CERTAIN VALUES OF HECKE L-FUNCTIONS AND GENERALIZED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. We compare two calculations due to Bloch and the author of the regulator of an elliptic curve with complex multiplication which is a quotient of a Fermat curve, and express the special value of its L-function at $s = 0$ in terms of special values of generalized hypergeometric functions.

1. INTRODUCTION

Let E be an elliptic curve over $\mathbb{Q}$ with complex multiplication by the integer ring $\mathscr{O}$ of an imaginary quadratic field K. Its L-function $L(E,s)$, or more precisely the L-function of the motive $h^1(E)$, is defined by an Euler product which converges absolutely for $\mathrm{Re}(s) > 3/2$. By results of Deuring, it is the L-function of a Hecke character of K, hence is analytically continued to the whole complex plane and satisfies a functional equation with respect to $s \leftrightarrow 2-s$. In particular, it has a simple zero at $s = 0$. In [4], [5], Bloch defined a regulator map for an elliptic curve and in the case of complex multiplication, constructed from torsion points a K-theory element whose regulator image gives the special value $L(E,2)$, or equivalently the value

$$L^*(E,0) := \lim_{s\to 0} s^{-1}L(E,s).$$

This is an analogue of the classical class number formula for number fields and motivated the general conjecture of Beilinson [2], [3] on special values of L-functions of motives.

Let X be a smooth projective curve over $\mathbb{Q}$. Beilinson's regulator map we consider is

$$r_{\mathscr{D}}\colon H^2_{\mathscr{M}}(X,\mathbb{Q}(2))_{\mathbb{Z}} \to H^2_{\mathscr{D}}(X_{\mathbb{R}},\mathbb{R}(2))$$

from the "integral part" of the motivic cohomology group to the real Deligne cohmology group (see §2.1). The Beilinson conjecture asserts firstly that $r_{\mathscr{D}} \otimes_{\mathbb{Q}} \mathbb{R}$ is an isomorphism of vector spaces of dimension the genus of X, and secondly that the value, i.e. the first non-vaniching Taylor coefficient, of $L(h^1(X),s)$ at $s = 0$ (although the analytic continuation in general is highly conjectural) is given by the "determinant" of $r_{\mathscr{D}}$ well-defined modulo $\mathbb{Q}^*$. When X is an elliptic curve, Bloch's result mentioned above is rephrased in this framework.

The regulator of the Femrat curve

$$X_N\colon x_0^N + y_0^N = z_0^N$$

of degree N over $\mathbb{Q}$ was studied by Ross [16], [17], Kimura [10] and the author [12]. In the category of pure motives over $\mathbb{Q}$ with $\mathbb{Q}$-coefficients, we have a decomposition ([12],

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see §3.1)

$$h^1(X_N) \simeq \bigoplus_{[a,b]\in H_N\backslash I_N} X_N^{[a,b]}$$

where we put $I_N = \{(a,b) \in (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \mid a, b, a+b \neq 0\}$, and $H_N = (\mathbb{Z}/N\mathbb{Z})^*$ with the natural action on I_N. The L-function of $X_N^{[a.b]}$ coincides with the L-function of the Jacobi-sum Hecke character $j_N^{a,b}$ of the N-cyclotomic field $\mathbb{Q}(\mu_N)$ (see §3.1). By projecting the element of Ross [17], the author [12] defines an element

$$e_N^{[a,b]} \in H^2_{\mathscr{M}}(X_N^{[a.b]}, \mathbb{Q}(2))_{\mathbb{Z}}$$

for each $[a,b]$, and expresses its regulator image by special values of generalized hypergeometric functions (see Theorem 3.2). The hypergeometric functions which apper are Appell's F_3 of two variables and Barnes' ${}_3F_2$ of one variable (see §3.2). The non-triviality of $e_N^{[a,b]}$ follows from the integral representation of the hypergeometric functions.

In this paper, we study the intersection of the above two stories. If $N = 3, 4$ or 6, then $K = \mathbb{Q}(\mu_N)$ is an imaginary quadratic field and $X_N^{[a,b]}$ is isomorphic to the motive $h^1(E)$ of an elliptic curve E over $\mathbb{Q}$ with complex multiplication by $\mathscr{O}$. Therefore, if we admit the Beilinson conjecture, then $H^2_{\mathscr{M}}(E, \mathbb{Q}(2))_Z = \mathbb{Q}$ and our element $e_N^{[a,b]}$ should be a non-zero rational multiple of Bloch's element, but we know neither the injectivity of the regulator map, nor the finite generation of the motivic cohomology group. Therefore, we compare directly these two elements when $N = 3$ (Proposition 5.1) and $N = 4$ with a restriction (Proposition 5.4). In fact, Ross' element is also supported on torsion points (in the Jacobian) and we compare the divisors of the functions defining those elements of the motivic cohomology groups. As a result, we obtain formulae expressing the special values of the Jacobi-sum Hecke L-functions in terms of special values of the generalized hypergeometric functions (Theorems 5.2 and 5.5).

