超局所解析とその周辺
Microlocal Analysis and Related Topics

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Summability of formal solutions for singular first-order linear PDEs with holomorphic coefficients II

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We study formal power series solutions for the following first-order linear partial differential equation with two complex variables:

$$[\{\alpha_0+\alpha_1 x+\beta(x,y)\}yD_x+\{ax+b(x,y)\}y^2D_y]u(x,y)=f(x,y), \tag{1}$$

where x, $y\in\mathbb{C}$, $D_x=\partial/\partial x$, $D_y=\partial/\partial y$. α_0, α_1 and a are constants, and α_0, $\alpha_1\neq 0$. β, b and f are holomorphic functions at the origin. Moreover β and b satisfy

$$\beta(x,0)\equiv b(x,0)\equiv 0. \tag{2}$$

We already know that the equation (1) has a unique formal power series solution $u(x,y)=\sum_{n=0}^{\infty}u_n(x)y^n\in\mathcal{O}[R][[y]]_2$ for some $R>0$. Here we say that $u(x,y)=\sum_{n=0}^{\infty}u_n(x)y^n$ belongs to $\mathcal{O}[R][[y]]_2$, if all $u_n(x)$ are holomorphic on $B(R)=\{x\in\mathbb{C};\ |x|\leq R\}$ ($R>0$ is independent of n), and there exist some positive constants C and K such that $\max_{|x|\leq R}|u_n(x)|\leq CK^n n!$ for all $n=0,1,2,\ldots$. Hence, the formal solution $u(x,y)$ diverges in general. (The suffix 2 of $\mathcal{O}[R][[y]]_2$ expresses the Gevrey index of power series.)

So we are concerned with the existence of the Gevrey asymptotic solutions for the above divergent solution $u(x,y)$, and in particular we are interested in the Borel summability of $u(x,y)$.

Definitions. (i) For $\theta\in\mathbb{R}$, $\kappa>0$ and $0<\rho\leq+\infty$, the sector $S(\theta,\kappa,\rho)$ in the universal covering space of $\mathbb{C}\setminus\{0\}$ is defined by

$$S(\theta,\kappa,\rho)=\left\{y;\ \left|\arg(y)-\theta\right|<\frac{\kappa}{2},\ 0<|y|<\rho\right\}. \tag{3}$$

We refer to θ, κ and ρ as the *direction*, the *opening angle* and the *radius* of $S(\theta,\kappa,\rho)$, respectively.

(ii) Let $u(x,y)=\sum_{n=0}^{\infty} u_n(x)y^n \in \mathcal{O}[R][[y]]_2$ and let $U(x,y)$ be a holomorphic function on $X = B(R) \times S(\theta,\kappa,\rho)$. Then we say that *$U(x,y)$ has $u(x,y)$ as an asymptotic expansion of the Gevrey order 2 in X* if the following asymptotic estimates hold: there exist some positive constants C and K such that

$$\max_{|x|\le R}\left|U(x,y)-\sum_{n=0}^{N-1} u_n(x)y^n\right| \le CK^N N!|y|^N, \tag{4}$$

for all $y \in S(\theta,\kappa,\rho)$ and $N = 1, 2, \ldots$. Then we write this as

$$U(x,y) \cong_2 u(x,y) \quad \text{in} \quad X. \tag{5}$$

(iii) Let $u(x,y)=\sum_{n=0}^{\infty} u_n(x)y^n \in \mathcal{O}[R][[y]]_2$. We say that *$u(x,y)$ is Borel summable in a direction θ* if there exists a holomorphic function $U(x,y)$ on $X = B(r) \times S(\theta,\kappa,\rho)$ for some $0 < r \le R$, $\rho > 0$ and $\underline{\kappa > \pi}$ which satisfies $U(x,y) \cong_2 u(x,y)$ in X. A given divergent power series $u(x,y) \in \mathcal{O}[R][[y]]_2$ is not necessarily Borel summable in general. However, when $u(x,y)$ is Borel summable in a direction θ, we see that the above holomorphic funtion $U(x,y)$ is unique. So we call this unique $U(x,y)$ the *Borel sum of $u(x,y)$ in a direction θ*.

Our purpose is to give conditions which the coefficients of the equation (1) should satisfy in order to assure the Borel summability of the formal solution in a given direction θ.

Assumptions. First let us consider the following initial value problem:

$$\frac{d\xi}{d\tau} = \exp\left[\frac{a\alpha_0}{{\alpha_1}^2}(e^{-\alpha_1\xi}-1+\alpha_1\xi)\right], \qquad \xi(0)=0. \tag{6}$$

Then we assume the following:

(Assumption 1) (6) has a holomorphic solution $\xi = F(\tau)$ on the region $B(r) \cup S(\theta,\kappa,+\infty)$ for some $r > 0$ and $\kappa > 0$.

Next, let us define the region $\Xi_{r,\theta,\kappa}$ consisting of the image of F by

$$\Xi_{r,\theta,\kappa} = \{\xi = F(\tau);\ \tau \in B(r) \cup S(\theta,\kappa,+\infty)\}, \tag{7}$$

and let us assume the following:

(Assumption 2) $\displaystyle\sup_{\xi\in\Xi_{r,\theta,\kappa}}\left|\exp\left[\frac{a\alpha_0}{{\alpha_1}^2}(e^{-\alpha_1\xi}-1+\alpha_1\xi)\right]\right| < \infty.$

Next, in order to state assumptions for coefficients, we define the region $\Omega_{r,\theta,\kappa}$ by

$$\Omega_{r,\theta,\kappa} = \left\{x = \frac{\alpha_0}{\alpha_1}\left(e^{-\alpha_1\xi}-1\right);\ \xi \in \Xi_{r,\theta,\kappa}\right\}, \tag{8}$$

and define the entire function $\mathscr{F}(\xi)$ by

$$\mathscr{F}(\xi)=\int_0^{\xi}\exp\left[-\frac{a\alpha_0}{{\alpha_1}^2}(e^{-\alpha_1\zeta}-1+\alpha_1\zeta)\right]d\zeta, \tag{9}$$

For the inhomogeneity term $f(x,y)$ we assume the following:

(Assumption 3) $f(x,y)$ can be continued analytically to $\Omega_{r,\theta,\kappa}\times\{y\in\mathbb{C};\ |y|\leq c\}$ for some $c>0$. Moreover, it has the following estimate there. There exist some positive constants C and δ such that

$$\max_{|y|\leq c}|f(x,y)|\leq C\exp\left[\delta\left|\mathscr{F}\left(-\frac{1}{\alpha_1}\log\left(1+\frac{\alpha_1}{\alpha_0}x\right)\right)\right|\right],\quad x\in\Omega_{r,\theta,\kappa}. \tag{10}$$

Finally, we impose the following conditions for the coefficients $\beta(x,y)$ and $b(x,y)$:

(Assumption 4) $\beta(x,y)$ and $b(x,y)$ can be continued analytically to $\Omega_{r,\theta,\kappa}\times\{y\in\mathbb{C};\ |y|\leq c\}$. Moreover, there exist some positive constants $K,\ L>0$ and $p>1$ such that

$$\left|\frac{1}{\alpha_0+\alpha_1 x}\frac{\partial^m\beta}{\partial y^m}(x,0)\right|\leq KL^m m!|E(x)|^m,\quad x\in\Omega_{r,\theta,\kappa},\quad m=1,2,\dots; \tag{11}$$

$$\begin{aligned}&\left|\frac{1}{\alpha_0+\alpha_1 x}\frac{\partial^m\beta}{\partial y^m}(x,0)\cdot ax\right|\\&\leq KL^m m!|E(x)|^{m+1}\cdot\left[1+\left|\mathscr{F}\left(-\frac{1}{\alpha_1}\log\left(1+\frac{\alpha_1}{\alpha_0}x\right)\right)\right|\right]^{-p},\\&\qquad x\in\Omega_{r,\theta,\kappa},\quad m=1,2,\dots;\end{aligned} \tag{12}$$

$$\begin{aligned}&\left|\frac{\partial^m b}{\partial y^m}(x,0)\right|\\&\leq KL^m m!|E(x)|^{m+1}\cdot\left[1+\left|\mathscr{F}\left(-\frac{1}{\alpha_1}\log\left(1+\frac{\alpha_1}{\alpha_0}x\right)\right)\right|\right]^{-p},\\&\qquad x\in\Omega_{r,\theta,\kappa},\quad m=1,2,\dots,\end{aligned} \tag{13}$$

where

$$E(x)=e^{-(a/\alpha_1)x}\left(1+\frac{\alpha_1}{\alpha_0}x\right)^{a\alpha_0/{\alpha_1}^2}, \tag{14}$$

If $x\in\Omega_{r,\theta,\kappa}$, it holds that $1+(\alpha_1/\alpha_0)x\neq 0$. Accordingly, all functions appearing in (10)—(13) are well-defined.

Let us state the main theorem.

Main Theorem. *Under assumptions* (Assumption 1)—(Assumption 4) *the formal solution $u(x,y)$ of* (1) *is Borel summable in the direction θ.*

On the Cauchy problem for hyperbolic operators of second order whose coefficients depend only on the time variable

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1. Main results

Notations:

the time variable: $t \in \mathbb{R} \overset{\text{dual}}{\longleftrightarrow} \tau \in \mathbb{R}$

the space variables: $x = (x_1, \cdots, x_n) \in \mathbb{R}^n \overset{\text{dual}}{\longleftrightarrow} \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$

$D_t = -i\partial_t$, $D_x = (D_1, \cdots, D_n) = -i(\partial_{x_1}, \cdots, \partial_{x_n})$

Consider hyperbolic operators of second order whose symbols have the form

$$P(t, x, \tau, \xi) = \tau^2 + \sum_{j=0}^{1} \sum_{|\alpha| \leq 2-j} a_{j,\alpha}(t, x)\tau^j \xi^\alpha,$$

where $a_{j,\alpha}(t, x) \in C^\infty([0, \infty) \times \mathbb{R}^n)$, and the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} P(t, x, D_t, D_x)u(t, x) = f(t, x) & \text{in } [0, \infty) \times \mathbb{R}^n, \\ D_t^j u(t, x)|_{t=0} = u_j(x) & \text{in } \mathbb{R}^n \quad (j = 0, 1). \end{cases}$$

We consider everything in the framework of C^∞.

