One-parameter families of multivariable operator means

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1 Means and paths

In this talk, we use operator means, in particular, the Kubo-Ando mean [10] plays a central role: A binary operation $\mathbf{m}$ on positive operators on a Hilbert space is called the *Kubo-Ando (operator) mean* if $\mathbf{m}$ satisfies the following axioms:

$$\begin{aligned} &\textbf{monotonicity}: && A \le C,\ B \le D \Longrightarrow A\,\mathbf{m}\,B \le C\,\mathbf{m}\,D. \\ &\textbf{semicontinuity}: && A_n \downarrow A,\ B_n \downarrow B \Longrightarrow A_n\,\mathbf{m}\,B_n \downarrow A\,\mathbf{m}\,B. \\ &\textbf{transformer inequality}: && T^*(A\,\mathbf{m}\,B)T \le T^*AT\,\mathbf{m}\,T^*BT. \\ &\textbf{normalization}: && A\,\mathbf{m}\,A = A. \end{aligned}$$

By semicontinuity, we may assume positive operators are invertible. The *representing function* $f_{\mathbf{m}}(x) = 1\,\mathbf{m}\,x$ for a Kubo-Ando mean $\mathbf{m}$ is operator monotone (concave) on $(0,\infty)$ and $\mathbf{m}$ is represented by

$$A\,\mathbf{m}\,B = A^{\frac{1}{2}} f_{\mathbf{m}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

A *path* $A\,\mathbf{m}_t B$ means parametrized operator means which is usually differentiable for t with $A\,\mathbf{m}_0 B = A$ and $A\,\mathbf{m}_1 B = 0$. A path is called *symmetric* if

$$A\,\mathbf{m}_t B = B\,\mathbf{m}_{1-t} A$$

holds for all $t \in [0,1]$. Typical example is (*quasi-arithmetic*) *power means* for $r \in [-1,1]$:

$$A\#_{r,t}B = A^{\frac{1}{2}}\left((1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\right)^{\frac{1}{r}} A^{\frac{1}{2}},$$

which include important means:

$$\begin{aligned} &\text{arithmetic mean:} && A\nabla_t B = A\#_{1,t}B = (1-t)A + tB \\ &\text{geometic mean:} && A\#_t B = A\#_{0,t}B \equiv \lim_{\varepsilon\to 0} A\#_{\varepsilon,t}B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}} \\ &\text{harmonic mean:} && A\,!_t B = A\#_{-1,t}B = ((1-t)A^{-1} + tB^{-1})^{-1}. \end{aligned}$$

Moreover the above paths are *interpolational* in the sense that

$$(A\#_{r,p}B)\#_{r,t}(A\#_{r,q}B) = A\#_{r,(1-t)p+tq}B$$

for all $p, q, t \in [0,1]$.

2 CPR geometry and Thompson metric

Here the *CPR geometry* represents the one on the Finsler manifold $\mathcal{A}^+$, the positive invertible elements in a unital C*-algebra $\mathcal{A}$, discussed by Corach-Porta-Recht [4, 5]. Corach himself reformulated it in [6]: The base manifold is $\mathcal{A}^+$ with the tangent vector bundle $\mathcal{A}^h$ (the tangent space at $A \in \mathcal{A}^+$ is $\mathcal{A}^+ - A$). For the invertible elements $\mathcal{G}$ in $\mathcal{A}$, the principal fibre bundle $\{\mathcal{G}, \mathcal{A}^+, \mathcal{U}_A, \pi_A\}$ for fixed $A \in \mathcal{A}$ is defined by

projection	$\pi_A\colon \mathcal{G} \to \mathcal{A}^+$, $G \mapsto GAG^*$
structure group	$\mathcal{U}_A = \{V \in \mathcal{G} \mid VAV^* = A\} = A^{1/2}\mathcal{U}A^{-1/2}$ with the action $L_V A = VAV^*$, which shows $\mathcal{A}^+$ is homogeneous.
fiber	$\pi_A^{-1}(B) = B^{1/2}A^{-1/2}\mathcal{U}_A = B^{1/2}\mathcal{U}A^{-1/2}$
tangent map	$\tau_G : \mathcal{A} \to \mathcal{A}^h, X \mapsto XG^* + GX^*$ (The tangent space for G is identified with $\mathcal{A}$ itself.)