This paper is constructed as follows. In §2, we recall necessary materials on the regulator of a curve. In §3, we recall the result of [12] on the regulator of a motive associated to a Fermat curve. In §4, we recall the result of Bloch on the regulator of an elliptic curve with complex multiplication. In fact, we need a slight modification; while Bloch uses the C-torsion points of E for a rational integer C divisible by the conductor $\mathbf{f}$ of the Hecke character, we only use the $\mathbf{f}$-torsion points, which is necessary for our comparisons and possible in our explicit cases. Finally in §5, we relate and compare the elements of motivic cohomology of a Fermat motive and an elliptic curve and prove main results.

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2. REGULATOR OF A CURVE

We recall briefly basic materials on the regulator of a curve.

2.1. **Definition.** Here we recall the Beilinson regulator map for curves [2] [3] (cf. [11], [13], [18]). For a projective smooth curve X over $\mathbb{Q}$, we consider the regulator map

$$r_{\mathscr{D}} \colon H^2_{\mathscr{M}}(X, \mathbb{Q}(2)) \to H^2_{\mathscr{D}}(X_{\mathbb{R}}, \mathbb{R}(2)).$$

The source is the motivic cohomology group defined via K-theory, for which we have an isomorphism

$$H^2_{\mathscr{M}}(X, \mathbb{Q}(2)) \simeq \operatorname{Ker}\left(\tau \otimes \mathbb{Q} \colon K_2^M(\mathbb{Q}(E)) \otimes \mathbb{Q} \to \bigoplus\nolimits_{x\in X^{(1)}} \kappa(x)^* \otimes \mathbb{Q}\right).$$

Here, $X^{(1)}$ is the set of closed points on X, $\kappa(x)$ is the residue field and $\tau = (\tau_x)$ is the tame symbol on the Milnor K-group

$$\tau_x(\{f, g\}) = (-1)^{\mathrm{ord}_x f \cdot \mathrm{ord}_x g} \left(\frac{f^{\mathrm{ord}_x g}}{g^{\mathrm{ord}_x f}} \right)(x).$$

The integral part $H^2_{\mathscr{M}}(X, \mathbb{Q}(2))_{\mathbb{Z}}$ is defined to be the image of the K-group of a regular model of X proper over $\mathbb{Z}$. For those curves which we study in this paper, the integral part is the whole.

The target is the real Deligne cohomology group (see [7]), for which we have an isomorphism

$$H^2_{\mathscr{D}}(X_{\mathbb{R}}, \mathbb{R}(2)) \simeq H^1(X(\mathbb{C}), \mathbb{R}(1))^+.$$

Here, $+$ denotes the part fixed by the "de Rham conjugation" $F_\infty \otimes c_\infty$, where the "infinite Frobenius" F_∞ is the complex conjugation acting on $X(\mathbb{C})$ and c_∞ is the complex conjugation on the coefficients.

The Poincaré duality induces a perfect pairing

$$\langle\ ,\ \rangle \colon H^1(X(\mathbb{C}), \mathbb{R}(1))^+ \otimes H^1(X(\mathbb{C}), \mathbb{R})^+ \to \mathbb{R}; \quad \langle \eta, \omega \rangle = \frac{1}{2\pi i} \int_{X(C)} \eta \wedge \omega$$

of $\mathbb{R}$-vector spaces. Further, by the isomorphism

$$H^0(X(\mathbb{C}), \Omega^1)^+ \xrightarrow{\sim} H^1(X(\mathbb{C}), \mathbb{R})^+; \quad \omega \mapsto \frac{1}{2}(\omega + \overline{\omega}),$$

where we write $\overline{\omega} = c_\infty \omega = F_\infty \omega$, we have

$$H^1(X(\mathbb{C}), \mathbb{R}(1))^+ \simeq \mathrm{Hom}_{\mathbb{R}}(H^0(X(\mathbb{C}), \Omega^1)^+, \mathbb{R}).$$

Under these identifications, the regulator is written as

$$\langle r_{\mathscr{D}}(\{f, g\}), \omega \rangle = -\frac{1}{2\pi i} \int_{X(\mathbb{C})} \log|f| \overline{d \log g} \wedge \omega.$$

2.2. **Elements from torsion points.** Fix a base point $x_0 \in X(\mathbb{Q})$ and embed X into its Jacobian variety by sending $x \in X$ to the class of $x - x_0$. Let $\mathrm{Div}^0(X_{\mathrm{tor}})$ denote the group of divisors on X (defined over $\mathbb{Q}$) supported on torsion points in the Jacobian. A homomorphism

$$e \colon \wedge^2 \mathrm{Div}^0(X_{\mathrm{tor}}) \otimes \mathbb{Q} \to H^2_{\mathscr{M}}(X, \mathbb{Q}(2))$$

is defined as follows (see [5], Lect. 10, [6], §5). For $\alpha \in \mathrm{Div}^0(X_{\mathrm{tor}})$, there exists a non-zero integer n and a rational function $f \in \mathbb{Q}(X)^*$ such that $\mathrm{div}(f) = n\alpha$. Then

$$f_\alpha = f \otimes \frac{1}{n} \in \mathbb{Q}(X)^* \otimes \mathbb{Q}$$

is well-defined modulo $\mathbb{Q}^* \otimes \mathbb{Q}$, independently of the choices of n and f. For α, $\beta \in \mathrm{Div}^0(X_{\mathrm{tor}})$, put

$$e_0(\alpha, \beta) = \{f_\alpha, f_\beta\} \in K_2^M(\mathbb{Q}(X)) \otimes \mathbb{Q}.$$

It is well-defined modulo the image of $(\mathbb{Q}(X)^* \otimes \mathbb{Q}^*) \otimes \mathbb{Q}$. Then there exist a number field k, $h_i \in k(X)^*$ and $c_i \in k^*$ such that

$$e(\alpha, \beta) := e_0(\alpha, \beta) + N_{k/\mathbb{Q}}\left(\sum \{h_i, c_i\}\right) \in \mathrm{Ker}(\tau \otimes \mathbb{Q}).$$

Since $K_2^M(k)$ of a number field is torsion, the element $e(\alpha, \beta) \in H^2_{\mathscr{M}}(X, \mathbb{Q}(2))$ is uniquely defined by α and β.