Def: We say that (CP) is C^∞ well-posed if

(E) $\forall f \in C^\infty([0, \infty) \times \mathbb{R}^n)$, $\forall u_j \in C^\infty(\mathbb{R}^n)$ $(j = 0, 1)$, $\exists u \in C^\infty([0, \infty) \times \mathbb{R}^n)$ satisfying (CP). **(Existence)**

(U) If $s > 0$, $u \in C^\infty([0, \infty) \times \mathbb{R}^n)$, $D_t^j u(t, x)|_{t=0} = 0$ in $\mathbb{R}^n$ $(j = 0, 1)$ & $\operatorname{supp} P(t, x, D_t, D_x)u \subset \{t \geq s\}$, then $\operatorname{supp} u \subset \{t \geq s\}$. **(Uniqueness)**

Throughout the talk we assume that

$$a_{j,\alpha}(t, x) \equiv a_{j,\alpha}(t) \quad \text{if } j + |\alpha| = 2.$$

By coordinate transformation we may assume that $P(t, x, \tau, \xi)$ has the following form:

$$P(t, x, \tau, \xi) = \tau^2 - a(t, \xi) + b_0(t, x)\tau + b(t, x, \xi) + c(t, x),$$

$$a(t, \xi) = \sum_{j,k=1}^{n} a_{j,k}(t)\xi_j\xi_k, \quad b(t, x, \xi) = \sum_{j=1}^{n} b_j(t, x)\xi_j, \quad a_{j,k}(t) = a_{k,j}(t).$$

Moreover, by Lax-Mizohata theorem we may assume that

(H) $a(t, \xi) \geq 0$ for $(t, \xi) \in [0, \infty) \times \mathbb{R}^n$

(see S. Mizohata, J. Math. Kyoto Univ. **1** (1961), 109–127). From Ivrii-Petkov's result we can assume without loss of generality that

(F) $a(t,\xi) \not\equiv 0$ in t for $\forall \xi \in \mathbb{R}^n \setminus \{0\}$

(see V. Ya. Ivrii and V. M. Petkov, Russian Math. Surveys **29** (1974), 1–70).

Sufficiency:

We assume in "Sufficiency" that

(A) the $a_{j,k}(t)$ are real analytic (for simplicity),

(B) the coefficients do not depend on x, *i.e.*,

$$b_0(t,x) \equiv b_0(t), \quad b(t,x,\xi) \equiv b(t,\xi), \quad c(t,x) \equiv c(t).$$

Let Ω be a neighborhood of $[0,\infty)$ in $\mathbb{C}$ such that the $a_{j,k}(t)$ are analytic in Ω, and put

$$\mathcal{R}(\xi) = \{(\mathrm{Re}\ \lambda)_+;\ \lambda \in \Omega \text{ and } a(\lambda,\xi) = 0\} \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\},$$

where $a_+ = \max\{a, 0\}$. The following condition is a kind of the Levi condition:

(L) $\forall T > 0$, $\exists C > 0$ s.t.

$$\min_{\tau \in \mathcal{R}(\xi)} |t-\tau|\,|b(t,\xi)| \le C\sqrt{a(t,\xi)} \quad \text{for } (t,\xi) \in [0,T] \times (\mathbb{R}^n \setminus \{0\}).$$

Thm 1: Under (A), (B) and (L) (CP) is C^∞ well-posed.

Remark: $\mathcal{R}(\xi)$ can be replaced in (L) by $\mathcal{R}'(\xi)$ satisfying

$$\sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \#(\mathcal{R}'(\xi) \cap \{t \le T\}) < \infty \text{ for } \forall T > 0,$$

where $\#A$ denotes the number of the elements of a set A.

Def: (i) Let f be a function on $\mathbb{R}$. We say that $f(t)$ is a semi-algebraic function if the graph of f is a semi-algebraic set, *i.e.*, the graph of f is a set defined by polynomial eauations and inequalities. (ii) Let $t_0 \in \mathbb{R}$, U be a neighborhood of t_0 and $f : U \to \mathbb{R}$. We say that f is semi-algebraic at t_0 if $\exists c > 0$ s.t. $\{(t,y) \in \mathbb{R}^2;\ y = f(t) \text{ and } |t - t_0| < c\}$ is a semi-algebraic set.

Necessity:

We assume in "Necessity" that

(A)$'$ the $a_{j,k}(t)$ and $b_j(t,x)$ ($1 \le j \le n$) are real analytic functions of $t \in [0,\infty)$.

Let $t_0 \ge 0$, $x_0 \in \mathbb{R}^n$ and $\xi^0 \in S^{n-1}$. If $n \ge 3$, we assume that

(A)$''_{(t_0,x^0)}$ the $a_{j,k}(t)$ and $b_j(t,x^0)$ ($1 \le j \le n$) are semi-analytic at t_0.

The following condition is very similar to the condition (L):

(L)$_{(t_0,x^0,\xi^0)}$ $\exists U$: nbd of t_0, $\exists \Gamma$: conic nbd of ξ^0, $\exists C > 0$ s.t.

$$\min_{\tau \in \mathcal{R}(\xi)} |t-\tau|\,|b(t,x^0,\xi)| \le C\sqrt{a(t,\xi)} \quad \text{for } (t,\xi) \in U \times \Gamma.$$

Thm 2: Assume that (A)$'$ and (B) are satisfied. Moreover, we assume that (A)$''_{(t_0,x^0)}$ is satisfied if $n \ge 3$. Then (L)$_{(t_0,x^0,\xi^0)}$ is necessary for C^∞ well-posedness.

Remark: Assume that (A)$'$ and (B) are satisfied, and that $(A)''_{(t_0,0)}$ is valid for any $t_0 \geq 0$ if $n \geq 3$. Then (CP) is C^∞ well-posed if and only if (L) is satisfied

related results:

- Colombini-Ishida-Orrú: Ark. Mat. **38** (2000), 223–230.
  (CP) is C^∞ well-posed if (B) is satisfied and the following conditions are satisfied:
  $\exists k \in \mathbb{N}$, $\exists C > 0$ s.t. $k \geq 2$ and $\sum_{j=0}^k |\partial_t^j a(t,\xi)| \neq 0$
  for $(t,\xi) \in [0,\infty) \times S^{n-1}$
  $|b(t,\xi)| \leq Ca(t,\xi)^{1/2-1/k}$ for $(t,\xi) \in [0,\infty) \times S^{n-1}$.
- Nishitani: J. Math. Kyoto Univ. **24** (1984), 91–104.

2. Outline of Proof of Thm 1

We consider the Cauchy problem for an ordinary differential operator with the parameters ξ and $\varepsilon \in [0,1]$:

$$\begin{cases} (P(t,D_t,\xi) - \varepsilon|\xi|^2)v_\varepsilon(t,\xi) = \hat{f}(t,\xi), \\ v_\varepsilon(0,\xi) = \hat{u}_0(\xi), \quad (D_t v_\varepsilon)(0,\xi) = \hat{u}_1(\xi). \end{cases}$$

We adopt the following energy forms:

$$\mathcal{E}_\varepsilon(t,\xi;\gamma) := E_\varepsilon(t,\xi)\exp[-\gamma\Phi(t,\xi)],$$

where $\gamma > 0$ and

$$\begin{aligned} E_\varepsilon(t,\xi) &:= |\partial_t v_\varepsilon(t,\xi)|^2 + (a(t,\xi) + \varepsilon|\xi|^2 + W(t,\xi)^2)|v_\varepsilon(t,\xi)|^2, \\ \Phi(t,\xi) &:= t + \sum_{\tau\in\mathcal{R}(\xi)} \log(\sqrt{(t-\tau)^2\langle\xi\rangle + 1} + (t-\tau)\langle\xi\rangle^{1/2}) \\ &\qquad + \log(\sqrt{t^2\langle\xi\rangle^{4/3} + 1} + t\langle\xi\rangle^{2/3}), \\ W(t,\xi) &:= \partial_t\Phi(t,\xi), \quad \langle\xi\rangle := (1+|\xi|^2)^{1/2}. \end{aligned}$$

Note that

$$\partial_t \log(\sqrt{\lambda(t,\xi)^2+1} + \lambda(t,\xi)) = \partial_t\lambda(t,\xi)/\sqrt{\lambda(t,\xi)^2+1}.$$

Then we have for $\gamma \gg 1$

$$\begin{aligned} \partial_t\mathcal{E}_\varepsilon(t,\xi;\gamma) &\leq |\hat{f}(t,\xi)|^2\exp[-\gamma\Phi(t,\xi)]/W(t,\xi), \\ \mathcal{E}_\varepsilon(t,\xi;\gamma) &\leq \mathcal{E}_\varepsilon(0,\xi;\gamma) + \int_0^t \exp[-\gamma\Phi(s,\xi)]|\hat{f}(s,\xi)|^2/W(s,\xi)\,ds, \end{aligned}$$

and Thm 1 can be proved by a standard argument.

Gevrey regularities of solutions of nonlinear singular partial differential equations

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§1. Introduction. For $\sigma \geq 1$ and an open subset V of $\mathbb{R}_x^n$ we denote by $\mathcal{E}^{\{\sigma\}}(V)$ the set of all functions $f(x) \in C^\infty(V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{x \in K} |\partial_x^\alpha f(x)| \leq C h^{|\alpha|} |\alpha|!^\sigma, \quad \forall \alpha \in \mathbb{N}^n.$$

A function in the class $\mathcal{E}^{\{\sigma\}}(V)$ is called an ultra-differentiable function of the Gevrey class of order σ.

For an interval $[0,T] = \{t \in \mathbb{R}\,;\, 0 \leq t \leq T\}$ we denote by $C^\infty([0,T], \mathcal{E}^{\{s\}}(V))$ the set of all infinitely differentiable functions $u(t,x)$ in $t \in [0,T]$ with values in $\mathcal{E}^{\{\sigma\}}(V)$ equipped with the usual local convex topology (see Komatsu [1]).

The class $\mathcal{E}^{\{\sigma\}}([0,T] \times V)$ can be defined in the same way.

We note the following example:

Example 1.1. *Let $(t,x) \in [0,T] \times \mathbb{R}$, $k \in \mathbb{N}^*(= \{1,2,\ldots\})$ and let us consider*

$$(t\partial_t + 1)^2 u - t^k \partial_x^2 u = f(t,x). \tag{1.1}$$

The following results are known:

(1) (1.1) is uniquely solvable in $C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(\mathbb{R}))$ for any $\sigma \geq 1$.

(2) If $k \geq 2$, (1.1) is also uniquely solvable in $\mathcal{E}^{\{\sigma\}}([0,T] \times \mathbb{R})$ for any $\sigma \geq 1$.

(3) But, in the case $k = 1$, the equation (1.1) is not uniquely solvable in $\mathcal{E}^{\{\sigma\}}([0,T] \times \mathbb{R})$ for any $\sigma > 1$.