Since $\mathcal{A}^+$ is a homogeneous space, we may usually assume the fixed element A is the identity I. Then the principal fibre bundle has a natural connection, which induces the covariant derivative D_t of a tangent field $X(t)$ along the curve $\gamma(t)$ in $\mathcal{A}^+$

$$D_t X = \dot{X} - \frac{1}{2}(\dot{\gamma}\gamma^{-1}X + X\gamma^{-1}\dot{\gamma}).$$

Then the geodesic equation

$$O = D_t\dot{\gamma} = \ddot{\gamma} - \dot{\gamma}\gamma^{-1}\dot{\gamma}$$

implies that the geodesic from A to B is the path of geometric Kubo-Ando means:

$$A\#_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}.$$

Moreover the above manifold $\mathcal{A}^+$ is the Finsler space with a Finsler metric

$$L(X; A) = \|X\|_A = \|A^{-1/2}XA^{-1/2}\| :$$

Since $\|X\|_A$ is an equvalent norm to $\|X\|$, it is a Finsler metric if

Finsler condition: $\quad \|P_t X\|_{\gamma(t)} = \|X\|_{\gamma(0)}$

holds for all curves γ and parallel transports P_t along γ [15]. In this case,

$$P_t X = \Gamma(t)\Gamma(0)^{-1}X(\Gamma(0)^*)^{-1}\Gamma(t)^*$$

for a parallel lift Γ for γ. For the case $A = I$ for simplicity, a lift Γ satisfies

$$\gamma = \pi_I(\Gamma) = \Gamma\Gamma^*,$$

so that $U_t = \gamma(t)^{-1/2}\Gamma(t)$ defines a unitary for each t. Therefore we show the Finsler condition by

$$\|P_t X\|_{\gamma(t)} = \|U_t U_0^* \gamma(0)^{-1/2}X\gamma(0)^{-1/2}U_0U_t^*\| = \|\gamma(0)^{-1/2}X\gamma(0)^{-1/2}\| = \|X\|_{\gamma(0)}.$$

Then the geodesic is the shortest path with respect to this metric: The length $\ell(\gamma)$ of path $\gamma(t)$ is defined by

$$\ell(\gamma) \equiv \int_0^1 L(\gamma'(t);\gamma(t))dt = \int_0^1 \|\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}\|dt.$$

If $\gamma(t)$ is a path from A to B, then

$$\begin{aligned} d(A,B) \equiv \inf_\gamma \ell(\gamma) &= \ell(A\#_t B) = \|\log(A^{-1/2}BA^{-1/2})\| \\ &= \log\left(\max\{\|A^{-1/2}BA^{-1/2}\|,\ \|B^{-1/2}AB^{-1/2}\|\}\right) \\ &= \log\left(\max\{r(A^{-1}B),\ r(B^{-1}A)\}\right). \end{aligned}$$

Also the homogeneity of $\mathcal{A}^+$ implies

$$d(A,B) = d(X^*AX, X^*BX) = d(I, A^{-1/2}BA^{-1/2})$$

for invertible X. The metric d makes $\mathcal{A}^+$ a complete metric space and it is called the *Thompson (part) one* [17, 14].
Remark. Batia-Holbrook [3] shows $A\#_t B$ is also the shortest for the metric $\|A^{-\frac{1}{2}}XA^{-\frac{1}{2}}\|_2$ for 2-norm. The essential part is the inequality

$$(*) \qquad \ell(\gamma) \equiv \int_0^1 \|\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}\|_2 dt \geqq \int_0^1 \|H'(t)\|_2 dt \geqq \|\log B - \log A\|_2$$

where $H(t) = \log\gamma(t)$. Though they showed the case for 2-norm, we can show the cases $(*)$ for all unitarily invariant norms by the logarithmic-geometric mean inequality in the Hiai-Kosaki means [9]:

$$\left|\!\left|\!\left| \int_0^1 H^t X K^{1-t} dt \right|\!\right|\!\right| \geqq |||H^{1/2}XK^{1/2}|||.$$

