On the triangle inequality in normed spaces

Kichi-Suke Saito

Department of Mathematics

Faculty of Science

Niigata University

saito@math.sc.niigata-u.ac.jp

1. Introduction

The triangle inequality is undoubtedly one of the most fundamental inequalities in mathematics. Let X be a normed (Banach) space. For any vectors $x, y \in X$,

$$||x + y|| \leq ||x|| + ||y|| \text{ (Triangle inequality).}$$

Several authors have been treating its generalizations and reverse inequalities (cf. Hudzik–Landes[7], S. Saitoh[16], Dragomir[2] and etc). Recently, Kato-Saito-Tamura [9] found the sharp triangle inequality and its reverse inequality with n elements in a normed space to study the geometrical structure of Banach spaces. After that, we have several papers about the triangle inequalities (cf. J. Pečarić–R. Rajić[15], Dragomir[3, 4] and Hsu–Shaw–Wong [6]). Very recently, Mitani-Saito-Kato-Tamura [13] proved the refinement of sharp triangle inequality and the reverse inequality.

Our aim in this talk is to present the recent results of sharp triangle inequalities in [8, 13].

2. Sharp triangle inequalities and the reverse

At first, we consider two non-zero vectors x, y of a normed space X. Then we have

Theorem 1 For two non-zero vectors $x, y \in X$ such that $\|x\| \geq \|y\|$,

$$\|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|y\|$$
$$\leq \|x\| + \|y\| \tag{1}$$
$$\leq \|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|x\|. \tag{2}$$

The first inequality with two elements (1) was given earlier in Hudzik and Landes [7]; the inequalites (1) and (2) are also found in a recent paper of Maligranda [10].

We next consider three non-zero vectors x, y, z of a normed space X. Then we have

Theorem 2. *For all nonzero elements x, y, z in a Banach space X with $\|x\| \geq \|y\| \geq \|z\|$,*

$$\begin{aligned}
&\|x+y+z\| + \left(3 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)\|z\| \\
&\quad + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)(\|y\| - \|z\|) \\
&\leq \|x\| + \|y\| + \|z\| \\
&\leq \|x+y+z\| + \left(3 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)\|x\| \\
&\quad - \left(2 - \left\|\frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)(\|x\| - \|y\|).
\end{aligned}$$

In general, we have the following triangle inequalities for n nonzero vectors $x_1, \cdots, x_n \in X$.

Theorem 3 ([13]). Let $n \geq 3$. For any non-zero vectors $x_1, \cdots, x_n$ of a normed space X,

$$\left\|\sum_{j=1}^{n} x_j\right\| + \sum_{k=2}^{n}\left(k - \left\|\sum_{j=1}^{k} \frac{x_j^*}{\|x_j^*\|}\right\|\right)(\|x_k^*\| - \|x_{k+1}^*\|)$$

$$\le \sum_{j=1}^{n} \|x_j\|$$

$$\le \Big\| \sum_{j=1}^{n} x_j \Big\| - \sum_{k=2}^{n} \left(k - \Big\| \sum_{j=n-(k-1)}^{n} \frac{x_j^*}{\|x_j^*\|} \Big\| \right) (\|x_{n-k}^*\| - \|x_{n-(k-1)}^*\|),$$

where $x_1^*, x_2^*, \cdots, x_n^*$ are the rearrangement of $x_1, x_2, \cdots, x_n$ satisfying $\|x_1^*\| \ge \|x_2^*\| \ge \cdots \ge \|x_n^*\|$, and $x_0^* = x_{n+1}^* = 0$.

In Theorem 3, we may assume that $\|x_1\| \ge \|x_2\| \ge \cdots \ge \|x_n\|$: that is, we have the following

Theorem 3a. Let $n \ge 3$. For all nonzero elements $x_1, \cdots, x_n$ in a normed space X such that $\|x_1\| \ge \|x_2\| \ge \cdots \ge \|x_n\|$,

$$\Big\| \sum_{j=1}^{n} x_j \Big\| + \left(n - \Big\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \Big\| \right) \|x_n\|$$

$$+ \sum_{k=2}^{n-1} \left(k - \Big\| \sum_{j=1}^{k} \frac{x_j}{\|x_j\|} \Big\| \right) (\|x_k\| - \|x_{k+1}\|)$$

$$\le \sum_{j=1}^{n} \|x_j\|$$

$$\le \Big\| \sum_{j=1}^{n} x_j \Big\| + \left(n - \Big\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \Big\| \right) \|x_1\|$$

$$- \sum_{k=2}^{n-1} \left(k - \Big\| \sum_{j=n-(k-1)}^{n} \frac{x_j}{\|x_j\|} \Big\| \right) (\|x_{n-k}\| - \|x_{n-(k-1)}\|).$$

Corollary 4([8]) For all nonzero elements $x_1, \cdots, x_n$ in a normed space X

$$\Big\| \sum_{j=1}^{n} x_j \Big\| + \left(n - \Big\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \Big\| \right) \min_{1 \le j \le n} \|x_j\|$$

$$\le \sum_{j=1}^{n} \|x_j\|$$

$$\leq \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \max_{1\leq j\leq n} \|x_j\|.$$

3. Equality of Corollary 4

In [8], Kato-Saito-Tamura considerd equality attainedness for each of our inequalities in a strictly convex Banach space. The following lemma is quite powerful in our subsequent discussions.

