Generalizations and extensions of Furuta inequality

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1 Furuta inequality

In what follows, an operator means a bounded linear operator on a Hilbert space H. An operator T is positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$.

The following result was published in 1987 and has been applied in a lot of papers.

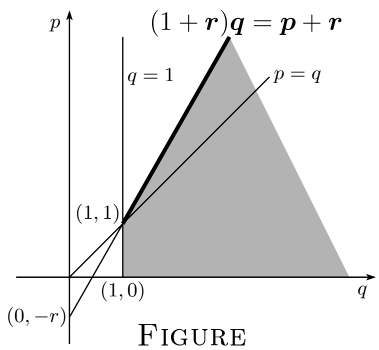
Theorem F (Furuta inequality [2]).
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



FIGURE

We remark that Löwner-Heinz theorem "$A \geq B \geq 0 \Longrightarrow A^\alpha \geq B^\alpha$ *for any* $\alpha \in [0,1]$" is the case $r = 0$ of Theorem F. Other proofs are given in [1][4] and also an elementary one-page proof in [3]. It is shown in [5] that the domain of p, q and r in Theorem F is the best possible for the inequalities (i) and (ii) to hold under the assumption $A \geq B$.

2 M. Uchiyama's results

A real-valued continuous function f defined on an interval $I \subseteq \mathbb{R}$ is operator monotone if $A \geq B$ implies $f(A) \geq f(B)$ for any self-adjoint operators A and B such that $\sigma(A), \sigma(B) \subseteq I$. Let $\mathbb{P}_+[a,b)$ and $\mathbb{P}_+^{-1}[a,b)$ be the families of functions defined as follows:

- $\mathbb{P}_+[a,b)$ is the set of all non-negative operator monotone functions defined on $[a,b)$.
- $\mathbb{P}_+^{-1}[a,b)$ is the set of all increasing functions h defined on $[a,b)$ such that $h([a,b)) = [0,\infty)$ and its inverse h^{-1} is operator monotone on $[0,\infty)$.

Uchiyama [6] introduced a new concept of majorization and showed a quite interesting result on operator monotone functions.

Definition ([6]). Let h be a non-decreasing function on I and k an increasing function on J. Then

$$h \preceq k \iff J \subseteq I \text{ and the composite } h \circ k^{-1} \text{ is operator monotone on } k(J).$$

Theorem A (Product theorem [6]). *Suppose $-\infty < a < b \leq \infty$. Then*

$$\mathbb{P}_+[a,b) \cdot \mathbb{P}_+^{-1}[a,b) \subseteq \mathbb{P}_+^{-1}[a,b), \quad \mathbb{P}_+^{-1}[a,b) \cdot \mathbb{P}_+^{-1}[a,b) \subseteq \mathbb{P}_+^{-1}[a,b).$$

Further, let $h_i \in \mathbb{P}_+^{-1}[a,b)$ for $1 \leq i \leq m$, and let g_j be a finite product of functions in $\mathbb{P}_+[a,b)$ for $1 \leq j \leq n$. Then for $\psi_i, \phi_j \in \mathbb{P}_+[0,\infty)$

$$\prod_{i=1}^m h_i(t) \prod_{j=1}^n g_j(t) \in \mathbb{P}_+^{-1}[a,b), \quad \prod_{i=1}^m \psi_i(h_i(t)) \prod_{j=1}^n \phi_j(g_j(t)) \preceq \prod_{i=1}^m h_i(t) \prod_{j=1}^n g_j(t).$$

He also obtained generalizations of Theorem F as applications of Theorem A.

Proposition B ([6]). *Let $h \in \mathbb{P}_+^{-1}[0,\infty)$, and let $\tilde{h}$ be a non-negative non-decreasing function on $[0,\infty)$ such that $\tilde{h} \preceq h$. Let g be a finite product of functions in $\mathbb{P}_+[0,\infty)$. Then for the function φ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$*

$$A \geq B \geq 0 \Longrightarrow \begin{cases} \varphi(g(B)^{\frac{1}{2}} h(A) g(B)^{\frac{1}{2}}) \geq g(B)^{\frac{1}{2}} \tilde{h}(A) g(B)^{\frac{1}{2}}, \\ \varphi(g(A)^{\frac{1}{2}} h(B) g(A)^{\frac{1}{2}}) \leq g(A)^{\frac{1}{2}} \tilde{h}(B) g(A)^{\frac{1}{2}}. \end{cases}$$