3. FERMAT CURVE

We recall the definition of a Fermat motive, its L-function and the result of [12] on the regulator.

3.1. Fermat motive and L-function. Let X_N be the Fermat curve of degree N over $\mathbb{Q}$ as in the introduction and

$$x^N + y^N = 1$$

be its affine equation. Put $K = \mathbb{Q}(\mu_N)$ and $X_{N,K} = X_N \times_{\mathbb{Q}} K$. Fix an embedding $K \hookrightarrow \mathbb{C}$ and put $\zeta = \exp(2\pi i/N)$. Let the group $G_N := (\mathbb{Z}/N\mathbb{Z})^{\oplus 2}$ act on $X_{N,K}$ by

$$(r,s).(x,y) = (\zeta^r x, \zeta^s y).$$

If we denote by $p_N^{a,b}$ the projector corresponding to the character $(r,s) \mapsto \zeta^{ar+bs}$, the pair $X_N^{a,b} = (X_{N,K}, p_N^{a,b})$ defines a motive over K with coefficients in K. The group $H_N = (\mathbb{Z}/N\mathbb{Z})^*$ acts on G_N by multiplication, and let $[a,b]$ denote the H_N-orbit of $(a,b) \in G_N$. Then the projector

$$p_N^{[a,b]} = \sum_{(a',b') \in [a,b]} p_N^{a',b'}$$

is an element of $\mathbb{Q}[G_N]^{H_N}$, so defines a motive $X_N^{[a,b]} = (X_N, p_N^{[a,b]})$ over $\mathbb{Q}$ with $\mathbb{Q}$-coefficients. Then we have decompositions of motives

$$h^1(X_{N,K}) = \bigoplus_{(a,b)\in I_N} X_N^{a,b}, \quad h^1(X_N) = \bigoplus_{[a,b] \in H_N \backslash I_N} X_N^{[a,b]},$$

where $I_N \subset G_N$ is the subset as in the introduction. If (a,b) is primitive, i.e. $\gcd(a,b,N) = 1$, then $X_N^{[a,b]}$ is isomorphic to the Grothendieck scalar restriction of $X_N^{a,b}$ (i.e. a motive over K regarded as a motive over $\mathbb{Q}$ via $\operatorname{Spec} K \to \operatorname{Spec} \mathbb{Q}$).

For a prime ideal $\mathfrak{p}$ of K not dividing N, let

$$\chi_{\mathfrak{p}} : \mathbb{F}_{\mathfrak{p}}^* \to \mu_N(K)$$

be the N-th power residue symbol, where $\mathbb{F}_{\mathfrak{p}}$ denotes the residue field at $\mathfrak{p}$. For $(a,b) \in I_N$, the Jacobi sum is defined by

$$j_N^{a,b}(\mathfrak{p}) = - \sum_{x,y \in \mathbb{F}_{\mathfrak{p}}, x+y=1} \chi_{\mathfrak{p}}^a(x) \chi_{\mathfrak{p}}^b(y).$$

Weil [20] showed that $j_N^{a,b}$ defines a Hecke character of K with conductor dividing N^2, hence the L-function $L(j_N^{a,b}, s)$ satisfies all the desired analytic properties. By [12], §3, we have for primitive $(a,b) \in I_N$

$$L(X_N^{[a,b]}, s) = L(X_N^{a,b}, s) = L(j_N^{a,b}, s)$$

independently of the choice of embedding $K \hookrightarrow \mathbb{C}$.

3.2. Regulator. Recall that $H^1(X(\mathbb{C}), \mathbb{C})$ is generated by the differential forms

$$\omega^{a,b} = x^{\langle a \rangle} y^{\langle b \rangle - N} \frac{dx}{x}, \quad (a,b) \in I_N$$

where $\langle a \rangle \in \{1, 2, \dots, N-1\}$ denotes the representative of a, and $\omega^{a,b}$ is holomorphic if and only if $\langle a \rangle + \langle b \rangle < N$. Note that it is an eigenventor for the G-action:

$$(r,s)^* \omega^{a,b} = \zeta^{ar+bs} \omega^{a,b}.$$

On the other hand, $H_1(X_N(\mathbb{C}),\mathbb{Z})$ is a cyclic $\mathbb{Z}[G]$-module generated by an element κ [15]. If we normalize as

$$\widetilde{\omega}^{a,b} := \left(\tfrac{1}{N}B\left(\tfrac{\langle a\rangle}{N},\tfrac{\langle b\rangle}{N}\right)\right)^{-1}\omega^{a,b},$$

it has the period

$$\int_\kappa \widetilde{\omega}^{a,b} = (1-\zeta^a)(1-\zeta^b).$$

By the definition of κ, we have $F_\infty\kappa = (-1,-1)_*\kappa$. Therefore,

$$\int_\kappa F_\infty\widetilde{\omega}^{a,b} = \int_{F_\infty\kappa}\widetilde{\omega}^{a,b} = \int_\kappa (-1,-1)^*\widetilde{\omega}^{a,b} = \zeta^{-a-b}\int_\kappa \widetilde{\omega}^{a,b} = (1-\zeta^{-a})(1-\zeta^{-b}).$$

