A role of new Stokes curves in the quantized Héon map

Akira Shudo

Department of Physics, Tokyo Metropolitan University,
1-1 Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan
shudo@phys.metro-u.ac.jp

A discrete analog of the Feynman-type path integral,

$$< q_n|U^n|q_0 >= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dq_1 dq_2 \cdots dq_{n-1} \exp\Big[\frac{i}{\hbar} S(q_0, \cdots, q_n)\Big] \quad (1)$$

gives quantum dynamics of the symplectic map. Here, $S(q_0, \cdots, q_n)$ represents the discretized Lagrangian given as

$$S(q_0, \cdots, q_n) = \sum_{j=1}^{n} \frac{1}{2}(q_j - q_{j-1})^2 - \sum_{j=1}^{n-1} V(q_j). \quad (2)$$

The form of $S(q_0, \cdots, q_n)$is derived so that applying the variational principle generates the symplectic map. In fact, we can easily see that the condition $\partial S(q_0, \cdots, q_n)/\partial q_j = 0 \quad (1 \leq j \leq n-1)$ yields the classical map in the Lagrangian form. The simplest possible choice of the potential function would be,

$$V(q) = -\frac{q^3}{3} - cq, \quad (3)$$

where c denotes a real parameter controlling qualitative features of the underlying area-preserving map, and possibly the corresponding quantum dynamics. This system is related to the so-called *Hénon family* via an appropriate affine transformation with a change of parameter. There is a classification theorm claiming that non-trivial polynomial diffeomorphisms from $\mathbb{C}^2$ to $\mathbb{C}^2$ are written as composition of the Hénon map, and the others, which are either elementary or affine map, are the mappings easily analyzed [1]. For this reason, the Hénon map can be regarded as a simplest possible polynomial diffeomorphism creating non-trivial dynamics.

Here, we fix the initial coordinate q_0 and regard the quantum propagator $< q_n|U^n|q_0 >$ as a function of the final coordinate q_n. We then discuss Stokes geometry of $I(q_n) \equiv < q_n|U^n|q_0 >$ in applying the saddle point method. In order to analyze Stokes geometry where more than 2 solutions appear as saddle point contribution, we refer to the work by Aoki, Kawai and Takei, in which a prescription to analyze Stokes phenomena in higher-order differential equations has been provided within the exact WKB framework [2]. The work contains not only mathematical justification of the preceding work [3] in which necessity of introducing *new Stokes curves* around crossing points of ordinary Stokes curves was pointed out in an ad-hoc way, but also claims that *virtual turning points* should first be taken into account to construct complete Stokes geometry. They also clarified that new Stokes curves play essentially the same part in Stokes geometry.

Here, we discuss Stokes geometry of quantum propagator $I(q_n)$ on the same line, especially focusing on Stokes geomety in the horseshoe limit. Since a simple symbolic dynamics (binary full shift) completely controlling the underlying classical dynamics in the horseshoe limit, it can be viewed as the simplest possible situation in the sense of dynamical systems and thus would be the first object to be investigated. We shall emphasize that even in such simple cases virtual turning points and new Stokes curves are crucial ingredients to obtain the transition matrices that describes the propagation of WKB solutions from $q_n = -\infty$ to $q_n = +\infty$.

In the horseshoe limit, all the turning points are located on the real plane, but as the nonlinear paremeter c decreases, some of them fall into purely imaginary plane. Such events occur as a result of coalescence of turning points. If we know how Stokes graphs change when such bifurcations occur, Stokes geometry in generic parameter values can be traced from the horseshoe limit in principle. This is exactly the same strategy to study the *pruning* of the horseshoe structure [4].

References

[1] S. Friedland and J. Milnor, Ergod. Theor. Dyn. Sys. **9**(1989) 67-99.

[2] T. Aoki, T. Kawai and Y. Takei, in Analyse algébrique des perturbations singulières. I. (ed. by L. Boutet de Monvel) Hermann, (1994) 69-84.

[3] H.L. Berk, W.M. Nevins and K.V. Roberts, J. Math. Phys. **23**(1982) 988-1002.

[4] P. Cvitanović, G. Gunaratne and I. Procaccia, Phys. Rev. A **38** (1988) 1503-1520.