leftrightarrow & (\mathbf{M}, w) \sim_0 (\mathbf{N}, v) \text{ and } (\mathbf{M}, w) \equiv_A (\mathbf{N}, v).
\end{array}
$$

Let $|w|_A = \{v \in M : w \sim_A v\}$.
A model $\mathbf{M}$ is $A$-simple if there is finite $V \subset M$ such that

- $V$ covers $|w|_A$ for each $w \in M$, and
- for all $u, t \in M - V$, $(\{v \in V : u \leq v\} = \{v \in V : t \leq v\}$ and $\{v \in V : uRv\} = \{v \in V : tRv\}) \Rightarrow u \sim_0 t$.

**Lemma 3.1** *Suppose $(M, \leq, R)$ is an* **I***-model of finite $\leq$-width. If a formula $A$ is true in all weak $A$-simple models $(M, \leq, R, \models)$ then $A$ is valid in $(M, \leq, R)$.*

**Proposition 3.2** *Suppose $\mathbf{M} = (M, \leq, R, \models)$ is a natural model of finite $\leq$-width. Then there exist formula $D_w$ such that $w \models D_w$, $w \in M$ and $v \not\models D_w$, $v \in M(w \not\leq v)$.*

**Lemma 3.3** *Suppose $\mathbf{M} = (M, \leq, R, \models_M)$ is a natural model of finite $\leq$-width and $\mathbf{N} = (M, \leq, R, \models_N)$ is a simple weak model. Then there exist formulas $A_1$, $A_2$, $\cdots$, $A_n$ such that $w \models_N p_i \Leftrightarrow w \models_M A_i$, for $i = 0, 1, \cdots n$.*

**Lemma 3.4** *Let* **L** *be a logic of finite $\leq$-width and* **L**$'$ *be the weak restriction of* **L**. *Then* $\mathbf{F}_{\mathbf{L}'}$ *is a frame for* **L**.

**Theorem 3.5** *Each logic of finite $\leq$-width is complete.*

# References


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