By (1), for any $\sigma \geq 1$ and any $f(t,x) \in \mathcal{E}^{\{\sigma\}}([0,T] \times \mathbb{R})$ we have a unique solution $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(\mathbb{R}))$ of (1.1); therefore whether or not (1.1) has a solution in $\mathcal{E}^{\{\sigma\}}([0,T] \times \mathbb{R})$ is reduced to the following problem:

Problem 1.2. If $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(\mathbb{R}))$ is a solution of (1.1), can we have the result $u(t,x) \in \mathcal{E}^{\{\sigma\}}([0,T] \times \mathbb{R})$?

§2. In the linear case. About this problem, the author has given in [3] a sufficient condition for the problem to be affirmative: the result was as follows. Let

$$P = (t\partial_t)^m + \sum_{j+|\alpha| \leq m, j<m} a_{j,\alpha}(t,x)(t\partial_t)^j \partial_x^\alpha \tag{2.1}$$

where $m \in \mathbb{N}^*$ and $a_{j,\alpha}(t,x) \in \mathcal{E}^{\{\sigma\}}([0,T] \times V)$ $(j + |\alpha| \leq m,\ j < m)$. We denote by $k_{j,\alpha}$ the order of zero point $t = 0$ of $a_{j,\alpha}(t,x)$: that is,

$$k_{j,\alpha} = \inf\{\, k \in \mathbb{N}\,;\, (\partial_t^k a_{j,\alpha})(0,x) \not\equiv 0 \text{ on } V\}. \tag{2.2}$$

Theorem 2.1 ([3]). *We set* $\Delta = \{(j,\alpha)\,;\, k_{j,\alpha} < |\alpha|\}$, *and suppose:*
1) $k_{j,\alpha} > 0$ *holds if* $|\alpha| > 0$, *and*
2) σ *satisfies*

$$1 \le \sigma \le 1 + \min\left[\infty,\ \min_{(j,\alpha)\in\Delta}\left(\frac{m-j-|\alpha|}{|\alpha|-k_{j,\alpha}}\right)\right]. \tag{2.3}$$

Then, if $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ *satisfies* $(Pu)(t,x) \in \mathcal{E}^{\{\sigma\}}([0,T]\times V)$, *we have* $u(t,x) \in \mathcal{E}^{\{\sigma\}}([0,T]\times V)$.

§3. In the nonlinear case. Let $m \in \mathbb{N}^*$, $\Lambda = \{(j,\alpha) \in \mathbb{N}\times\mathbb{N}^n\,;\, j+|\alpha| \le m, j < m\}$ and $d = \#\Lambda$ be fixed. In this section we will consider the nonlineat equation

$$(t\partial_t)^m u = F(t,x,D^m u) \quad \text{with } D^m u = \{(t\partial_t)^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda}. \tag{3.1}$$

We denote by $z = \{z_{j,\alpha}\}_{(j,\alpha)\in\Lambda}$ the variable in $\mathbb{R}^d$ (which corresponds to $D^m u = \{(t\partial_t)^j\partial_x^\alpha u\}_{(j,\alpha)\in\Lambda}$). Let Ω be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$, and let $F(t,x,z)$ be a C^∞ function on Ω. Let $\sigma \ge 1$, $T > 0$, and let V be an open subset of $\mathbb{R}^n$.

The main assumptions are as follows.

c_1) $m \ge 1$ is an integer:
c_2) $F(t,x,z) \in \mathcal{E}^{\{\sigma\}}(\Omega)$:
c_3) $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ is a solution of (3.1) on $[0,T]\times V$.

The condition c_3) includes the property: $(t,x) \in [0,T]\times V \Longrightarrow (t,x,D^m u(t,x)) \in \Omega$. We set

$$k_{j,\alpha} = \inf\{\, k \in \mathbb{N}\,;\, (\partial_t^k(\partial F/\partial z_{j,\alpha}))(0,x,D^m u(0,x)) \not\equiv 0 \text{ on } V\},$$
$$\Delta = \{(j,\alpha)\,;\, k_{j,\alpha} < |\alpha|\}.$$

Theorem 3.1 ([4]). *Suppose the conditions* $c_1) \sim c_3)$, *and*
1) $k_{j,\alpha} > 0$ *holds if* $|\alpha| > 0$,
2) σ *satisfies*

$$1 \le \sigma \le 1 + \min\left[\infty,\ \min_{(j,\alpha)\in\Delta}\left(\frac{m-j-|\alpha|}{|\alpha|-k_{j,\alpha}}\right)\right]. \tag{3.2}$$

Then we have $u(t,x) \in \mathcal{E}^{\{\sigma\}}([0,T]\times V)$.

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On the Deficiency Index of Even Order Symmetric Differential Expressions with Essential Spectrum

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Abstract

We consider real symmetric expressions of order $m = 2n$ which are of the form

$$My = \sum_{j=0}^{n} (-1)^j (p_j y^{(j)})^{(j)},$$

where $p_j \in C^j(I, \mathbb{R})$ for $j = 0, \ldots, n$ and $p_n > 0$ on $I = [1, \infty)$. The spectral properties of M we shall be considering consist of the essential spectrum of M, denoted by $\sigma_e(M)$, and the nullities of M, denoted by $\mathrm{nul}(M - \lambda)$ which are defined as follows :

$$\begin{aligned} \sigma_e(M) &= \{\lambda \in \mathbb{C} \mid \text{range } T_1(M - \lambda I_{\mathcal{L}^2}) \text{ is not closed}\} \\ \mathrm{nul}(M - \lambda) &= \dim \ker(T_1(M - \lambda I_{\mathcal{L}^2})) \quad \text{for } \lambda \in \mathbb{C}, \end{aligned}$$

where $I_{\mathcal{L}^2}$ is the identity on $\mathcal{L}^2(I)$ and $T_1(M - \lambda I_{\mathcal{L}^2})$ is the maximal operator generated by $M - \lambda I_{\mathcal{L}^2}$. Since the nullities of M are constant in $\mathbb{C} \setminus \mathbb{R}$, we consider the nullity only for $\lambda = i$. We refer to this nullity as the deficiency index of M and denote it by $d(M)$.

In this paper, we shall show that for all n, $k \in \mathbb{N}$ with $n \leq k < 2n$, there exist expressions M of order $2n$ with nonempty essential spectrum such that $d(M) = k$. A brief historical background on the deficiency index problem will also be discussed.

A note on wave equations in Einstein & de Sitter spacetime

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The current note is concerned with the wave propagating in the universe modeled by the cosmological models with expansion. We are motivated by the significant importance of the solutions of the partial differential equations arising in the cosmological problems for our understanding of the universe. The covariant wave equation in Einstein & de Sitter spacetime is given by

$$\partial_t^2 \psi - t^{-\frac{4}{3}} \Delta \psi + 2t^{-1}\partial_t \psi = f(t,x). \tag{1}$$

One can not expect the wellposedness of the initial valued problem for this equation due to the singular coefficient $t^{-\frac{4}{3}}$. Therefore, we shall impose a different kind of initial condition on the equation as

$$\lim_{t\to 0} t\psi(t,x) = \varphi_0(x), \quad \lim_{t\to 0} \left(t\partial_t\psi(t,x) + \psi(t,x) + 3t^{-\frac{1}{3}}\Delta\varphi_0(x) \right) = \varphi_1(x). \tag{2}$$

We denote the unit ball and the unit sphere by B^n and S^n respectively, and denote the area of S^n by ω_n, and define the operator M_n by

$$M_n[h](r;x) := \begin{cases} 2^{-1}\{h(x+r) + h(x-r)\} \quad \text{for } n = 1, \\ \dfrac{1}{\omega_n (n-1)!!} \partial_r \left(\dfrac{1}{r}\partial_r \right)^{\frac{n-2}{2}} r^{n-1} \displaystyle\int_{B^n} \frac{h(x+ry)}{\sqrt{1-|y|^2}} dy \qquad \text{for } n = 2, 4, \cdots . \\ \dfrac{1}{\omega_{n-1}(n-2)!!} \partial_r \left(\dfrac{1}{r}\partial_r \right)^{\frac{n-3}{2}} r^{n-2} \displaystyle\int_{S^{n-1}} h(x+ry) dS_y \quad \text{for } n = 3, 5, \cdots . \end{cases}$$

Then we can get the following:

Theorem 1 *Assume that* $\varphi_0,\ \varphi_1 \in C^\infty(\mathbf{R}^n_x)$ *and* $f \in C^\infty([0,\infty)\times\mathbf{R}^n_x)$, *and that with some* $\varepsilon > 0$ *one has for* α, $|\alpha| \le [(n+1)/2]$

$$|\partial_x^\alpha f(t,x)| + |t\partial_t\partial_x^\alpha f(t,x)| \le C_\alpha t^{\varepsilon-2} \text{ for all } x\in\mathbf{R}^n_x \text{ and for all small } t>0.$$

Then, the solution $\psi = \psi(t,x)$ *to the problem (1) and (2) is represented by*

$$\psi(t,x)=\frac{1}{t}M_n[\varphi_0](3t^{1/3},x)-\frac{3}{t^{2/3}}\frac{\partial M_n[\varphi_0]}{\partial r}(3t^{1/3},x)+\frac{3}{2}\int_0^1(1-s^2)M_n[\varphi_1](3t^{1/3}s,x)ds$$
$$+\frac{3t^2}{2}\int_0^1\int_0^{1-b^{1/3}} b(1+b^{2/3}-s^2)M_n[f(\cdot,tb)](3t^{1/3}s;x)dsdb.$$

Remark: The original equation in Einstein & de Sitter spacetime is considered only for the problem in the spatial dimension $n=3$.

In fact, by putting $u(t,x) = t\psi(t,x)$, the problem (1) and (2) can be reduced to the problem

$$\partial_t^2 u - t^{-\frac{4}{3}}\Delta u = tf(t,x)$$

and

$$\lim_{t\to 0} u(t,x)=\varphi_0(x),\ \lim_{t\to 0}\left(\partial_t u(t,x)+3t^{-\frac{1}{3}}\Delta\varphi_0(x)\right)=\varphi_1(x).$$

We solve this problem for the proof of Theorem 1.