Lemma 5. *Let X be a strictly convex Banach space. Let $x_1, x_2, \ldots, x_n$ be nonzero elements in X. Then the following are equivalent.*

(i) $\left\| \sum_{j=1}^{n} \alpha_j x_j \right\| = \sum_{j=1}^{n} \alpha_j \|x_j\|$ *with any positive numbers* $\alpha_1, \alpha_2, \ldots, \alpha_n$.

(ii) $\left\| \sum_{j=1}^{n} \alpha_j x_j \right\| = \sum_{j=1}^{n} \alpha_j \|x_j\|$ *with some positive numbers* $\alpha_1, \alpha_2, \ldots, \alpha_n$.

(iii) $\frac{x_1}{\|x_1\|} = \frac{x_2}{\|x_2\|} = \cdots = \frac{x_n}{\|x_n\|}$.

Theorem 6. *Let X be a strictly convex Banach space and $x_1, x_2, \ldots, x_n$ nonzero elements in X. Let $\|x_{j_0}\| = \min\{\|x_j\| : 1 \leq j \leq n\}$ and $\|x_{j_1}\| = \max\{\|x_j\| : 1 \leq j \leq n\}$. Let $J_0 = \{j : \|x_j\| = \|x_{j_0}\|,\ 1 \leq j \leq n\}$. Then*

$$\left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1\leq j\leq n} \|x_j\| = \sum_{j=1}^{n} \|x_j\| \tag{3}$$

if and only if either

(a) $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$

or

(b) $\frac{x_j}{\|x_j\|} = \frac{x_{j_1}}{\|x_{j_1}\|}$ *for all* $j \in J_0^c$ *and* $\sum_{j=1}^{n} \frac{x_j}{\|x_j\|} = \| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \| \frac{x_{j_1}}{\|x_{j_1}\|}$.

Theorem 7. *Let X be a strictly convex Banach space and $x_1, x_2, \ldots, x_n$ nonzero elements in X. Let $\|x_{j_0}\| = \min\{\|x_j\| : 1 \leq j \leq n\}$ and $\|x_{j_1}\| = \max\{\|x_j\| : 1 \leq j \leq n\}$. Let $J_1 = \{j : \|x_j\| = \|x_{j_1}\|, 1 \leq j \leq n\}$. Then*

$$\sum_{j=1}^{n} \|x_j\| = \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \max_{1\leq j\leq n} \|x_j\| \tag{4}$$

if and only if either

(a) $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$

or

(b) $\frac{x_j}{\|x_j\|} = \frac{x_{j_0}}{\|x_{j_0}\|}$ *for all* $j \in J_1^c$ *and* $\sum_{j=1}^n x_j = \| \sum_{j=1}^n x_j \| \frac{x_{j_0}}{\|x_{j_0}\|}$.

Theorem 8. *Let* X *be a strictly convex Banach space and* $x_1, x_2, \ldots, x_n$ *nonzero elements in* X. *Then the equality*

$$\left\| \sum_{j=1}^n x_j \right\| + \left(n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \le j \le n} \|x_j\| \tag{5}$$

$$= \sum_{j=1}^n \|x_j\| = \left\| \sum_{j=1}^n x_j \right\| + \left(n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right) \max_{1 \le j \le n} \|x_j\| \tag{6}$$

holds if and only if

(a) $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$

or

(b) $\dfrac{x_1}{\|x_1\|} = \dfrac{x_2}{\|x_2\|} = \cdots = \dfrac{x_n}{\|x_n\|}$.

4. Applications

For non-zero vectors $x, y \in X$, we define the angular distance $\alpha[x, y]$ between x and y by

$$\alpha[x, y] = \| \frac{x}{\|x\|} - \frac{y}{\|y\|} \|.$$

Then the well-known Dunkl-William inequality [5] states that for any two non-zero elements x, y,

$$\alpha[x, y] \le \frac{4\|x - y\|}{\|x\| + \|y\|}.$$

The refinment established by Maligranda [10] is

$$\alpha[x, y] \le \frac{\|x - y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}}.$$

More generally, J. Pečarić and R. Rajić [15] showed that, for n nonzero elements $x_1, \cdots, x_n$,

$$\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \| \le \min_{1 \le i \le n} \{ \frac{1}{\|x_i\|} (\| \sum_{j=1}^n x_j \| + \sum_{j=1}^n |\|x_j\| - \|x_i\||) \}.$$

We next apply Corollary 4 to a geometric property of Banach spaces. Recall that a Banach space X is called *uniformly non-ℓ_1^n* provided there exists ε $(0 < \varepsilon < 1)$ such that for any $x_1, \cdots, x_n$ in the *unit sphere* of X there exists $\theta = (\theta_j)$ of n signs ± 1 for which

$$(7) \qquad \left\| \sum_{j=1}^{n} \theta_j x_j \right\| \leq n(1-\varepsilon).$$

When $n = 2$, X is called *uniformly non-square.* By virtue of Corollary 4 we immediately have the following fact.