Also, Corach et.al. [5, 2] showed the convexity for the metric: For geodesics γ and δ, the followings are equivalent:

(i) $F(t) = d(\gamma(t),\delta(t)) = \log\|\gamma(t)^{-1/2}\delta(t)\gamma(t)^{-1/2}\|$: convex:

(ii) $d(\gamma(t),\delta(t)) \leqq (1-t)d(\gamma(0),\delta(0)) + td(\gamma(1),\delta(1))$.

The above equivalence is guaranteed by the interpolationality for the path $A\#_t B$. This convexity suggests that the curvature of $\mathcal{A}^+$ is negative. In Riemannian geometry, the above convexity implies exactly the negativity of the curvature. But, in Finsler geometry, the notion of it has not been completely established yet.

3 Lawson-Lim's operator mean

Recently, Lawson-Lim [12, 13, 11] defines multivariable operator means parametrized by $t \in [0,1]$ which is an extension of Ando-Li-Mathius' geometric operator mean [1]: For a symmetric path $\mathbf{m}_t$ in Kubo-Ando means, it is defined inductively:

$(n=2)$: $\mathbf{m}[2,t](A_1,A_2) = A_1 \, \mathbf{m}_t \, A_2$

$(n+1)$: $\mathbf{m}[n+1,t](A_1,\cdots,A_{n+1}) = \lim_{r\to\infty} A_{\mathbf{m}}(r)_k$ if the limit exists

where $\begin{cases} A_{\mathbf{m}}(r)_k = \mathbf{m}[n,t]((A_{\mathbf{m}}(r-1)_j)_{j\neq k}) \\ (A_{\mathbf{m}}(1)_k = A_k). \end{cases}$

Then they showed that $\#[n,t](A_1,\cdots,A_n)$ always exists making use of the Thompson metric and that it coincides with Ando-Li-Mathius' one for $t = 1/2$. In [8], we pointed out that the arithmetic mean plays an essential part. In fact, it is expressed by the weight $\{t[n]_k\}$:

$$\nabla[n,t](A_1,\cdots,A_n) = \sum_{k=1}^{n} t[n]_k A_k.$$

Also the harmonic mean is

$$![n,t](A_1,\cdots,A_n) = \left(\sum_{k=1}^{n} t[n]_k A_k^{-1}\right)^{-1}.$$

If A_k are commuting, then the geometric mean is

$$\#[n,t](A_1,\cdots,A_n) = \prod_{k=1}^{n} A_k^{t[n]_k}.$$

Moreover we extend the convexity

$$d(A_1\#_t B_1, A_2\#_t B_2) \leqq d(A_1,B_1)\nabla_t d(A_2,B_2)$$

of the Thompson metric:

$$\begin{aligned} d(\#[n,t](A_1,\cdots,A_n), \#[n,t](B_1,\cdots,B_n) &\leqq \nabla[n,t](d(A_1,B_1),\cdots,d(A_n,B_n)) \\ &= \sum_{k=1}^{n} t[n]_k d(A_k,B_k), \end{aligned}$$

which shows the existence of the Lawson-Lim geometric mean.

Then we obtain the formulae for $t[n]_k$ in [8]:

Lemma.

$$t[n]_n = \frac{t}{1+(n-2)t}$$

$$t[n]_1 = \frac{1-t}{1+(n-2)(1-t)} = \frac{1-t}{(n-1)-(n-2)t}$$

Theorem.

