作用素の三角不等式の等号成立条件について

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In this talk I would like to explain my paper which is a joint work with Tsuyoshi Ando.

First I would like to fix some notations. We denote by H an infinite dimensional complex Hilbert space. The set of all bounded linear operators on H is denoted by $B(H)$. For $X \in B(H)$, we define its "absolute value" by $|X| = (X^*X)^{1/2}$.

We would like to consider a triangle inequality for this absolute value. It is well-known that the inequality

$$|A + B| \leq |A| + |B|$$

is **wrong**. But we have

Theorem 0.1. *(Thompson, '76 PJM) For two $n \times n$matrices $A, B \in M_n(\mathbb{C})$, we can find two unitaries $U, V \in M_n(\mathbb{C})$ s.t.*

$$|A + B| \leq U|A|U^* + VBV^*.$$

Moreover

Theorem 0.2. *(Thompson, '79 PJM) For some unitaries $U, V \in M_n(\mathbb{C})$, the equality*

$$|A + B| = U|A|U^* + V|B|V^*$$

holds if and only if there exists a unitary $W \in M_n(\mathbb{C})$ s.t. $A = W|A|$ and $B = W|B|$. (That is, A and B have a common phase part W.)

Therefore in both cases A and B satisfy

$$|A + B| = |A| + |B|.$$

The aim of this talk is to generalize the second theorem for infinite dimensional setting. Our main result is:

Theorem 0.3. *For two bounded linear operators $A, B \in B(H)$, the triangle equality $|A+B| = |A| + |B|$ holds if and only if there exists a partial isometry W such that $A = W|A|$ and $B = W|B|$.*

Remark 0.1. (1) Why do we consider $|A+B| = |A| + |B|$ instead of $|A+B| = U|A|U^* + V|B|V^*$?

Take a projection $P \in B(H)$ and subprojection $Q \leq P$ s.t. four projections

$$P, Q, P-Q, 1-P$$

are infinite rank. Then it is easy to find unitaries U, V s.t.

$$|P + (-Q)| = U|P|U^* + V|(-Q)|V^*.$$

However P and $-Q$ cannot have a common phase part. Hence Thompson's theorem does not hold for this triangle equality.
(2) The proof of our theorem is simple if we have some faithful finite trace or H is finite dimensional as follows. (The following argument does work for the equality $|A_1 + \cdots A_n| = |A_1| + \cdots |A_n|$.)

Take polar decompositions

$$A = U|A|, \ B = V|B|, \ A + B = W|A+B|.$$

We may assume that U, V, W are unitaries. By using the triangle equality

$$|A+B| = |A| + |B|,$$

we have

$$\begin{aligned} W^*U|A| + W^*V|B| &= W^*(A+B) \\ &= |A+B| = |A| + |B|. \end{aligned}$$

Then

$$
\begin{aligned}
|A| + |B| =& \frac{1}{4}\{(1 + W^*U)|A|(1 + W^*U)^* \\
&- (1 - W^*U)|A|(1 - W^*U)^* \\
&+ (1 + W^*V)|B|(1 + W^*V)^* \\
&- (1 - W^*V)|B|(1 - W^*V)^*\} \\
\leq& \frac{1}{4}\{(1 + W^*U)|A|(1 + W^*U)^* \\
&+ (1 + W^*V)|B|(1 + W^*V)^*\} \\
=& \frac{1}{4}\{(W^*U|A| + W^*V|B|) \\
&+ (W^*U|A| + W^*V|B|)^* \\
&+ (|A| + |B|) \\
&+ (W^*U|A|U^*W + W^*V|B|V^*W)\} \\
=& \frac{1}{4}\{3(|A| + |B|) \\
&+ (W^*U|A|U^*W + W^*V|B|V^*W)\}.
\end{aligned}
$$

That is,

$$|A| + |B| \leq W^*U|A|U^*W + W^*V|B|V^*W.$$

Since we have a faithful trace, this inequality implies

$$|A| + |B| = W^*U|A|U^*W + W^*V|B|V^*W$$

and hence

$$(1 - W^*U)|A|(1 - W^*U)^* = 0,$$

$$(1 - W^*V)|B|(1 - W^*V)^* = 0,$$

in other words

$$W^*A = W^*U|A| = |A|$$

and

$$W^*B = W^*V|B| = |B|.$$

So we are done.

This argument heavily depends on **finiteness**. We need another method for general cases.