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Reconstruction of penetrable obstacles in acoustics

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Let D be an unknown obstacle with an unknown index of refraction subset of a larger domain Ω with a homogeneous index of refraction. Assume that D is penetrable. We send an acoustic wave from the boundary of Ω. Suppose that we are given all possible Cauchy data or the Dirichlet-to-Neumann measured on $\partial\Omega$. The inverse problem we consider in this talk is to determine the shape of D using the boundary measurements.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^2-boundary. Let $D \Subset \Omega$ be an open set with C^2-boundary. We assume $\gamma_D \in C^2(\overline{D})$ satisfies $\gamma_D \geq c_\gamma$ for some positive constant c_γ. We define $\widetilde{\gamma} := 1 + \gamma_D \chi_D$, where χ_D is the characteristic function of D. Let $k > 0$. We consider the Dirichlet problem

$$\begin{cases} \nabla \cdot (\widetilde{\gamma} \nabla v) + k^2 v = 0 \text{ in } \Omega, \\ v = f \text{ on } \partial\Omega. \end{cases} \tag{$\sharp$}$$

We remark that when $D = \emptyset$ (that is, $\widetilde{\gamma} \equiv 1$ on Ω) the problem ($\sharp$) is the following boundary value problem for the Helmholtz equation:

$$\begin{cases} \Delta v_0 + k^2 v_0 = 0 \text{ in } \Omega, \\ v_0 = f \text{ on } \partial\Omega. \end{cases} \tag{$\natural$}$$

We now assume that k^2 is neither a Dirichlet eigenvalue of the operator $-\nabla \cdot (\widetilde{\gamma} \nabla \bullet)$ in Ω nor it of the operator $-\Delta$ in Ω. Then we can define the Dirichlet-to-Neumann (DN) maps $\Lambda_D, \Lambda_\emptyset : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ as

$$\Lambda_D f := \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} (v \text{ is the solution to } (\sharp)), \quad \Lambda_\emptyset f := \left.\frac{\partial v_0}{\partial \nu}\right|_{\partial\Omega} (v_0 \text{ is the solution to } (\natural))$$

since ($\sharp$) and ($\natural$) have the unique solution, respectively. We remark that the operator $\Lambda_\emptyset$ is known data because it is determined only by Ω and k. The operator Λ_D corresponds to observation data on $\partial\Omega$. On the other hand, D and γ_D are unknown data. We consider the inverse problem that one reconstructs D from the DN map Λ_D. In this talk, we try to reconstruct D by the enclosure method with complex geometrical optics (CGO) solutions having polynomial-type phase functions for the Helmholtz equation.

The reconstruction procedures by the enclosure method for the case of *impenetrable* obstacles are given by Ikehata [I] and Nakamura-Yoshida [NY]. However, we need more precise analysis in the case c etrable obstacles. Indeed, although the coefficient of the equation is smooth in the case of impenetrable obstacles, it only has piecewise smoothness in the case of penetrable obstacles. Hence we need to analyze more precisely, for example apply the result of Li-Vogelius [LV] and so on.

We now state CGO solutions we use. We construct them by using the idea of [UW] (which gives CGO solutions for the Laplace equation) and the Vekua transformation. We first fix a number $c_* \in \mathbb{C}$ with $|c_*| = 1$ and a point $x_* = (x_{*1}, x_{*2}) \in \mathbb{R}^2 \setminus \overline{\Omega}$, and choose $\eta(x) := c_*\big((x_1 - x_{*1}) + i(x_2 - x_{*2})\big)^N$ as a phase function. We may assume $x_* = 0$ without loss of generality by a parallel translation. We choose an open cone Γ with the cone point $x_*(= 0)$ and the opening angle π/N satisfying $\mathrm{Re}\,\eta(x) > 0$ for all $x \in \Gamma$. We denote the Vekua transformation by T_k (see (13.9) on page 58 in [V]):

$$T_k V(x) := V(x) - k|x| \int_0^1 V\big((1 - s^2)x\big)\, J_1\big(k|x|s\big)\, ds,$$

where J_m is the Bessel function of the first kind of order m. It is known that the Vekua transformation T_k maps harmonic functions to solutions to the Helmholtz equation. We define $V_h^\sharp(x) := T_k(\exp(\eta(\cdot)/h))(x)$. We want to use the following functions $V_{t,h}(x)$ as CGO solutions:

$$\begin{aligned} V_{t,h}(x) &:= \phi_t(x) \exp\Big(-\frac{1}{ht}\Big) V_h^\sharp(x) \\ &= \exp\left(\frac{1}{h}\Big[-\frac{1}{t} + \eta(x)\Big]\right) \big(\phi_t(x) + O(h)\big) \quad (h \to 0), \end{aligned}$$

where $\phi_t(x)$ is the suitable cut-off function which is identically equal to zero outside of Γ. Unfortunately, the function $V_{t,h}$ itself does not satisfy the Helmholtz equation at the hands of the cut-off function. Then we define $f_{t,h} := V_{t,h}|_{\partial\Omega}$, and consider the solution $v_{0,t,h} = v_0$ to the Dirichlet problem ($\natural$) with the Dirichlet data $f_{t,h}$. The CGO solution we use here is $v_{0,t,h}$.

Now we define the indicator function $E(t, h)$ by

$$E(t, h) := \int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f_{t,h} \overline{f}_{t,h}\, d\sigma \qquad \text{for } t > 0,\ h > 0$$

with the CGO solution $v_{0,t,h}$ (We remark $v_{0,t,h}|_{\partial\Omega} = f_{t,h}$). We remark that $E(t, h)$ is determined only by known data and observation data. On the other hand, we define Θ_D by

$$\Theta_D := \sup_{x \in D \cap \Gamma} \mathrm{Re}\,\eta(x)$$

(see Figure). This corresponds to a "support function". We remark that Θ_D is determined by unknown data D. However, **if we can reconstruct $\boldsymbol{\Theta_D}$ from the observation data (or $\boldsymbol{E(t, h)}$)** then we can obtain some information about the shape of D. In particular, for example if we know a priori that D is star-shaped then the shape of D is completely reconstructed. Briefly speaking, our result is

$$\lim_{h \to 0} E(t, h) = \begin{cases} 0 & \text{if } \dfrac{1}{t} > \Theta_D, \\ +\infty & \text{if } \dfrac{1}{t} \le \Theta_D, \end{cases}$$

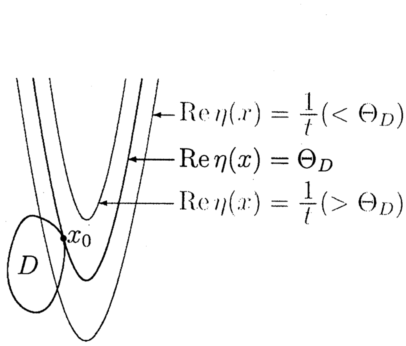

Figure.

in particular,

$$\Theta_D = \frac{1}{\sup\left\{t > 0 : \lim_{h \to 0} E(t,h) = 0\right\}}.$$

Hence **we can reconstruct Θ_D from observation data**. The detail is as follows:

Theorem. *Let $D \cap \Gamma \neq \emptyset$. We assume that the set $\{x \in \Gamma : \mathrm{Re}\,\eta(x) = \Theta_D\} \cap \partial D$ consists only of one point x_0, and the "relative curvature" to $\eta_R(x) = \Theta_D$ of ∂D at x_0 does not vanish. Then the following holds:*

- *For $1/t > \Theta_D$, there exist $C, c_1 > 0$ such that $|E(t,h)| \leq Ce^{-c_1/h}$.*
- *For $1/t = \Theta_D$, there exists $c > 0$ such that $E(t,h) \geq ch^{-1/2}$.*
- *For $1/t < \Theta_D$, there exist $c, c_2 > 0$ such that $E(t,h) \geq ce^{c_2/h}$.*

We omit to state the definition of "relative curvature" here. However we remark that the curvature assumption is always satisfied as long as N is large enough. Hence we can say that the curvature assumption is minor.

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Existence of classical solutions near characteristic points of first order nonlinear partial differential equations in real domains

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Let

$$F(x, u, u_x) = 0 \tag{1}$$

be a nonlinear partial differential equation in a neighbourhood of $\Omega \subset \mathbb{R}^d$ of $x = 0$, where $F(x, u, p)$ is a real-valued smooth function in a neighborhood of $(x, u, p) = (0, 0, 0) \in \mathbb{R}^{2d+1}$ such that $F(0, 0, 0) = 0$.

If $F_{p_i}(0, 0, 0) \neq 0$ for some i, then it follows from the theory of first order partial differential equations that we have solutions by solving noncharacteristic Cauchy problem with the method of characteristic (see [1], [4]).

If

$$F_{p_i}(0, 0, 0) = 0 \quad \text{for } \ i = 1, 2, \cdots, d, \tag{2}$$

the above theory is not applicable. If $F(x, u, p)$ is analytic, the existence of analytic solutions is studied under the condition (2), for example [2], [3] and [5]. In these papers firstly they construct a solutions of formal power series and next show the convergence under a condition, so called, Poincaré's condition.

However if $F(x, u, p)$ is *not analytic*, it seems that general existence results are not known. The purpose of this lecture is that we construct a solution $u(x)$ of $F(x, u, u_x) = 0$ in a neighborhood of $x = 0$ with $u(0) = 0$ and $u_{x_i}(0) = 0$ $(1 \leq i \leq d)$ under condition (2).

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A family of K3 surfaces and a GKZ differential equation induced from a Fano polytope

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0 Overview

In this talk, we deal with a relationship between a family of elliptic K3 surfaces and a GKZ hypergeometric differential equation which are both induced from a Fano polytope. They are connected each other in the sense that the periods of the former constitute the basis of the solution space of the latter and they share the monodromy group which is isomorphic to the Hilbert modular group for $\sqrt{2}$. We will give an elliptic fibration to the family of K3 surfaces. In the generic case, a member surface of the family has singular fibres consisting of one of the type IV^* ,one of the type I_{10}, and six of the type I_1, following the Kodaira's notation. Confluences of the singular fibers occur for the special subfamilies, which correspond to the singular loci of the GKZ differential equation. The main result is that the monodromies for the closed paths around some certain singular loci generate the entire monodromy group. As a tool to analyse the monodromy, we will use a numerical computation for the monodromy representation relative to a certain basis.
As a starting point, the family of K3 surfaces and the GKZ differential equation are induced below from the Fano polytope. The rest of the story will be told in the talk.

1 A family of K3 surfaces

1.1 Polytope P_3

We first derive a family of surfaces from a Fano polytope.