From these, it follows that

$$c_\infty\widetilde{\omega}^{a,b} = F_\infty\widetilde{\omega}^{a,b} = \widetilde{\omega}^{-a,-b},$$

hence $H^0(X_N(\mathbb{C}),\Omega^1)^+$ is generated by $\widetilde{\omega}^{a,b}$ with $\langle a\rangle+\langle b\rangle < N$, and $H^1(X_N(\mathbb{C}),\mathbb{R}(1))^+$ is generated by $\widetilde{\omega}^{a,b}-\widetilde{\omega}^{-a,-b}$.

Let

$$(\alpha,n) = \alpha(\alpha+1)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$$

be the Pochhammer symbol. Appell's hypergeometric function F_3 of two variables [1] is defined by

$$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y) = \sum_{m,n\geq 0}\frac{(\alpha,m)(\alpha',n)(\beta,m)(\beta',n)}{(\gamma,m+n)(1,m)(1,n)}x^my^n.$$

On the other hand, Barnes' hypergeometric function ${}_3F_2$ of one variable (cf. [19]) is defined by

$${}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};x\right) = \sum_{n\geq 0}\frac{(a,n)(b,n)(c,n)}{(d,n)(e,n)(1,n)}x^n.$$

For real numbers α, $\beta > 0$, we put

$$\widetilde{F}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}F_3(\alpha,\beta,1,1,\alpha+\beta+1;1,1)$$

(see [12] for the convergence). By comparing integral representations of F_3 and ${}_3F_2$, and using Dixon's formula (cf. [19]) on the value of ${}_3F_2$ at 1 several times, we have also ([12], §4.10)

$$\widetilde{F}(\alpha,\beta) = \left(\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right)^2 {}_3F_2\left(\begin{matrix}\alpha,\beta,\alpha+\beta-1\\ \alpha+\beta,\alpha+\beta\end{matrix};1\right).$$

Remark 3.1. Recall that the Beta function is related to Gauss' hypergeometric function as

$$F(\alpha,\beta,\alpha+\beta+1;1) = \frac{\alpha+\beta}{\alpha\beta}B(\alpha,\beta)^{-1}.$$

Now, consider the element of Ross [17]

$$e_N = \{1-x,1-y\} \in K_2^M(\mathbb{Q}(X_N)).$$

Since $\tau(e_N)$ is torsion ([17], Theorem 1), it defines an element of $H^2_{\mathscr{M}}(X_N,\mathbb{Q}(2))_\mathbb{Z}$. Put as in [12]

$$e_N^{[a,b]} = p_N^{[a,b]}(e_N) \in H^2_{\mathscr{M}}(X_N^{[a,b]},\mathbb{Q}(2))_\mathbb{Z}.$$

Then, [12], Theorem 4.14 implies the following:

Theorem 3.2. *If* $(a,b) \in I_N$ *is primitive, i.e.* $\gcd(a,b,N)=1$, *then we have*

$$r_{\mathscr{D}}(e_N^{[a,b]}) = -\frac{1}{2N^2} \sum_{(a',b')\in[a,b]/\{\pm 1\}} \Big(\widetilde{F}\big(\tfrac{\langle a'\rangle}{N}, \tfrac{\langle b'\rangle}{N}\big) - \widetilde{F}\big(\tfrac{\langle -a'\rangle}{N}, \tfrac{\langle -b'\rangle}{N}\big)\Big)\big(\widetilde{\omega}^{a',b'} - \widetilde{\omega}^{-a',-b'}\big)$$

and $\widetilde{F}\big(\frac{\langle a\rangle}{N}, \frac{\langle b\rangle}{N}\big) - \widetilde{F}\big(\frac{\langle -a\rangle}{N}, \frac{\langle -b\rangle}{N}\big) \neq 0$ *for any* $(a,b) \in I_N$.

4. ELLIPTIC CURVE WITH COMPLEX MULTIPLICATION

We recall the result of Bloch ([4], Lect. 8-9, [5], cf. [6]) in a modified form which only uses $\mathbf{f}$-torsion points where $\mathbf{f}$ is the conductor of the Hecke character.