Proof of Main Result

Take polar decompositions

$$A = U|A|, \;\; B = V|B|.$$

By the triangle equality we have

$$\begin{aligned}(|A| + |B|)^2 &= |A + B|^2 \\ &= (U|A| + V|B|)^*(U|A| + V|B|)\end{aligned}$$

and hence

$$|A|(U^*V - 1)|B| + |B|(V^*U - 1)|A| = 0.$$

That is

$$i|A|(U^*V - 1)|B|$$

is self-adjoint. Since $|A|, |B| \leq |A| + |B|$, we can find two contractions K, L satisfying

$$\begin{aligned}|A|^{1/2} = K(|A| + |B|)^{1/2} = K|A + B|^{1/2}, \\ |B|^{1/2} = L(|A| + |B|)^{1/2} = L|A + B|^{1/2}\end{aligned}$$

and the support of $K^*K + L^*L$ is dominated by that of $|A| + |B|$.

Then we have

$$|A| + |B| = (|A| + |B|)^{1/2}(K^*K + L^*L)(|A| + |B|)^{1/2}.$$

Thus $K^*K + L^*L$ is equal to the support projection of $|A| + |B|$ and hence

$$K^*KL^*L = L^*LK^*K.$$

Direct computations show

$$\begin{aligned}&i|A|(U^*V - 1)|B| \\ =&|A + B|^{1/2} \\ &\times \{iK^*K|A + B|^{1/2}(U^*V - 1)|A + B|^{1/2}L^*L\} \\ &\times |A + B|^{1/2}.\end{aligned}$$

Since the left-hand side is self-adjoint, we conclude that

$$iK^*K|A + B|^{1/2}(U^*V - 1)|A + B|^{1/2}L^*L$$

is a self-adjoint operator. In particular

$$\sigma(iK^*K|A + B|^{1/2}(U^*V - 1)|A + B|^{1/2}L^*L) \subset \mathbb{R}$$

We define a positive operator D by

$$D = [|A+B|^{1/2}(L^*L)(K^*K)|A+B|^{1/2}]^{1/2}.$$

Then we have

$$\begin{aligned}
&\mathbb{R} \supset \\
&\sigma(iK^*K|A+B|^{1/2}(U^*V-1)|A+B|^{1/2}L^*L) \setminus \{0\} \\
&\quad = i\sigma(D(U^*V-1)D) \setminus \{0\} \\
&\quad \subset i\mathbb{W}(D(U^*V-1)D) \text{ (the numerical range)} \\
&\quad \subset i\{z \in \mathbb{C};\ |z+||D||^2| \leq ||D||^2\} \\
&\quad = \text{the circle with center } -i||D|| \text{ and radius } ||D||.
\end{aligned}$$

Therefore we conclude

$$iK^*K|A+B|^{1/2}(U^*V-1)|A+B|^{1/2}L^*L = 0$$

and hence

$$i|A|(U^*V-1)|B| = 0.$$

Let

$$W(|A|\xi + |B|\eta) = A\xi + B\eta.$$

$(\xi, \eta \in H)$

Since

$$\begin{aligned}
&||A\xi + B\eta||^2 - |||A|\xi + |B|\eta||^2 \\
&\quad = 2Re\langle |A|(U^*V-1)|B|\eta, \xi\rangle \\
&\quad = 0,
\end{aligned}$$

W is a partial isometry. Obviously W satisfies

$$A = W|A|, \quad B = W|B|.$$

<u>Final Remark</u>

Since we deal with only two operators, we can get some commutativity. This is the crucial point in our argument. I have no idea to attack this problem for 3 (or n) operators.

References

[1] C.A. Akemann, J. Anderson and G.K. Pedersen, Triangle inequalities in operator algebras, *Linear and Multilinear Algebra*, **11**(1982), no. 2, 167–178.

[2] T. Ando and T. Hayashi, A characterization of the operator-valued triangle equality, to appear in JOT

[3] P.R. Halmos, *A Hilbert space problem book*, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London 1967

[4] R.C. Thompson, Convex and concave functions of singular values of matrix sums, *Pacific J. Math.*, **66**(1976), no. 1, 285–290.

[5] ______, Matrix type metric inequalities, *Linear and Multilinear Algebra*, **5**(1977/78), no. 4, 303–319.

[6] ______, The case of equality in the matrix-valued triangle inequality, *Pacific J. Math.*, **82** (1979), no. 1, 279–280.

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