Definition 1.1. *A three dimensional polytope in $\mathbb{R}^3$ is a Fano polytope if and only if*

1. $(0,0,0)$ *is an interior point.*
2. *any vertex is in $\mathbb{Z}^3$.*
3. *any face is a triangle and its three vertices generate $\mathbb{Z}^3$.*

It is known that there are 18 Fano polytopes upto isomorphism. Out of those, we pick up a 5 vertexed Fano polytope

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix},$$

where each column represents a vertex.
We enlarge P_3 by adding a column and a row as follows.

$$A = \left[\begin{array}{c|c} 1 & 1\ldots 1 \\ \hline 0 & \\ 0 & P_3 \\ 0 & \end{array}\right]. \tag{1.1}$$

Let the elements of the first row of A represent 6 parameters $x_1, \ldots, x_6$ with exponent 1. Let the second through forth elements of each column of A represent the terms t_1, t_2, t_3 in this order with the exponent being the value of the elements. Thus each column of A represent the product of the parameter and the terms corresponding to its elements. Following these rules, we derive the equation with parameters $x_1, \ldots, x_6$:

$$x_1 + x_2t_1 + x_3t_2 + x_4t_3 + x_5t_3^{-1} + x_6t_1^{-1}t_2^{-1}t_3^{-1} = 0.$$

Multiplying $t_1t_2t_3$, we have

$$t_1t_2t_3(x_1 + x_2t_1 + x_3t_2 + x_4t_3) + x_5t_1t_2 + x_6 = 0.$$

Put $x = x_2t_1, y = x_3t_2$, $z = x_4t_3$. We get

$$\frac{1}{x_2x_3x_4}xyz(x_1 + x + y + z) + \frac{x_5}{x_2x_3}xy + x_6 = 0.$$

Put

$$\lambda = \frac{x_4x_5}{{x_1}^2}, \mu = \frac{x_2x_3x_4x_6}{{x_1}^4}. \tag{1.2}$$

Then the equation is reduced to the following which is an affine representation of a K3 surface $S_{\lambda,\mu}$;

$$xyz(x + y + z + 1) + \lambda xy + \mu = 0, \text{with} \quad \lambda, \mu \in \mathbb{C}. \tag{1.3}$$

So we obtain a two parameter family of K3 surfaces $\mathcal{F} = \{S_{\lambda,\mu}\}$.

2 A GKZ-hypergeometric differential equation

In addition to the K3 surface family $\mathcal{F}$, the matrix A of (1.1) equipped with a column vector $\beta \in \mathbb{Z}^4$ as a parameter induces a GKZ-hypergeometric differential equation. We write up A again and fix β as follows:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix}, \quad \beta = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

In order to determine the differential equation from this data, we need the following preparations; Let the columns of A correspond to 6 variables $x_1, \ldots, x_6$ of the differential equation. $\mathbb{Z}$-solutions of $A\nu = 0$ are

$$\nu(l,n) = {}^t(\nu_1(l,n), \ldots, \nu_6(l,n)) = l\,{}^t(-2,0,0,1,1,0) + n\,{}^t(-4,1,1,1,0,1) \quad l,n \in \mathbb{Z}.$$

Let $I(l,n) = \{i \in \mathbb{Z} \mid 1 \leq i \leq 6, \nu_i(l,n) > 0\}$ and $J(l,n) = \{j \in \mathbb{Z} \mid 1 \leq j \leq 6, \nu_j(l,n) < 0\}$. Let $\nu_0 = {}^t(-1,0,0,0,0,0)$ which is a solution of $A\nu_0 = \beta$. Let $\theta_{x_i} = \frac{\partial}{\partial x_i}(i = 1, \ldots, 6)$, the Euler operators.

Then the GKZ-hypergeometric differential equation consists of the following two systems;

1st system of equations:

$$\begin{aligned}&\left(\sum_{i=1}^{6}\theta_{x_i}\right)\varphi=-\varphi,\\&(\theta_{x_2}-\theta_{x_6})\varphi=0,\\&(\theta_{x_3}-\theta_{x_6})\varphi=0,\\&(\theta_{x_4}-\theta_{x_5}-\theta_{x_6})\varphi=0.\end{aligned}\tag{2.1}$$

2nd system of equations:

$$\prod_{i\in I(l,n)}\left(\frac{\partial}{\partial x_i}\right)^{\nu_i(l,n)}\varphi=\prod_{j\in J(l,n)}\left(\frac{\partial}{\partial x_j}\right)^{-\nu_j(l,n)}\varphi\quad\text{for all } l,n\in\mathbb{Z}.\tag{2.2}$$

The 1st system (2.1) is equivalent to the claim that φ is of the form

$$\varphi=\frac{1}{x_1}\tilde{\varphi}(\lambda,\mu),\qquad\text{where }\lambda=\frac{x_4x_5}{x_1^2},\quad\mu=\frac{x_2x_3x_4x_6}{x_1^4}.\tag{2.3}$$

Note that these λ,μ correspond to λ,μ in (1.2). Thus we have

$$\begin{aligned}&\theta_{x_5}=\theta_\lambda,\qquad\theta_{x_6}=\theta_\mu,\\&\theta_{x_2}=-\theta_{x_6}=\theta_\mu,\qquad\theta_{x_3}=\theta_{x_6}=\theta_\mu,\qquad\theta_{x_4}=\theta_{x_5}+\theta_{x_6}=\theta_\lambda+\theta_\mu,\\&\theta_{x_1}=-(\theta_{x_2}+\ldots+\theta_{x_6})-1=-2\theta_\lambda-4\theta_\mu-1.\end{aligned}\tag{2.4}$$

The 2nd system (2.2) is reduced to

$$\begin{aligned}\frac{\partial^2\varphi}{\partial x_4\partial x_5}&=\frac{\partial^2\varphi}{\partial x_1^2},\\\frac{\partial^3\varphi}{\partial x_2\partial x_3\partial x_6}&=\frac{\partial^3\varphi}{\partial x_4\partial x_5^2}.\end{aligned}\tag{2.5}$$

From (2.4) we have

$$\begin{aligned}\frac{\partial^2}{\partial x_4\partial x_5}&=\frac{1}{x_4}\theta_{x_4}\frac{1}{x_5}\theta_{x_5}=\frac{1}{x_4x_5}(\theta_\lambda+\theta_\mu)\theta_\lambda,\\\frac{\partial^2}{\partial x_1^2}&=\frac{1}{x_1}\theta_{x_1}\frac{1}{x_1}\theta_{x_1}=\frac{1}{x_1^2}\theta_{x_1}(\theta_{x_1}-1)=\frac{1}{x_1^2}(2\theta_\lambda+4\theta_\mu+1)(2\theta_\lambda+4\theta_\mu+2),\\\frac{\partial^3}{\partial x_2\partial x_3\partial x_6}&=\frac{1}{x_2x_3x_6}\theta_{x_2}\theta_{x_3}\theta_{x_6}=\frac{1}{x_2x_3x_6}\theta_\mu^3=\frac{x_4}{x_1^4}\frac{1}{\mu}\theta_\mu^3,\\\frac{\partial^3}{\partial x_4\partial x_5^2}&=\frac{1}{x_4x_5}\theta_{x_4}\theta_{x_5}\frac{1}{x_5}\theta_{x_5}=\frac{1}{x_4x_5^2}\theta_{x_4}\theta_{x_5}(\theta_{x_5}-1)=\frac{1}{x_4x_5^2}(\theta_\lambda+\theta_\mu)\theta_\lambda(\theta_\lambda-1).\end{aligned}$$

Hence we can reduce the number of the variables and rewrite the GKZ-hypergeometric differential equation.

Let L_1,L_2 be differential operators as

$$\begin{aligned}L_1&=\lambda(2\theta_\lambda+4\theta_\mu+1)(2\theta_\lambda+4\theta_\mu+2)-\theta_\lambda(\theta_\lambda+\theta_\mu),\\L_2&=\lambda^2\theta_\mu^3-\mu(\theta_\lambda+\theta_\mu)\theta_\lambda(\theta_\lambda-1).\end{aligned}\tag{2.6}$$

Then the equations (2.1),(2.2) are equivalent to

$$L_1\varphi=0,\quad L_2\varphi=0.\tag{2.7}$$

Period differential equations for families of $K3$ surfaces derived from some reflexive polytopes

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The concept of reflexive polytope is introduced by Batyrev ([B]). Among them three dimensional ones are listed by Ohtsuka ([O]). We have the following 5 three dimensional reflexive polytopes with 5 verteces having at most terminal singularities:

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}, P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}, P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix},$$

$$P_5 = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}, P_r = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}.$$

By using the vertices like a Newton polygon, we obtain families of $K3$ surfaces with 2 complex parameters from each of them. For P_3 there is a detailed study by Ishige ([I]).

For the four rest cases we obtained:

(1) power series expressions of a period of the families of $K3$ surfaces,
(2) a system of defferential equations for these periods,
(3) the Pfaffian systems corresponding to those systems and their singular loci, and that
(4) all these Pfaffians are integrable with 4-dimensional space of solutions.

Remark 0.1. *Recently Takayama and Nakayama ([T-N]) have made a computer calculation for the Groebner basis to get the system of differential equations for 3-dimentional Fano polytopes with at most 6 verteces.*

Remark 0.2. *We expect that we shall be able to define the lattice structure of our $K3$ surfaces, the period domain, the monodromy group. We restrict our attention to the aspect of differential equation, and omit all these subsequent arguments in our talk.*

Case P_4

By using the vertices and the origin as a Newton polygon, we obtain a family of affine complex algebraic surfaces with parameters $a_1, \cdots, a_6$:

$$xyz^2(a_1x + a_2y + a_3z + a_4 + a_5z^{-1} + a_6x^{-1}y^{-1}z^{-2}) = 0.$$

By choosing only effective parameters, we obtain the following family $\mathcal{F}_4$ with 2 parameters λ, μ:

$$S(\lambda, \mu) : xyz^2(x + y + z + 1) + \lambda xyz + \mu = 0. \tag{0.1}$$

For a generic parameters (λ, μ), the minmal compact nonsingular model is a $K3$ surface. So we have unique (up to a constant factor) holomorphic 2-form on it. That is given by

$$\omega = \frac{zdz \wedge dx}{\partial F / \partial y}, \tag{0.2}$$

where $F = xyz^2(x+y+z+1) + \lambda xyz + \mu$ and F_y means the partial derivative for y. The period of $S(\lambda,\mu)$ is obatined by the integration of ω on some 2-cycle. For sufficiently small λ, μ and ε, we have a lifting of a real torus $\{|x| = \varepsilon\} \times \{|z| = \varepsilon\}$ to a 2-cycle Γ near the origin on $S(\lambda,\mu)$.

