Logarithmic trace of Szegö and Toeplitz projectors

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In my book with V. Guillemin "The Spectral Theory of Toeplitz Operators", we defined Toeplitz projectors on a compact contact manifold, which are analogues of the Szegö projector on a strictly pseudo-convex boundary. If X is a contact manifold of dimension $2n-1$, the kernel of a Toeplitz projector, just as the Szegö kernel, has a holonomic singularity including a polar and a logarithmic term, of the form

$$S = \phi\,(q+0)^{-n} + \psi\,\mathrm{Log}\,(q+0)$$

where $\phi = \phi(x,y), \psi = \psi(x,y)$ are smooth kernels (densities with respect to y) and $q = q(x,y)$ is a suitable phase function: it vanishes on the diagonal, $\mathrm{Re}\,q$ is positive outside ($\mathrm{Re}\,q \geq c\ \mathrm{dist}(x,y)^2$), and on the diagonal $-d_x\mathrm{Im}q = d_y\mathrm{Im}q$ is the contact form (up to a positive factor).

The coefficient ψ of the logarithmic term is well defined (at least its Taylor expansion along the diagonal), so as its trace, the integral over the diagonal:

$$L(S) = \int_X \psi(x,x)$$

In the holomorphic case, X is the boundary of a strictly pseudo-convex domain, S is the Szegö kernel, the kernel of the orthogonal projector on the space of boundary values of holomorphic functions $\ker\bar{\partial}_b \subset L^2(X, dv)$ for some smooth volume element dv. In this case, K. Hirachi has shown that the trace $L(S)$ only depends on the CR structure of X (not on dv) and is invariant under deformations of the CR structure (Logarithmic singularity of the Szegö kernel and a global invariant of strictly pseudo-convex domains, math.CV/0309176, to appear in Ann. Math.).

Here we show that the trace $L(S)$ of a Toeplitz operator only depends on the contact structure of X. If X is the three sphere equipped with any contact form, this invariant vanishes (Y. Eliashberg has shown that, up to isomorphism, there is only one (taut) contact form coming from an embeddable CR structure, but there are many other "over-twisted" ones); this makes it not unlikely that $L(S)$ vanishes identically.