4.1. L**-function.** Let E be an elliptic curve over $\mathbb{Q}$ with complex multiplication by the whole integer ring $\mathscr{O}$ of an imaginary quadratic field K. Then the class number of K is one. Write $\mu_K = \mathscr{O}^*$ for the roots of unity in K. We view K as a subfield of $\mathbb{C}$. By results of Deuring, there exists a (quasi-) Hecke character χ from the group of ideals of K prime to the conductor of E such that

$$L(h^1(E), s) = L(\chi, s) = L(\overline{\chi}, s).$$

Note that $\overline{\chi}(\mathbf{a}) = \chi(\overline{\mathbf{a}})$ since E is defined over $\mathbb{Q}$. Let $\mathbf{f}$ be the conductor of χ. We choose χ so that $\chi((\alpha)) = \overline{\alpha}$ for $\alpha \in \mathscr{O}$ with $\alpha \equiv 1 \pmod{\mathbf{f}}$. Let

$$\chi_f \colon (\mathscr{O}/\mathbf{f})^* \to K^*$$

be the finite character associated to χ, that is,

$$\chi((\alpha)) = \overline{\alpha}\chi_f(\alpha)$$

for any $\alpha \in \mathscr{O}$ prime to $\mathbf{f}$. As usual, for an ideal $\mathbf{a}$ (resp. an integer α) which is not prime to $\mathbf{f}$, we set $\chi(\mathbf{a}) = 0$ (resp. $\chi_f(\alpha) = 0$). Note that $\chi_f(\alpha) = \alpha$ for $\alpha \in \mu_K$, and $\overline{\chi}_f(\alpha) = \chi_f(\overline{\alpha})$ for any $\alpha \in \mathscr{O}$.

Let $\omega_E \in H^0(E(\mathbb{C}), \Omega^1)^+$ be a real holomorphic differential form normalized so that

$$\frac{1}{2\pi i}\int_{E(\mathbb{C})} \omega_E \wedge \overline{\omega}_E = -1, \tag{4.1}$$

and let $\varGamma \subset \mathbb{C}$ be its period lattice. Then we have an analytic isomorphism

$$E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/\varGamma; \quad x \mapsto \int_o^x \omega$$

between $(E(\mathbb{C}), F_\infty, \omega)$ and $(\mathbb{C}/\varGamma, ^-, dz)$, and we have the covolume $\mathrm{Covol}(\varGamma) = \pi$. The isomorphism is compatible with the $\mathscr{O}$-actions via the isomorphism

$$\theta : \mathscr{O} \xrightarrow{\sim} \mathrm{End}_K(E_K)$$

such that $\theta(\alpha)^*\omega = \alpha\omega$ for any $\alpha \in \mathscr{O}$.

Let

$$(\ ,\ )\colon \mathbb{C}/\varGamma \otimes \varGamma \to \mathbb{C}^*;\ (z,\gamma) \mapsto \exp(z\overline{\gamma} - \overline{z}\gamma)$$

be the Pontrjagin duality pairing. Choose $\Omega \in \varGamma$ such that $\mathscr{O}\Omega = \varGamma$, and $\nu \in \mathscr{O}$ such that $\mathbf{f} = (\nu)$; note that $\mathbf{f} = \overline{\mathbf{f}} = (\overline{\nu})$. Then it induces a perfect pairing

$$\langle\ ,\ \rangle\colon \mathscr{O}/\mathbf{f} \otimes \mathscr{O}/\mathbf{f} \to \mathbb{C}^*; \quad \langle u, v\rangle = \left(\frac{\Omega}{\overline{\nu}}u, \Omega v\right).$$

For $\alpha \in \mathscr{O}/\mathbf{f}$, define the Gauss sum by

$$G_\alpha = \sum_{u\in\mathscr{O}/\mathbf{f}} \chi_f(u)\langle u, \alpha\rangle.$$

Let $E_{\mathbf{f}}$ denote the $\mathbf{f}$-torsion points on E. By the identification $E_{\mathbf{f}} \simeq \mathscr{O}/\mathbf{f}$; $x \mapsto x\overline{\nu}/\Omega$, we view χ_f also as a character of $E_{\mathbf{f}}$.

By the standard argument as in [6], we have

$$\begin{aligned} L(\chi,s) &= {\sum_{\mathbf{a}\subset\mathscr{O}}}' \frac{\chi(\mathbf{a})}{N(\mathbf{a})^s} = \frac{1}{\sharp\mu_K} {\sum_{\alpha\in\mathscr{O}}}' \frac{\chi_f(\alpha)\overline{\alpha}}{|\alpha|^{2s}} \\ &= \frac{1}{\sharp\mu_K G_1} {\sum_{\alpha\in\mathscr{O}}}' \frac{G_\alpha\overline{\alpha}}{|\alpha|^{2s}} = \frac{1}{\sharp\mu_K G_1} \sum_{u\in\mathscr{O}/\mathbf{f}} {\sum_{\alpha\in\mathscr{O}}}' \frac{\chi_f(u)\langle u,\alpha\rangle\overline{\alpha}}{|\alpha|^{2s}} \\ &= \frac{|\Omega|^{2s}}{\sharp\mu_K G_1\overline{\Omega}} \sum_{x\in E_{\mathbf{f}}} {\sum_{\gamma\in\varGamma}}' \frac{\chi_f(x)(x,\gamma)\overline{\gamma}}{|\gamma|^{2s}} = \frac{|\Omega|^{2s}}{G_1\overline{\Omega}} \sum_{x\in E_{\mathbf{f}}/\mu_K} {\sum_{\gamma\in\varGamma}}' \frac{(\overline{\chi_f(x)}x,\gamma)\overline{\gamma}}{|\gamma|^{2s}} \end{aligned}$$

where $\sum'$ denotes the sum except for 0. Here we used the fact that ${\sum'}_{\gamma\in\varGamma} \frac{\overline{\gamma}}{|\gamma|^{2s}} = 0$, which follows from $-\varGamma = \varGamma$, and that $\overline{\chi_f(x)}x$ depends only on the class of x in $E_{\mathbf{f}}/\mu_K$.