We have

$$\eta(\lambda,\mu) = \int_\Gamma \omega = (2\pi i)^2 \sum_{n,m=0}^{\infty} (-1)^m \frac{(5m+2n)!}{n!(m!)^3(2m+n)!}\lambda^n\mu^m. \tag{0.3}$$

Set

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -2 \end{pmatrix}, \beta = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

By this data we obtain a GKZ hypergeometric differential equation $L_1 = L_2 = 0$ with

$$\begin{cases} L_1 &= \theta_\lambda(\theta_\lambda + 2\theta_\mu) - \lambda(2\theta_\lambda + 5\theta_\mu + 1)(2\theta_\lambda + 5\theta_\mu + 2), \\ L_2 &= \lambda^2\theta_\mu^3 + \mu\theta_\lambda(\theta_\lambda - 1)(2\theta_\lambda + 5\theta_\mu + 1), \end{cases} \tag{0.4}$$

where $\theta_\lambda = \lambda\frac{\partial}{\partial\lambda}, \theta_\mu = \mu\frac{\partial}{\partial\mu}$ (see [H]), and $\eta(\lambda,\mu)$ satisfies this system. It has a 6-dimensional solution space, and it is not irreducible. By some geometric observation we expect η satisfies a 4-dimensional system. By a direct calculation we obtain the system $L_1 = L_3 = 0$ for η with

$$\begin{cases} L_1 &= \theta_\lambda(\theta_\lambda + 2\theta_\mu) - \lambda(2\theta_\lambda + 5\theta_\mu + 1)(2\theta_\lambda + 5\theta_\mu + 2), \\ L_3 &= \lambda^2(4\theta_\lambda^2 - 2\theta_\lambda\theta_\mu + 5\theta_\mu^2) - 8\lambda^3(1 + 3\theta_\lambda + 5\theta_\mu + 2\theta_\lambda^2 + 5\theta_\lambda\theta_\mu) + 25\mu\theta_\lambda(\theta_\lambda - 1). \end{cases} \tag{0.5}$$

We can transform the system (0.5) to an explicit Pfaffian system, and we know that satisfies the integrability condition. So we can say that the system (0.5) has a 4-dimensional solution space as expected. According to this calculation we get the singular locus of the system (0.5):

$$\lambda\mu(\lambda^2(4\lambda - 1)^3 - 2(2 + 25\lambda(20\lambda - 1))\mu - 3125\mu^2) = 0. \tag{0.6}$$

Note that it is composed of three rational curves.

The other cases

We can make similar calculations for the rest of our list. We give only the results.

(i) For P_2, we have the family of $K3$ surfaces: $\mathcal{F}_2 : xyz(x+y+z+1) + \lambda z + \mu xy = 0$ and the holomorphic 2-form $\omega = \dfrac{dz \wedge dx}{\partial F_2/\partial y}$, $(F_2 = xyz(x+y+z+1) + \lambda z + \mu xy)$. The period of a 2-cycle Γ near the origin is given by

$$\int_\Gamma \omega = (2\pi i)^2 \sum_{n,m=0}^{\infty} (-1)^n \frac{(3m+2n)!}{(m!)^2(n!)^3}\lambda^n\mu^m.$$

The corresponding GKZ equation $L_1 = L_2 = 0$ is given by

$$\begin{cases} L_1 &= \theta_\lambda^2 - \mu(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2), \\ L_2 &= \theta_\lambda^3 + \lambda(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2)(3\theta_\lambda + 2\theta_\mu + 3). \end{cases} \tag{0.7}$$

The 4-dimensional system $L_1 = L_3 = 0$ is given by

$$\begin{cases} L_1 &= \theta_\lambda^2 - \mu(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2), \\ L_3 &= \theta_\lambda(3\theta_\lambda - 2\theta_\mu) + 9\lambda(3\theta_\lambda + 2\theta_\mu + 1)(3\theta_\lambda + 2\theta_\mu + 2) + 4\mu\theta_\lambda(3\theta_\lambda + 2\theta_\mu + 1). \end{cases} \tag{0.8}$$

The singular locus is given by $\lambda\mu(729\lambda^2 - (4\mu - 1)^3 + 54\lambda(1 + 12\mu)) = 0$.

(ii)For P_5, we have the family of $K3$ surfaces: $\mathcal{F}_5 : xyz(x+y+z+1)+\lambda z+\mu = 0$ and the holomorphic 2-form $\omega = \dfrac{dz\wedge dx}{\partial F_5/\partial y}$, $(F_5 = xyz(x+y+z+1)+\lambda z+\mu)$. The period of a 2-cycle Γ near the origin is given by

$$\int_\Gamma \omega = (2\pi i)^2 \sum_{n,m=0}^{\infty} (-1)^n \frac{(4m+3n)!}{(m!)^2 n!((m+n)!)^2}\lambda^n\mu^m$$

The corresponding GKZ equation $L_1 = L_2 = 0$ is given by

$$\begin{cases} L_1 &= \lambda\theta_\mu^2 + \mu\theta_\lambda(3\theta_\lambda+4\theta_\mu+1),\\ L_2 &= \theta_\lambda(\theta_\lambda+\theta_\mu)^2 + \lambda(3\theta_\lambda+4\theta_\mu+1)(3\theta_\lambda+4\theta_\mu+2)(3\theta_\lambda+4\theta_\mu+3).\end{cases} \tag{0.9}$$

The 4-dimensional system $L_1 = L_3 = 0$ is given by

$$\begin{cases} L_1 &= \lambda\theta_\mu^2 + \mu\theta_\lambda(3\theta_\lambda+4\theta_\mu+1),\\ L_3 &= \lambda\theta_\lambda(3\theta_\lambda+2\theta_\mu) + \mu\theta_\lambda(1-\theta_\lambda) + 9\lambda^2(3\theta_\lambda+4\theta_\mu+1)(3\theta_\lambda+4\theta_\mu+2).\end{cases} \tag{0.10}$$

The singular locus is given by $\lambda\mu(\lambda^2(1+27\lambda)^2 - 2\lambda\mu(1+189\lambda) + (1+576\lambda)\mu^2 - 256\mu^3) = 0$.

(iii)For the case P_r, we have the family $\mathcal{F}_r : xyz(x+y+z+1)+\lambda x+\mu y = 0$ and the holomorphic 2-form $\omega = \dfrac{dx\wedge dy}{\partial F_r/\partial z}$, ($F_r = xyz(x+y+z+1)+\lambda x+\mu y)$. The period of a 2-cycle Γ near the origin is given by

$$\eta(\lambda,\mu) = \int_\Gamma \omega = (2\pi i)^2 \sum \frac{(3m+3n)!}{(n!)^2(m!)^2(m+n)!}\lambda^n\mu^m. \tag{0.11}$$

In this case our period is reduced to the Appell hypergeometric function F_4 (see Koike [K]):

$$\eta(\lambda,\mu) = F_4(1/3,2/3,1,1;27\lambda,27\mu) = F(1/3,2/3,1;x)F(1/3,2/3,1,;y),$$

where F is Gauss hypergeometric function and $x(1-y) = 27\lambda, y(1-x) = 27\mu$.

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A notion of boundedness at infinity for univariate hyperfunctions

Yasunori OKADA (Chiba University) (岡田靖則)

Massera studied periodic solutions to periodic ordinary differential equations in several situations, and in the linear setting, gave the following result ([2, Theorem 4]).

Theorem 1. *Consider a system of ordinary differential equations*

$$\frac{d}{dt}x = a(t)x + b(t),$$

where $a(t)$ *and* $b(t)$ *are* $\mathbb{R}^{m\times m}$*-valued and* $\mathbb{R}^m$*-valued continuous functions. Assume that* $a(t)$ *and* $b(t)$ *are* 1*-periodic. Then the existence of a solution which is bounded in the future implies the existence of a* 1*-periodic solution.*

We were interested in the question whether there is a counterpart to this phenomenon in the framework of hyperfunctions. Since usual hyperfunctions admit no notion of inequality nor boundedness, our problem might be translated into

Problem. Can we introduce a notion of "boundedness at $+\infty$" into univariate hyperfunctions, admitting a Massera type theorem?

Note that Chung-Kim-Lee [1] constructed and studied the space $\mathcal{B}_{L^\infty}$ of bounded hyperfunctions in the several variables by duality, using the heat kernel method.

On the other hand, in order to give an answer to the problem, we introduce the sheaf $\mathscr{B}_{L^\infty}$ on $\mathbb{D}^1 := \mathbb{R} \sqcup \{\pm\infty\} = [-\infty, \infty]$ of univariate hyperfunctions bounded at infinity, (refer to [3]), in a similar manner to the original definitions of hyperfunctions and Fourier hyperfunctions in one-dimensional case given in Sato [4]. We also give the relation between $\mathcal{B}_{L^\infty}$ and $\mathscr{B}_{L^\infty}$.

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Microlocal analysis of a vortex sheet

hyperfunction = vortex layer
— *I. Imai*

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Vortex sheet is a 2D inviscid and incompressible fluid whose velocity field $\vec{u}(t,x,y) = (u(t,x,y), v(t,x,y))$ is discontinuous on a curve $\Gamma(t) = \{(x,y) \in \mathbf{R}^2 : \ y = f(t,x)\}$, which is called the interface. The time evolution of the interface is described by Birkoff-Rott equation. Althogh this equation has a long history, the corresponding Cauchy problem is not solved in a general framework. On the other hand, I. Imai [3] said that a hyperfunction is a vortex sheet, and we want to show that this idea is successfully applicable to solve Birkoff-Rott equation.

We assume that the velocity field $\vec{u}$ is incompressible everywhere, and is irrotational outside of $\Gamma(t)$. The velocity field may be discontinuous on $\Gamma(t)$, and its vorticity is concentrated there. Let $f(t,x)$ be the above function which defines $\Gamma(t)$, and let $g(t,x)$ be a function which indicates the strength of the vorticity at each point $(x, f(t,x)) \in \Gamma(t)$. Then they satisfy the following Birkoff-Rott equation:

$$f_t = -U f_x + V, \tag{1}$$

$$g_t = -(Ug)_x. \tag{2}$$

Here $U(t,x)$, $V(t,x)$ are determined by

$$\begin{aligned}
&U(t,x) - \sqrt{-1}V(t,x) \\
=&\frac{\sqrt{-1}}{2\pi}\int_{-\infty}^{+\infty}\frac{g(t,x')}{x - x' + \sqrt{-1}f(t,x) - \sqrt{-1}f(t,x')}dx' \\
=&\frac{\sqrt{-1}}{2\pi}\lim_{\substack{\varepsilon\to+0\\ R\to+\infty}}\left(\int_{x-R}^{x-\varepsilon} + \int_{x+\varepsilon}^{x+R}\right)
\end{aligned} \tag{3}$$

This is a singular integral which one usually regards as determined by f_x, g. Differentiating (1) by x, we have the following Cauchy problem for f_x and g:

$$
\begin{cases} f_{xt} = -U_x f_x - U f_{xx} + V_x, \\ g_t = -(Ug)_x, \\ f_x(0,x) = f_*(x). \end{cases} \tag{4}
$$

The difficulty arises from the integration (3). R. E. Caflisch, O. F. Orellana [1] and C. Sulem, P. L. Sulem, C. Bardos, U. Frisch [4] considered the case of an analytic f_*. J. Duchon, R. Robert [2] considered the case of a small f_*. S. Wu [5] considered the case of á Sobolev function, but she could not assign the complete value of $f_x(0,x)$. We want to show that if f_* is periodic and has Hölder continuity, then (4) has a solutuion.