Theorem C ([6]). *Let $h \in \mathbb{P}_+^{-1}[0,\infty)$, and let $\tilde{h}$ be a non-negative non-decreasing function on $[0,\infty)$ such that $\tilde{h} \preceq h$. Let g_n be a finite product of functions in $\mathbb{P}_+[0,\infty)$ for each n, and let the sequence $\{g_n\}$ converge pointwise to g. Suppose $g \neq 0$ and $g(0+) = g(0)$. Then for the function φ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$*

$$A \geq B \geq 0 \Longrightarrow \begin{cases} \varphi(g(B)^{\frac{1}{2}} h(A) g(B)^{\frac{1}{2}}) \geq g(B)^{\frac{1}{2}} \tilde{h}(A) g(B)^{\frac{1}{2}}, \\ \varphi(g(A)^{\frac{1}{2}} h(B) g(A)^{\frac{1}{2}}) \leq g(A)^{\frac{1}{2}} \tilde{h}(B) g(A)^{\frac{1}{2}}. \end{cases}$$

3 Our results

We obtain extensions of Proposition B and Theorem C. In fact, Theorem 2 yields Theorem C by putting $\hat{h}(t) = t$ and $g(t) = 1$.

Proposition 1. *Let f_i be non-negative non-decreasing functions on $[0,\infty)$ and $g_j(t) = \prod_{i=1}^{j} f_i(t)$. Let h, $\hat{h}$ and $\tilde{h}$ be non-negative non-decreasing functions on $[0,\infty)$ such that $f_n(t) \preceq \hat{h}(t)g_{n-1}(t)$, $\tilde{h} \preceq h$ and $h(0)g_{n-1}(0) = 0$. Then for the functions ψ_j and φ_j defined by*

$$\psi_j(h(t)g_j(t)) = \hat{h}(t)g_j(t) \quad \text{and} \quad \varphi_j(h(t)g_j(t)) = \tilde{h}(t)g_j(t),$$

if $A, B \geq 0$ satisfy

$$\psi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}}) \geq \hat{h}(B)g_{n-1}(B),$$

then

$$\varphi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) \geq f_n(B)^{\frac{1}{2}}\varphi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}})f_n(B)^{\frac{1}{2}}$$

holds. Furthermore,

$$\psi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) \geq \hat{h}(B)g_n(B)$$

holds if $\hat{h} \preceq h$.

Theorem 2. *Let $\hat{h} \in \mathbb{P}_+^{-1}[0,\infty)$, and let h and $\tilde{h}$ be non-negative non-decreasing functions on $[0,\infty)$ such that $\tilde{h} \preceq h$ and $\hat{h} \preceq h$. Let g be a finite product of functions in $\mathbb{P}_+[0,\infty) \cup \mathbb{P}_+^{-1}[0,\infty)$ and γ_n a finite product of functions in $\mathbb{P}_+[0,\infty)$ for each n, and let the sequence $\{g(t)\gamma_n(t)\}$ converge pointwise to $\bar{g}(t)$. Suppose $\bar{g} \neq 0$ and $\bar{g}(0+) = \bar{g}(0)$. Then for the functions ψ, $\bar{\psi}$, φ and $\bar{\varphi}$ defined by*

$$\psi(h(t)g(t)) = \hat{h}(t)g(t), \quad \bar{\psi}(h(t)\bar{g}(t)) = \hat{h}(t)\bar{g}(t),$$

$$\varphi(h(t)g(t)) = \tilde{h}(t)g(t), \quad \bar{\varphi}(h(t)\bar{g}(t)) = \tilde{h}(t)\bar{g}(t),$$

if $A, B \geq 0$ satisfy

$$\psi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \geq \hat{h}(B)g(B),$$

then

$$g(B)^{\frac{1}{2}}\bar{\varphi}(\bar{g}(B)^{\frac{1}{2}}h(A)\bar{g}(B)^{\frac{1}{2}})g(B)^{\frac{1}{2}} \geq \bar{g}(B)^{\frac{1}{2}}\varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})\bar{g}(B)^{\frac{1}{2}}$$

and

$$\bar{\psi}(\bar{g}(B)^{\frac{1}{2}}h(A)\bar{g}(B)^{\frac{1}{2}}) \geq \hat{h}(B)\bar{g}(B)$$

holds.

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