Now, we assume that we can choose Ω and ν so that

$$\frac{\Omega}{\nu} \in \mathbb{R}. \tag{4.2}$$

Then, since $G_1/\overline{G_1} = \overline{\nu}/\nu = \Omega/\overline{\Omega}$, we have $G_1\overline{\Omega} \in \mathbb{R}$ and hence $G_1\overline{\Omega} = \pm N(\mathbf{f})^{\frac{1}{2}}|\Omega|$. On the other hand,

$$|\Omega|^2 = \frac{\operatorname{Covol}(\varGamma)}{\operatorname{Covol}(\mathscr{O})} = \frac{2\pi}{|d_K|^{\frac{1}{2}}}$$

where d_K denotes the discriminant of K. Hence we have

$$L(\chi,s) = \pm\frac{(2\pi)^s}{|d_K|^{\frac{s}{2}} N(\mathbf{f})^{\frac{1}{2}}|\Omega|} \sum_{x\in E_{\mathbf{f}}/\mu_K} {\sum_{\gamma\in\varGamma}}' \frac{(\overline{\chi_f(x)}x,\gamma)\overline{\gamma}}{|\gamma|^{2s}}.$$

Let $E(\mathbb{R})^0$ be the connected component of the origin with the orientation such that the real period

$$\Omega_{\mathbb{R}} = \int_{E(\mathbb{R})^0} \omega_E$$

is positive, and let $\Omega_{\mathbb{R}} = h\Omega$ with $h \in \mathscr{O}$. Then, by the functional equation $\varLambda(\chi,s) = \pm\varLambda(\chi,2-s)$ with

$$\varLambda(s) = (|d_K|N(\mathbf{f}))^{\frac{s}{2}}\,\frac{\varGamma(s)}{(2\pi)^s}\,L(\chi,s)$$

(cf. [8]), we obtain

$$L^*(\chi,0) = \pm\frac{N(\mathbf{f})^{\frac{1}{2}}|h|}{\Omega_{\mathbb{R}}} \sum_{x\in E_{\mathbf{f}}/\mu_K} {\sum_{\gamma\in\varGamma}}' \frac{(\overline{\chi_f(x)}x,\gamma)\overline{\gamma}}{|\gamma|^4}. \tag{4.3}$$

4.2. **Regulator.** Let E and ω_E be as above. Recall that $\overline{\omega}_E := c_\infty\omega_E = F_\infty\omega_E$. Then, $H^1(E(\mathbb{C}),\mathbb{R}(1))^+$ is generated by $\omega_E - \overline{\omega}_E$ and since

$$\langle \omega_E - \overline{\omega}_E, \tfrac{1}{2}(\omega_E + \overline{\omega}_E)\rangle = -1$$

by our normalization (4.1), the $\mathbb{Q}$-structure is given by $\Omega_{\mathbb{R}}(\omega_E - \overline{\omega}_E)$. Bloch's theorem, which in fact is valid for any elliptic curve over $\mathbb{R}$, is written as follows.

Theorem 4.1 (Bloch [4], [5], cf. [6], (3.2))**.** *For* $\alpha = \sum_x \alpha_x x$, $\beta = \sum_x \beta_x x \in \operatorname{Div}^0(E_{\mathrm{tor}})$, *we have*

$$r_{\mathscr{D}}(e(\alpha,\beta)) = -\frac{1}{2}\sum_{x,y}\alpha_x\beta_y {\sum_{\gamma\in\varGamma}}' \frac{(x-y,\gamma)\overline{\gamma}}{|\gamma|^4}\cdot(\omega_E - \overline{\omega}_E).$$

We apply the theorem to the divisors

$$\alpha = \sum_{x \in E_{\mathbf{f}}} (x - o), \quad \beta = \sum_{x \in E_{\mathbf{f}}/\mu_K} (\chi_f(\overline{x})x - o). \tag{4.4}$$

Since $\overline{\chi}_f(x) = \chi_f(\overline{x})$ by the assumption (4.2), both α and β are defined over $\mathbb{R}$, hence over $\mathbb{Q}$. If we put

$$e_E := e(\alpha, \beta) \in H^2_{\mathscr{M}}(E, \mathbb{Q}(2))_{\mathbb{Z}},$$

we have by the theorem and (4.3) :

Corollary 4.2. *Under the assumption* (4.2)*, let* $e_E \in H^2_{\mathscr{M}}(E, \mathbb{Q}(2))_{\mathbb{Z}}$ *be as above. Then we have*

$$r_{\mathscr{D}}(e_E) = \pm \frac{N(\mathbf{f})^{\frac{1}{2}}}{2|h|} L^*(\chi, 0) \cdot \Omega_{\mathbb{R}}(\omega_E - \overline{\omega}_E).$$

Remark 4.3. The Beilinson conjecture implies that $N(\mathbf{f})^{\frac{1}{2}}/|h| \in \mathbb{Q}$, which will be the case in our examples.

5. COMPARISONS

Now we relate and compare the results of preceding sections and prove main results.