Let $\omega = [0,T] \times \mathbf{R}$ for $T > 0$. We define

$$
\mathcal{F}^p(\omega) = \{h(t,x) \in C^0(\omega);\ h(t, x+2\pi) = h(t,x),\ ||h||_p < \infty\},
$$

where

$$
||h||_p = \sup_{(t,x)\in\omega} |h(t,x)| + \sup_{(t,x_1,x_2)\in\Delta} \frac{|h(t,x_1) - h(t,x_2)|}{|x_1 - x_2|^p},
$$

$$
\Delta = \{(t, x_1, x_2) \in [0,T] \times \mathbf{R}^2;\ x_1 \neq x_2\}.
$$

We also define $\mathcal{F}^{p+1}(\omega) = \{h(t,x) \in \mathcal{F}^p(\omega);\ h_x(t,x) \in \mathcal{F}^p(\omega)\}$. Then we have the following

Theorem. *If $f_*(x) \in \mathcal{F}^{p+1}(\omega)$ satisfies $\int_{-\pi}^{\pi} f_*(x)dx = 0$ and $T > 0$ is small enough, then there exist $f_x, g \in \mathcal{F}^{p+1}(\omega)$ satifying $f_{xt}, g_t \in \mathcal{F}^p(\omega)$ and (4).*

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Geometric properties and decay estimates in crystal theory in the nearly cubic case (Results in collaboration with C.Melotti)

Otto Liess

We study algebraic and geometric properties of the slowness surface of the system of crystal optics for tetragonal crystals in the nearly cubic case. A first group of results refers to the location and nature of the singular points of the surfaces under consideration, whereas other results relate to curvature properties of these surfaces. It is only for these curvature properties that we need to remain in the nearly cubic case.

These results are needed in the study of the asymptotic behaviour at infinity of the solutions of the system. We also study the general form of quartic and sextic surfaces of "slowness type" which are quadratic in their variables and compare them with the quartics and sextics which appear in the theory of crystal optics and of crystal acoustics for cubic and tetragonal crystals.

We finally discuss optimality of the decay estimates in the case of an example of a system which has a number of features which relates it to crystal theory, but is simpler than the system of crystal acoustics. Since decay estimates are more robust for first order systems than for scalar equations, we shall apply a result on a conjecture of P.Lax on the situations under which a scalar constant coefficient linear partial differential operator is the determinant of a first order symmetric hyperbolic system.

Phase space Feynman path integrals via piecewise bicharacteristic paths and their semiclassical approximations[1]

Naoto Kumano-go [2] (Kogakuin University)

Let $T > 0$ and $x \in \mathbf{R}^d$. Let $U(T,0)$ be the fundamental solution for the Schrödinger equation

$$(i\hbar\partial_T - H(T,x,\frac{\hbar}{i}\partial_x))U(T,0) = 0\,, \quad U(0,0) = I \tag{1}$$

with the Planck parameter $0 < \hbar < 1$. By the Fourier transform, we can write

$$\begin{aligned} H(T,x,\frac{\hbar}{i}\partial_x)v(x) &= \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{R}^d} e^{ix\cdot\xi_0} H(T,x,\hbar\xi_0)\widehat{v}(\xi_0)d\xi_0 \\ &= \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbf{R}^{2d}} e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0} H(T,x,\xi_0)v(x_0)dx_0d\xi_0 \end{aligned} \tag{2}$$

with x_0, $\xi_0 \in \mathbf{R}^d$. Now we consider the function $U(T,0,x,\xi_0)$ satisfying

$$U(T,0)v(x) = \left(\frac{1}{2\pi\hbar}\right)^d \int_{\mathbf{R}^{2d}} e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0} U(T,0,x,\xi_0)v(x_0)dx_0d\xi_0\,. \tag{3}$$

Using the phase space path integral introduced by R. P. Feynman, we formally write

$$e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0) = \int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q,p]\,. \tag{4}$$

Here $(q,p) : [0,T] \to \mathbf{R}^d \times \mathbf{R}^d$ are the paths with $q(0) = x_0$, $q(T) = x$ and $p(0) = \xi_0$ in the phase space, $\phi[q,p]$ is the action of Hamiltonian type defined by

$$\phi[q,p] = \int_{[0,T)} p(t)\cdot dq(t) - \int_{[0,T)} H(t,q(t),p(t))dt\,, \tag{5}$$

and the phase space path integral $\int \sim \mathcal{D}[q,p]$ is a sum over all the paths (q,p). However, in the sense of mathematics, the measure $\mathcal{D}[q,p]$ does not exist. Furthermore, in the uncertainty principle, we can not obtain the position $q(t)$ and the momentum $p(t)$ at the same time t.

In this talk, using piecewise bicharacteristic paths, we prove the existence of the phase space Feynman path integrals

$$\int e^{\frac{i}{\hbar}\phi[q,p]}F[q,p]\mathcal{D}[q,p]$$

with general functional $F[q,p]$ as integrand. More precisely, we give a fairly general class $\mathcal{F}$ such that for any $F[q,p] \in \mathcal{F}$, the time slicing approximation converges uniformly on compact subsets with respect to the final position x and the initial momentum ξ_0.

Assumption 1 *$H(t,x,\xi)$ is a real-valued function of $(t,x,\xi) \in \mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d$. For any multi-indices α, β, $\partial_x^\alpha\partial_\xi^\beta H(t,x,\xi)$ is continuous and $|\partial_x^\alpha\partial_\xi^\beta H(t,x,\xi)| \le C_{\alpha,\beta}(1+|x|+|\xi|)^{\max(2-|\alpha+\beta|,0)}$.*

[1] *Naoto Kumano-go and Daisuke Fujiwara, Bull. Sci. math. 132 (2008) 313-357.*

[2] supported by JSPS. KAKENHI(C)21540196

Let $\Delta_{T,0} = (T_{J+1}, T_J, \ldots, T_1, T_0)$ be any division of the interval $[0, T]$, i.e.,

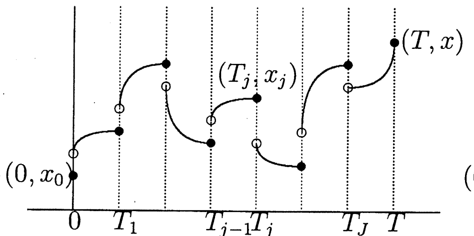
$$\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0 \,. \tag{6}$$

Let $t_j = T_j - T_{j-1}$ and $|\Delta_{T,0}| = \max_{1 \le j \le J+1} t_j$. Set $x_{J+1} = x$. Let $x_j \in \mathbf{R}^d$ and $\xi_j \in \mathbf{R}^d$ for $j = 1, 2, \ldots, J$. Assume that $\kappa_2 d |\Delta_{T,0}| < 1/2$. Then we can obtain the bicharacteristic paths $\bar{q}_{T_j,T_{j-1}} = \bar{q}_{T_j,T_{j-1}}(t, x_j, \xi_{j-1})$ and $\bar{p}_{T_j,T_{j-1}} = \bar{p}_{T_j,T_{j-1}}(t, x_j, \xi_{j-1})$, $T_{j-1} \le t \le T_j$ by Hamilton's canonical equation

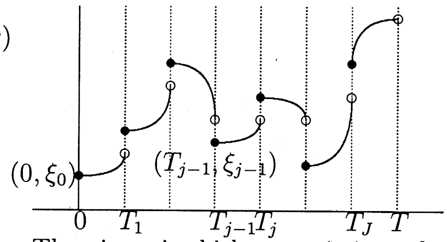
$$\begin{aligned} &\partial_t \bar{q}_{T_j,T_{j-1}}(t) = (\partial_\xi H)(t, \bar{q}_{T_j,T_{j-1}}, \bar{p}_{T_j,T_{j-1}}) \,, \\ &\partial_t \bar{p}_{T_j,T_{j-1}}(t) = -(\partial_x H)(t, \bar{q}_{T_j,T_{j-1}}, \bar{p}_{T_j,T_{j-1}}) \,, \quad T_{j-1} \le t \le T_j \end{aligned} \tag{7}$$

with $\bar{q}_{T_j,T_{j-1}}(T_j) = x_j$ and $\bar{p}_{T_j,T_{j-1}}(T_{j-1}) = \xi_{j-1}$. We define the piecewise bicharacteristic paths $q_{\Delta_{T,0}} = q_{\Delta_{T,0}}(t, x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0)$ and $p_{\Delta_{T,0}} = p_{\Delta_{T,0}}(t, x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0)$ by

$$\begin{aligned} &q_{\Delta_{T,0}}(t) = \bar{q}_{T_j,T_{j-1}}(t, x_j, \xi_{j-1}) \,, \quad T_{j-1} < t \le T_j, \quad q_{\Delta_{T,0}}(0) = x_0 \,, \\ &p_{\Delta_{T,0}}(t) = \bar{p}_{T_j,T_{j-1}}(t, x_j, \xi_{j-1}) \,, \quad T_{j-1} \le t < T_j, \quad j = 1, 2, \ldots, J, J+1 \,. \end{aligned} \tag{8}$$



The piecewise bicharacteristic path $q_{\Delta_{T,0}}$



The piecewise bicharacteristic path $p_{\Delta_{T,0}}$

Then the functionals $\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$, $F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ become functions, i.e.,

$$\begin{aligned} \phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] &= \phi_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0) \,, \\ F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] &= F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0) \,. \end{aligned} \tag{9}$$

Definition 1 *We say that $F[q,p] \in \mathcal{F}$ if $F_{\Delta_{T,0}} = F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ satisfies Assumption 2.*

Assumption 2 *Let $m \ge 0$. Let $u_j \ge 0$, $j = 1, 2, \ldots, J, J+1$ are non-negative parameters depending on the division $\Delta_{T,0}$ such that $\sum_{j=1}^{J+1} u_j \equiv U < \infty$. For any integer $M \ge 0$, there exist positive constants A_M, X_M such that for any $\Delta_{T,0}$, any multi-indices α_j, β_{j-1} with $|\alpha_j|$, $|\beta_{j-1}| \le M$, $j = 1, 2, \ldots, J, J+1$ and any $1 \le k \le J$,*

$$\begin{aligned} &|(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}}) F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0)| \\ &\le A_M (X_M)^{J+1} (\prod_{j=1}^{J+1} (t_j)^{\min(|\beta_{j-1}|,1)})(1 + \sum_{j=1}^{J+1} (|x_j| + |\xi_{j-1}|) + |x_0|)^m \,, \end{aligned} \tag{10}$$

$$|(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j}\partial_{\xi_{j-1}}^{\beta_{j-1}})\partial_{x_k}F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\dots,\xi_1,x_1,\xi_0,x_0)|$$

$$\leq A_M(X_M)^{J+1}u_k(\prod_{j\neq k}(t_j)^{\min(|\beta_{j-1}|,1)})(1+\sum_{j=1}^{J+1}(|x_j|+|\xi_{j-1}|)+|x_0|)^m. \tag{11}$$

Theorem 1 *Let T be sufficiently small. Then, for any $F[q,p]\in\mathcal{F}$,*

$$\int e^{\frac{i}{\hbar}\phi[q,p]}F[q,p]\mathcal{D}[q,p] \equiv \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]}F[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]\prod_{j=1}^{J} dx_j d\xi_j \tag{12}$$

converges uniformly on compact sets of $\mathbf{R}^{3d}$ with respect to (x,ξ_0,x_0), i.e., (12) is well-defined.