(i) $$t[n]_{n-m} = \frac{m(m+1) + 2m(n-2m-2)t + \left(n^2-(4m+1)n+4m(m+1)\right)t^2}{(n-1)\,(m+(n-2m)t)\,(m+1+(n-2(m+1))t)}$$

(ii) $\displaystyle\sum_{j>n-m-1} t[n]_j = t[n]_n + \cdots + t[n]_{n-m} = \frac{(m+1)\,(m+(n-2m-1)t)}{(n-1)\,(m+1+(n-2m-2)t)}.$

Here we give another short proof of the above to show the probability distribution distribution function

$$F_n(k) = \sum_{j<k+1} t[n]_j = 1 - \frac{(n-k)\,(n-k-1+(2k-n+1)t)}{(n-1)\,(n-k+(2k-n)t)}.$$

Proof. Suppose the formula for $F_N(k)$ is valid for all k. Putting $v = F_N(k-1)$ and $w = F_N(k)$, we have

$$a_{n+1} = va_n + (1-v)b_n \quad \text{and} \quad b_{n+1} = wa_n + (1-w)b_n.$$

Thereby

$$a_{n+1} - b_{n+1} = (v-w)a_n + (w-v)b_n = (v-w)(a_n - b_n) = \cdots = (v-w)^n,$$

and hence $b_n = a_n - (v-w)^{n-1}$. Then we have $a_{n+1} - a_n = -(1-v)(v-w)^{n-1}$ and

$$a_{n+1} = a_1 - (1-v)\sum_{k=0}^{n-1}(v-w)^k \longrightarrow 1 - \frac{1-v}{1-v+w},$$

which coincides with $F_{N+1}(k)$. THerefore, the formulae $F_n(k)$ are valid by induction. Thus (ii) in Theorem is obtained by $1 - F_n(k)$ and (i) by $t[n]_k = F_n(k) - F_n(k-1)$. $\square$

Now we give the table for the density function $t[n]_k$:

$1-t$	t					
$\frac{1-t}{2-t}$	$\frac{1-t+t^2}{(2-t)(1+t)}$	$\frac{t}{1+t}$				
$\frac{1-t}{3-2t}$	$\frac{3-4t+2t^2}{3(3-2t)}$	$\frac{1+2t^2}{3(1+2t)}$	$\frac{t}{1+2t}$			
$\frac{1-t}{4-3t}$	$\frac{6-9t+4t^2}{2(4-3t)(3-t)}$	$\frac{3-2t+2t^2}{2(3-t)(2+t)}$	$\frac{1+t+4t^2}{2(2+t)(1+3t)}$	$\frac{t}{1+3t}$		
$\frac{1-t}{5-4t}$	$\frac{10-16t+7t^2}{5(5-4t)(2-t)}$	$\frac{2-2t+t^2}{5(2-t)}$	$\frac{1+t^2}{5(1+t)}$	$\frac{1+2t+7t^2}{5(1+t)(1+4t)}$	$\frac{t}{1+4t}$	
$\frac{1-t}{6-5t}$	$\frac{15-25t+11t^2}{3(5-3t)(6-5t)}$	$\frac{10-12t+5t^2}{3(4-t)(5-3t)}$	$\frac{2-t+t^2}{(4-t)(3+t)}$	$\frac{3+2t+5t^2}{3(3+t)(2+3t)}$	$\frac{1+3t+11t^2}{3(2+3t)(1+5t)}$	$\frac{t}{1+5t}$

The table for $t[n]_k$

4 Appendix 1: binomial mean $\mathbf{m}[n]_t$ for $\mathbf{m}_t$

From the viewpoint of probability distribution, a simple one-parameter extension of symmetric path can be defined inductively:

$$\begin{aligned}
\mathbf{m}[2]_t(A_1, A_2) &= A_1\, \mathbf{m}_t A_2 \\
\mathbf{m}[3]_t(A_1, A_2, A_3) &= (\,\mathbf{m}[2]_t(A_1, A_2))\, \mathbf{m}_t(\,\mathbf{m}[2]_t(A_2, A_3)) \\
\mathbf{m}[n+1]_t(A_1, \cdots, A_{n+1}) &= (\,\mathbf{m}[n]_t(A_1, \cdots, A_n))\, \mathbf{m}_t(\,\mathbf{m}[n]_t(A_2, \cdots, A_{n+1})).
\end{aligned}$$