5.1. **Preliminaries.** On X_N, we have $3N$ points

$$P_n = (0 : \zeta^n : 1), \quad Q_n = (\zeta^n : 0 : 1), \quad R_n = (\xi\zeta^n : 1 : 0), \quad (n \in \mathbb{Z}/N\mathbb{Z})$$

where we put $\xi = \exp(\pi i/N)$. If we choose P_0 as the base point, then the above points are torsion in the Jacobian.

We have $\langle \widetilde{\omega}^{a,b}, \widetilde{\omega}^{a',b'} \rangle \neq 0$ if and only if $(a', b') = (-a, -b)$, and one calculates

$$\langle \widetilde{\omega}^{a,b}, \widetilde{\omega}^{-a,-b} \rangle = \frac{N^2}{2\pi i} \cdot \frac{(1 - \zeta^a)(1 - \zeta^b)}{1 - \zeta^{a+b}}. \tag{5.1}$$

5.2. **The case $N = 3$.** If $N = 3$, then $X_3 = E$ itself is an elliptic curve with the origin P_0. It has complex multiplication by the integer ring $\mathscr{O}$ of $K = \mathbb{Q}(\mu_3)$, induced by the multiplication of μ_3 on x. One sees that

$$X_3^{[1,1]} = h^1(E)$$

and the Hecke character χ associated with E equals $j_3^{2,2} = \overline{j}_3^{1,1}$. By (5.1), the real holomorphic form normalized as (4.1) is

$$\omega_E = \frac{1}{3}\sqrt{\frac{2\pi}{\sqrt{3}}} \cdot \widetilde{\omega}_3^{1,1}.$$

Since $\int_\kappa \widetilde{\omega}_3^{1,1} = -3\zeta$, we have $\mathit{\Gamma} = \mathscr{O}\Omega$ with

$$\Omega = \Omega_{\mathbb{R}} = \sqrt{\frac{2\pi}{\sqrt{3}}}\,.$$

By Hasse [9], Satz 2, we have $\mathbf{f} = (3)$, so if we set $\nu = 3$, the assumption (4.2) is satisfied.

Proposition 5.1. *Let* $e_3^{[1,1]}$ *and* e_E *be the elements defined in* §*3 and* §*4, respectively. Then we have* $r_{\mathscr{D}}(e_E) = r_{\mathscr{D}}(e_3^{[1,1]})$.

Proof. Since $E_{\mathbf{f}} = \{P_n, Q_n, R_n \mid n \in \mathbb{Z}/3\mathbb{Z}\}$, $(\mathscr{O}/\mathbf{f})^*/\mu_K = \{1\}$, and Q_0 corresponds to 1 under $E_{\mathbf{f}} \simeq \mathscr{O}/\mathbf{f}$, the divisors (4.4) are:

$$\alpha = \sum_{n \in \mathbb{Z}/3\mathbb{Z}} (P_n + Q_n + R_n) - 9P_0, \quad \beta = Q_0 - P_0.$$

We can take

$$f_\alpha = \frac{xy}{(1-y)^3}, \quad f_\beta = \frac{1-x}{1-y} \otimes \frac{1}{3}.$$

Then we have

$$3e_0(\alpha, \beta) = -\{x, 1-y\} + \{y, 1-x\} - 3\{1-y, 1-x\}.$$

Since $3\{x, 1-y\} = \{x^3, 1-y\} = \{1-y^3, 1-y\}$ comes from a quotient rational curve, it does not contribute to the regulator, and similarly for $\{y, 1-x\}$. Hence the proposition is proved. □

Theorem 5.2. *Let $\chi = j_3^{1,1}$ or $j_3^{2,2}$ be the Jacobi-sum Hecke character of $\mathbb{Q}(\mu_3)$. Then we have*

$$L^*(\chi, 0) = \frac{1}{6\sqrt{3}\pi}\Big(\widetilde{F}\big(\tfrac{1}{3}, \tfrac{1}{3}\big) - \widetilde{F}\big(\tfrac{2}{3}, \tfrac{2}{3}\big)\Big).$$

Proof. By Theorem 3.2, Corollary 4.2 and Proposition 5.1, we obtain the equality except the sign. Since $L(\chi, 2)$ is positive and the root number (the sign of the functional equation) is 1, $L^*(\chi, 0)$ is positive. On the other hand, since $\widetilde{F}(\alpha, \beta)$ is monotonously decreasing with respect to each parameter ([12], Proposition 4.25), the right-hand side is also positive. □

Remark 5.3. The theorem verifies and refines [12], Corollary 4.21. The regulator of X_3 as an elliptic curve is also calculated by Rohrlich [14], §3, Example. The definition of the regulator map in loc. cit. is -2 times ours and the element considered there equals $3e_3$.

5.3. **The case $N = 4$.** If $N = 4$, then $H_4 \backslash I_4 = \{[1, 2], [2, 1], [1, 1]\}$. Here we study the first two cases; by the symmetry, it suffices to treat the first case. Let $K = \mathbb{Q}(\mu_4)$ and $\mathscr{O}$ be its integer ring.