Remark Even when $F[q,p]\equiv 1$, each integral of the right-hand side of (12) does not converge absolutely. We treat integrals of this type as oscillatory integrals.

Example We give an example when $d=1$, $H(x,\xi)=x^2/2+\xi^2/2$ and $F[q,p]\equiv 1$:
Assume that $|T_j-T_{j-1}|<\pi/2$. By Hamilton's canonical equation $\partial_t\bar{q}_{T_j,T_{j-1}}(t)=\bar{p}_{T_j,T_{j-1}}(t)$, $\partial_t\bar{p}_{T_j,T_{j-1}}(t)=-\bar{q}_{T_j,T_{j-1}}(t)$, $T_{j-1}\leq t\leq T_j$ with $\bar{q}_{T_j,T_{j-1}}(T_j)=x_j$ and $\bar{p}_{T_j,T_{j-1}}(T_{j-1})=\xi_{j-1}$, we obtain the bicharacteristic paths

$$\bar{q}_{T_j,T_{j-1}}(t)=\frac{x_j\cos(t-T_{j-1})-\xi_{j-1}\sin(T_j-t)}{\cos(T_j-T_{j-1})},\ \bar{p}_{T_j,T_{j-1}}(t)=\frac{-x_j\sin(t-T_{j-1})+\xi_{j-1}\cos(T_j-t)}{\cos(T_j-T_{j-1})}.$$

Using the piecewise bicharacteristic paths $q_{\Delta_{T,0}}$ and $p_{\Delta_{T,0}}$, we can write the functional $\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]$ as the function

$$\phi_{\Delta_{T,0}}=\int_{[0,T)}p_{\Delta_{T,0}}\cdot dq_{\Delta_{T,0}}(t)-\int_{[0,T)}H(t,q_{\Delta_{T,0}},p_{\Delta_{T,0}})dt=\sum_{j=1}^{J+1}\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1}),$$

where

$$\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1})=-x_{j-1}\cdot\xi_{j-1}+\frac{2x_j\cdot\xi_{j-1}-(x_j^2+\xi_{j-1}^2)\sin(T_j-T_{j-1})}{2\cos(T_j-T_{j-1})}.$$

Performing the oscillatory integration with respect to (ξ_1,x_1) in (12), we have

$$\left(\frac{1}{2\pi\hbar}\right)\int_{\mathbf{R}^2}e^{\frac{i}{\hbar}\phi_{T,T_1}(x_2,\xi_1,x_1)+\frac{i}{\hbar}\phi_{T_1,0}(x_1,\xi_0,x_0)}dx_1d\xi_1=e^{\frac{i}{\hbar}\phi_{T,0}(x_2,\xi_0,x_0)}\left(\frac{\cos(T_2-T_1)\cos(T_1-T_0)}{\cos T}\right)^{1/2}.$$

Using the above relation inductively and taking $|\Delta_{T,0}|=\max_{1\leq j\leq J+1}|T_j-T_{j-1}|\to 0$ in (12), we get the function $U(T,0,x,\xi_0)$ of the fundamental solution for the Schrödinger equation (1).

$$e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)=\int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q,p]$$
$$\equiv\lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^J\int_{\mathbf{R}^{2J}}e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1})}\prod_{j=1}^{J}dx_jd\xi_j$$
$$=\lim_{|\Delta_{T,0}|\to 0}e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)}\left(\frac{\prod_{j=1}^{J+1}\cos(T_j-T_{j-1})}{\cos T}\right)^{1/2}$$
$$=\frac{1}{(\cos T)^{1/2}}\exp\frac{i}{\hbar}\Big(-x_0\cdot\xi_0+\frac{2x\cdot\xi_0-(x^2+\xi_0^2)\sin T}{2\cos T}\Big).\quad\square$$

On a Schrödinger operator with a merging pair of a simple pole and a simple turning point — WKB theoretic transformation to the canonical form

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We talk about the transformation of a Schrödinger equation with a merging pair of a simple pole and a simple turning point (an MPPT equation, for short) to its canonical form and some properties of its WKB solutions. This is a joint work with T. Kawai (Kyoto univ.), T. Koike (Kobe univ.) and Y. Takei (Kyoto univ.).

In [AKT] an MPT equation, i.e., a Schrödinger equation with a merging pair of two simple turning points, was discussed and some properties of its WKB solutions were explicitly described through a WKB theoretic transformation of an MPT equation to its canonical equation (the ∞-Weber equation in this case). On the other hand, as is shown in [Ko1] and [Ko2], a simple pole of the potential can be also thought of as a kind of turning points. Thus it is a natural problem to consider an MPPT equation, i.e., a Schrödinger equation with a merging pair of a simple pole and a simple turning point. The purpose of this talk is to discuss the WKB theoretic transformation of an MPPT equation to its canonical equation.

Let us first give the specific definition of an MPPT equation. An MPPT equation is a Schrödinger equation with a large parameter η of the following form:

$$(*) \qquad \left(\frac{d^2}{d\tilde{x}^2} - \eta^2\left(\frac{\tilde{Q}_0(\tilde{x},a)}{\tilde{x}} + \eta^{-1}\frac{\tilde{Q}_1(\tilde{x},a)}{\tilde{x}} + \eta^{-2}\frac{\tilde{Q}_2(\tilde{x},a)}{\tilde{x}^2}\right)\right)\tilde{\psi} = 0,$$

where $\tilde{Q}_j(\tilde{x},a)$ $(j = 0,1,2)$ are holomorphic near $(\tilde{x},a) = (0,0)$ and $\tilde{Q}_0(\tilde{x},a)$ satisfies the following conditions:

$$\tilde{Q}_0(0,a) \neq 0 \text{ if } a \neq 0$$
$$\tilde{Q}_0(\tilde{x},0) = c\tilde{x} + O(\tilde{x}^2) \text{ with } c \neq 0.$$

For such an MPPT equation $(*)$ we can construct a transformation of the form

$$x(\tilde{x},a,\eta) = \sum_{k=0}^{\infty} \eta^{-k} x_k(\tilde{x},a)$$

that transforms $(*)$ to its canonical equation (the ∞-Whittaker equation in this case)

$$(**) \qquad \left(\frac{d^2}{dx^2} - \eta^2\left(\frac{1}{4} + \frac{\alpha(a,\eta)}{x} + \eta^{-2}\frac{\tilde{Q}_2(0,a)}{x^2}\right)\right)\psi = 0$$

where $\alpha(a,\eta) = \sum_k \eta^{-k}\alpha_k(a)$ is an infinite series. To be more precise, $x(\tilde{x},a,\eta)$ and $\alpha(a,\eta)$ can be constructed so that they satisfy the following relation:

$$\frac{\tilde{Q}_0(\tilde{x},a)}{\tilde{x}} + \eta^{-1}\frac{\tilde{Q}_1(\tilde{x},a)}{\tilde{x}} + \eta^{-2}\frac{\tilde{Q}_2(\tilde{x},a)}{\tilde{x}^2} = \left(\frac{\partial x}{\partial \tilde{x}}\right)^2 \left(\frac{1}{4} + \frac{\alpha(a,\eta)}{x} + \eta^{-2}\frac{\tilde{Q}_2(0,a)}{x^2}\right) - \frac{1}{2}\eta^{-2}\{x;\tilde{x}\},$$

where $\{x;\tilde{x}\}$ stands for the Schwarzian derivative. Here we note that $x(\tilde{x},a,\eta)$ and $\alpha(a,\eta)$ are Borel transformable series in a neighborhood of $(\tilde{x},a) = (0,0)$ with coefficients depending holomorphically on a. Furthermore, by using $x(\tilde{x},a,\eta)$ and $\alpha(a,\eta)$, we can formally represent a WKB solution $\tilde{\psi}(\tilde{x},a,\eta)$ of $(*)$ with an appropriately chosen WKB solution $\psi(x,\eta,\alpha(a,\eta))$ of $(**)$ in a neighborhood of $(\tilde{x},a) = (0,0)$ as follows:

$$\tilde{\psi}(\tilde{x},a,\eta) = \left(\frac{\partial x}{\partial \tilde{x}}\right)^{1/2} \psi(x(\tilde{x},a,\eta),\eta,\alpha(a,\eta)).$$

Let L and M be the Borel transform of the operators that appear in $(*)$ and $(**)$ respectively, that is, the operators obtained by replacing η in $(*)$ and $(**)$ by ∂_y. Note that L and M are well-defined microdifferential operators since $\alpha(a,\eta)$ are Borel transformable. Then we can reinterpret the above transformation from the microdifferential viewpoint as follows (cf. [AY]): there exist invertible microdifferential operators $\mathcal{X}$ and $\mathcal{Y}$ that satisfy

$$L\mathcal{X} = \mathcal{Y}M.$$

The operators $\mathcal{X}$ and $\mathcal{Y}$ can be explicitly written in terms of $x(\tilde{x},a,\eta)$. Furthermore, we can describe the action of $\mathcal{X}$ upon the Borel transform $\tilde{\psi}_B(\tilde{x},a,y)$ as an integro-differential operator in a concrete manner, where the Borel transformability of $x(\tilde{x},a,\eta)$ is effectively used. The details will be explained in the talk.

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