

q -numerical range of a matrix
–the projection of a convex body in a
4-dimensional space onto a plane–

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1. numerical range of a matrix and the equation of its boundary

Definition Suppose that T is a bounded linear on a complex Hilbert space H . The range

$$W(T) = \{\langle Tx, x \rangle : x \in H, \|x\| = 1\}, \quad (1.1)$$

is called the *numerical range* of T .

It is known that the spectrum $\sigma(T)$ of T is contained in $W(T)$. The range $W(T)$ is a bounded convex subset of \mathbf{C} (Hausdorff, 1919 [H];[T],[AL]). If H is finite-dimensional, then the set $W(T)$ is compact and the boundary of $W(T)$ lies on an algebraic curve. We assume that H is finite-dimensional. Set

$$\begin{aligned} M_R &= \max \sigma(\Re(T)), & m_R &= \min \sigma(\Re(T)), \\ M_I &= \max \sigma(\Im(T)), & m_I &= \min \sigma(\Im(T)). \end{aligned}$$

In 1902, Bendixson showed the inclusion

$$\sigma(T) \subset \{x + iy : (x, y) \in \mathbf{R}^2, m_R \leq x \leq M_R, m_I \leq y \leq M_I\}.$$

This relation was refined as

$$W(T) = \bigcap_{0 \leq \theta \leq \pi} \{z \in \mathbf{C} : \min \sigma(\Re(\exp(-i\theta)T)) \leq \Re(z \exp(-i\theta))\}$$

$$\leq \max \sigma(\Re(\exp(-i\theta)T)).$$

A concrete procedure to determine the equation of $\partial W(T)$ is provided in the following. We use an algebraic or a convex analytic method. Suppose that Δ is a compact convex set in $\mathbf{C} \cong \mathbf{R}^2$ and 0 is an interior point of Δ . Then the set

$$\Delta^\perp = \{x + iy : (x, y) \in \mathbf{R}^2, xu + yv + 1 \geq 0 \text{ for every } u + iv \in \Delta\}$$

is also a compact convex set. If p, q are positive real numbers satisfying $1/p + 1/q = 1$, then the sets

$$B_p = \{x + iy : (x, y) \in \mathbf{R}^2, |x|^p + |y|^p \leq 1\}, B_q = \{x + iy : (x, y) \in \mathbf{R}^2, |x|^q + |y|^q \leq 1\}$$

satisfy

$$B_p^\perp = B_q, \quad B_q^\perp = B_p.$$

If $W(T)$ contains 0 as an interior point, then

$$\partial(W(T)^\perp) \subset \{x + iy : \det(I_n + x\Re(T) + y\Im(T)) = 0\}.$$

Even if 0 is not an interior point of $W(T)$, the boundary of $W(T)$ is the dual curve of the algebraic curve:

$$\{x + iy : \det(I_n + x\Re(T) + y\Im(T)) = 0\},$$

and hence the degree of the equation of $\partial W(T)$ is at most $n(n-1)$.

2. the q -numerical range of a matrix

Definition Suppose that T is a bounded linear operator on a complex Hilbert space H , and q is a real number $0 \leq q \leq 1$. The q -numerical range $W_q(T)$ is defined as

$$W_q(T) = \{\langle Tx, y \rangle : x, y \in H, \|x\| = \|y\| = 1, \langle x, y \rangle = q\}. \quad (2.1)$$

We assume that H is finite-dimensional. In 1984, N. K. Tsing proved the following formula and proved the convexity of $W_q(T)$:

$$W_q(T) = \{qz + \sqrt{1 - |q|^2}w : z \in W(T), w \in \mathbf{C}, |w| \leq \phi(z)\},$$

where

$$\phi(z) = \max\{\langle T^*Tx, x \rangle : x \in H, \|x\| = 1, \langle Tx, x \rangle = z\}$$

is convex on $W(T)$ as a result of Binding in 1985 (cf.[B],[LP]). (Tsing used a slightly different method.) $W_1(T) = W(T)$.

Tsing's formula provides a basis to compute the boundary of the equation. We set

$$\Gamma(T) = \{(x, y, u, v) \in \mathbf{R}^2, x+iy \in W(T), x^2+y^2+u^2+v^2 \leq \phi(x+iy)\}. \quad (2.2)$$

Then $\Gamma(T)$ is a compact convex set in \mathbf{R}^4 . For $0 < q < 1$, we consider an orthogonal projection $\Pi_q : \mathbf{R}^4 \rightarrow \mathbf{R}^2 \cong \mathbf{C}$ defined by

$$\Pi_q(x, y, u, v) = (qx + \sqrt{1-q^2}u, qy + \sqrt{1-q^2}v).$$

Then the following equation holds:

$$W_q(T) = \Pi_q(\Gamma(T)) \quad (2.3)$$

(cf. [CN1]). The boundary of $\Gamma(T)$ lies on an algebraic hypersurface in the 4-dimensional space \mathbf{R}^4 . The set $W_q(T)$ is also the image of this boundary by Π_q .

The order of the equation of the boundary of $\Gamma(T)$ is at most $2n(n-1)^2$. The equation $F(x, y, u, v) = 0$ is obtained by

$$F(x, y, u, v) = G(x, y, x^2 + y^2 + u^2 + v^2),$$

where $G(x, y, z) = 0$ is the equation of the dual algebraic surface of the algebraic surface defined by

$$\det(I_n + X \Re(T) + Y \Im(T) + Z T^*T) = 0.$$

If T, S are respective $n \times n, m \times m$ matrices. Then the following two conditions are mutually equivalent

- (i) $\Gamma(T) \subset \Gamma(S)$,
- (ii) $W_q(T) \subset W_q(S)$ for $0 \leq q \leq 1$.

In this talk, we provide a concrete method to compute the equation of the boundary of $W_q(T)$ for an irreducible 3×3 matrix , for example

$$T = \begin{pmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{pmatrix}.$$

REFERENCES

- [AL] T. Ando and C. K. Li(editors), special issue: The Numerical Range and Numerical Radius,*Linear and Multilinear Algebra* 37,(1994).
- [B] P. Binding, Hermitian forms and the fibration of spheres.*Proc. Amer. Math. Soc.*, 94 (1985), no. 4, 581-584.
- [BW]J. Bruce and C. T. Wall, On the classification of cubic surfaces, J. London Math. Soc., 19(1979), 245-256.
- [CN1] M. T. Chien and H. Nakazato,Davis-Wielandt shell and q -numerical range, *Linear Algebra Appl.*, 340(2002), 15-31.
- [CN2] M. T. Chien and H. Nakazato, The boundary of the q -numerical range of a reducible matrix, *Linear and Multilinear Algebra*, 55(2007), 275-292.
- [CNP] M. T. Chien, H. Nakazato and P. Psarrakos, The q -numerical range and the Davis-Wielandt shell of reducible 3×3 matrices, *Linear and Multilinear Algebra*, 54(2006), 79-112.
- [H] F. Hausdorff, Das Wertevorrat einer Bilinearform,*Math. Zeit.*3(1919),314-316.
- [LP] C. K. Li and Y. T. Poon, Convexity of the joint numerical range , *SIAM J. Matrix Analysis Appl.* 21(1999), 668–678.
- [T] O. Toeplitz, Das algebraische Analogue zu einer Satze von Fejér, *Math. Zeit.*2(1918),187-197.
- [Ts] N. K. Tsing, The constrained bilinear form and the C -numerical range, *Linear Algebra Appl.* 56(1984), 195-206.