Let E be an elliptic curve over $\mathbb{Q}$ defined by (the normalization of)

$$u_0^4 + v_0^2 w_0^2 = w_0^4,$$

which is naturally a quotient of X by the morphism

$$p\colon X \to E; \quad (u_0 : v_0 : w_0) = (x_0 z_0 \colon y_0^2 \colon z_0^2).$$

Let P'_n, Q'_n and R'_n be the image of P_n, Q_n and R_n, respectively, and P'_0 be the origin. Then P'_n, R'_n (resp. Q'_n) depend exactly on n mod 2 (resp. mod 4). The action of $\mathscr{O}$ on X induced by the multiplication of μ_4 on x descends to an action on E, which defines complex multiplication of E by $\mathscr{O}$.

One sees that p^* induces an isomorphism

$$h^1(E) \simeq X_4^{[1,2]}$$

of motives over $\mathbb{Q}$ with $\mathbb{Q}$-coefficients. The Hecke character associated with E coincides with $j_4^{3,2} = \overline{j_4^{1,2}}$. Since the degree of p is 2, we have $\langle p^*\omega_E, p^*\overline{\omega}_E \rangle = 2\langle \omega_E, \overline{\omega}_E \rangle = -2$, so by (4.1) and (5.1) we have

$$p^*\omega_E = \sqrt{\frac{\pi}{8}} \cdot \widetilde{\omega}_4^{1,2}.$$

One can show that p induces a surjection on homology, hence $H_1(E(C),\mathbb{Z}) = \mathscr{O}\cdot p_*\kappa$. Since $\int_{p_*\kappa}\omega_E = \int_\kappa p^*\omega_E$ and $\int_\kappa \widetilde{\omega}_4^{1,2} = (1-i)(1-i^2) = 2(1-i)$, we obtain $\mathit{\Gamma} = \mathscr{O}\Omega$ with

$$\Omega = \sqrt{\frac{\pi}{2}}\cdot(1-i), \quad \Omega_{\mathbb{R}} = (1+i)\Omega = \sqrt{2\pi}.$$

By Hasse [9], Satz 3, we have the conductor $\mathbf{f} = (2(1+i))$, so if we let $\nu = 2(1+i)$, the assumption (4.2) is satisfied.

Proposition 5.4. *Let $e_4^{[1,2]}$ and e_E be the elements defined in §3 and §4, respectively. Then we have $r_{\mathscr{D}}(p^*e_E) = 2\cdot r_{\mathscr{D}}(e_3^{[1,2]})$.*

Proof. One sees that the eight points $\{P'_m, Q'_n, R'_m \mid m\in\mathbb{Z}/2\mathbb{Z}, n\in\mathbb{Z}/4\mathbb{Z}\}$ are exactly the $\mathbf{f}$-torsion poinst on E. We have $(\mathscr{O}/\mathbf{f})^*/\mu_K = \{1\}$, and under the identification $E_{\mathbf{f}} \xrightarrow{\sim} \mathscr{O}/\mathbf{f}$, Q'_0 corresponds to 1 since $\int_{P_0}^{Q_0}\widetilde{\omega}^{1,2} = 1$. Therefore, the divisors (4.4) are:

$$\alpha = \sum_{n\in\mathbb{Z}/2\mathbb{Z}}(P'_n + R'_n) + \sum_{n\in\mathbb{Z}/4\mathbb{Z}} Q'_n - 8P'_0, \quad \beta = Q'_0 - P'_0.$$

Since

$$p^*(\alpha) = \sum_{n\in\mathbb{Z}/4\mathbb{Z}}(P_n + 2Q_n + R_n) - 8P_0 - 8P_2, \quad p^*(\beta) = 2Q_0 - P_0 - P_2$$

we have

$$p^*f_\alpha = \frac{xy^2}{(1-y)^2(1+y)^2}, \quad p^*f_\beta = \frac{(1-x)^2}{(1-y)(1+y)}\otimes\frac{1}{4}.$$

By a calculation as before, we obtain

$$r_{\mathscr{D}}(p^*e_E) = r_{\mathscr{D}}(\{1-x,1-y\} + \{1-x,1+y\})$$

where $r_{\mathscr{D}}$ here is the regulator map for X_4. Now we apply the projector $p_4^{[1,2]} = p_4^{1,2} + p_4^{3,2}$. Note that $p_4^{[1,2]}p^* = p^*$. Since $\{1-x,1+y\} = (0,2)^*e_4$, where $(0,2)\in G_4$, and $p_4^{1,2}(0,2)^* = p_4^{1,2}$, $p_4^{3,2}(0,2)^* = p_4^{3,2}$, we have $p_4^{[1,2]}(0,2)^*e_4 = e_4^{[1,2]}$, hence the proposition is proved. □

Theorem 5.5. *Let $\chi = j_4^{1,2}$ or $j_4^{3,2}$ be the Jacobi-sum Hecke character of $\mathbb{Q}(\mu_4)$. Then we have*

$$L^*(\chi,0) = \frac{1}{8\pi}\Big(\widetilde{F}\big(\tfrac14,\tfrac12\big) - \widetilde{F}\big(\tfrac34,\tfrac12\big)\Big).$$

Proof. As in the proof of Theorem 5.2, it follows from Theorem 3.2, Corollary 4.2, Proposition 5.4, and the fact that the root number is 1. □

Remark 5.6. The theorem verifies and refines [12], Corollary 4.21. In the remaining case $N = 4$, $[a,b] = [1,1]$, we have $\mathbf{f} = (4)$ and our special points on X_4 are not sufficient to cover $E_{\mathbf{f}}$. We have the same difficulty for $N = 6$.

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