Proposition 9. *For a Banach space X the following are equivalent.*

(i) *X is uniformly non-ℓ_1^n.*

(ii) *There exists ε $(0 < \varepsilon < 1)$ such that for any $x_1, \cdots, x_n$ in the unit ball of X there exists $\theta = (\theta_j)$ of n signs ± 1 for which (8) holds true.*

Indeed, assume that there exists ε $(0 < \varepsilon < 1)$ such that for any $x_1, \cdots, x_n$ in the *unit sphere* of X there exist $\theta = (\theta_j)$ of n signs ± 1 for which (8) is valid. Take $x_1, \cdots, x_n$ from the *unit ball* of X. If $\|x_{j_0}\| := \min\{\|x_1, \| \ldots, \|x_n\|\} \leq 1/2$, we have

$$\left\| \sum_{j=1}^{n} \sigma_j x_j \right\| \;\leq\; \sum_{j \neq j_0} \|x_j\| + \|x_{j_0}\| \leq (n-1) + \frac{1}{2} \leq n(1 - \frac{1}{2n}).$$

Let $\|x_{j_0}\| \geq \frac{1}{2}$. According to our assumption there exists n signs (θ_j) for which (8) is valid for $x_1/\|x_1\|, \cdots, x_n/\|x_n\|$. Therefore by the first inequality of Corollary 4

$$\begin{aligned} \left\| \sum_{j=1}^{n} \theta_j x_j \right\| &\leq \sum_{j=1}^{n} \|x_j\| - \left(n - \left\| \sum_{j=1}^{n} \theta_j \frac{x_j}{\|x_j\|} \right\| \right) \|x_{j_0}\| \\ &\leq n - \frac{n\varepsilon}{2} = n\left(1 - \frac{\varepsilon}{2}\right). \end{aligned}$$

Consequently by letting $\varepsilon_0 = \min\{\frac{\varepsilon}{2}, \frac{1}{2n}\}$ we have the conclusion.

References

[1] J. B. Diaz and F. T. Metcalf, *A complementary triangle inequality in Hilbert and Banach spaces*, Proc. Amer. Math. Soc., **17** (1966), 88–97.

[2] S. S. Dragomir, *Reverses of the triangle inequality in Banach spaces,* J. Inequal. Pure and Appl. Math., **6**(5) (2005), Art. 129, pp. 46.

[3] S. S. Dragomir, *Bounded for the normalized Jensen functional,* preprint.

[4] S. S. Dragomir, *A generalization of the Pecaric-Rajic inequalities in normed linear spaces,* preprint.

[5] C. F. Dunkl and K. S. Williams, *A simple norm inequality*, Amer. Math. Monthly, **71** (1964), 53–54.

[6] C. Y. Hsu, S. Y. Shaw and H. J. Wong, *Refinments of generalized triangle inequalities,* preprint.

[7] H. Hudzik and T. R. Landes, *Characteristic of convexity of Köthe function spaces,* Math. Ann., **294** (1992), 117–124.

[8] M. Kato, K.-S. Saito and T. Tamura, *Sharp triangle inequality and its reverse in Banach spaces*, Math. Inequal. Appl., **10** (2007), 451–460.

[9] M. Kato, K.-S. Saito and T. Tamura, *Uniform non-ℓ_1^n-ness of ψ-direct sums of Banach spaces $X \oplus_\psi Y$*, submitted.

[10] L. Maligranda, *Simple norm inequalities,* Amer. Math. Monthly, **113** (2006), 256–260.

[11] J. L. Massera and J. J. Schäffer, *Linear differential equations and functional analysis I,* Ann. of Math., **67** (1958), 517–573.

[12] P. R. Mercer, *The Dunkl-Williams inequality in an inner product space*, Math. Inequal. Appl., **10** (2007), 447–450.

[13] K. -I. Mitani, K. -S. Saito, M. Kato and T. Tamura, *On sharp triangle inequalities in Banach spaces*, J. Math. Anal. Appl., **336** (2007), 1178–1186.

[14] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and new inequalities in analysis,* Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.

[15] J. Pečarić and R. Rajić, *The Dunkl-Williams inequality with n elements in normed linear spaces,* Math. Inequal. Appl., **10** (2007), 461–470.

[16] S. Saitoh, *Generalizations of the triangle inequality,* J. Inequal. Pure Appl. Math., **4**(3) (2003), no. 3, Art. 62, pp. 5.

[17] S. -E. Takahasi, J. M. Rassias, S. Saitoh and Y. Takahashi, Refined generalizations of the triangle inequality on a Banach spaces, preprint.