This path is *symmetric* in the sense of

$$\mathbf{m}[n]_t(A_1, \cdots, A_n) = \mathbf{m}[n]_{1-t}(A_n, \cdots, A_1)$$

The binomial arithmetic mean is

$$\nabla[n]_t(A_1, \cdots, A_n) = \sum_{k=1}^{n} {}_{n-1}C_{k-1}(1-t)^{n-k}t^{k-1}A_k,$$

and the barycenter is the usual arithmetic mean:

$$\int_0^1 \nabla[n]_t(A_1, \cdots, A_n) = \sum_{k=1}^{n} {}_{n-1}C_{k-1}B(n-k+1, k)A_k = \frac{1}{n}\sum_{k=1}^{n} A_k$$

where $B(p, q)$ is the beta function. As in [16], a multivariable extension of *logarithmic mean*

$$L[2](a, b) = \frac{b-a}{\log b - \log a}$$

is a fascinating one. Considering

$$L[2](A, B) = \int_0^1 A\#_t B\ dt$$

holds in Kubo-Ando means, we might define

$$L[n](A_1, \cdots, A_n) = \int_0^1 \#[n]_t(A_1, \cdots, A_n)dt.$$

5 Appendix 2: chaotic power mean $\mathfrak{m}[n; r]_t$

In [7], we extend the Kubo-Ando means: A sequence $\{A_n\}$ of positive (invertible) operators is called *chaotically decreasing* and denoted by $A_n \downdownarrows$ if $A_n \gg A_{n+1}$ for all n. If a chaotically decreasing sequence $\{A_n\}$ is lower bounded; $\log A_n \geq c$ for some scalar c, then it converges to some positive (invertible) operator A, which is denoted by $A_n \downdownarrows A$. Now, following the Kubo-Ando theory, we define a *chaotic mean* $\mathfrak{m}$ as a binary operation on positive operators satisfying:

monotonicity: $A \leq C$ and $B \leq D$ imply $A\,\mathfrak{m}B \ll C\,\mathfrak{m}D$.

semicontinuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \,\mathfrak{m} B_n \downdownarrows A \,\mathfrak{m} B$.

normalization: $A \,\mathfrak{m} A = A$.

Clearly, all Kubo-Ando means are chaotic ones.

Then the path for the (quasi-arithmetic) chaotic power means is

$$A \,\mathfrak{m}_{r,t} B = A^{\frac{1}{2}} \left((1-t)A^r + tB^r\right)^{\frac{1}{r}} A^{\frac{1}{2}},$$

and it is also interpolational

$$(A \,\mathfrak{m}_{r,p} B) \,\mathfrak{m}_{r,t} (A \,\mathfrak{m}_{r,q} B) = A \,\mathfrak{m}_{r,(1-t)p+tq} B.$$

In particular, the chaotically geometric mean is

$$A \,\mathfrak{m}_{0,t} B = \exp((1-t)\log A + t\log B).$$

Now consider the metric $d_r(A,B) = \|A^r - B^r\|/r$. Then,

$$d_r(A \,\mathfrak{m}_{r,t} B, C \,\mathfrak{m}_{r,t} D) \leqq (1-t)d_r(A,C) + td_r(B,D),$$

that is, $D_r(t) = d_r(A \,\mathfrak{m}_{r,t} B, C \,\mathfrak{m}_{r,t} D)$ is convex. So we can define a multivariable chaotic operator mean parametrized by $t \in [0,1]$ like the Lawson-Lim construction. Then we have

$$(**) \qquad \mathfrak{m}[n;r]_t(A_1, \cdots, A_n) = \left(\sum_{k=1}^n t[n]_k A_k^r\right)^{1/r}.$$

In fact, to get $\mathfrak{m}[3;r]_t(A,B,C)$, we have

$$\begin{aligned} A_{\mathfrak{m}}(2)_k &= \mathfrak{m}[2;r]_t(\mathfrak{m}[2;r]_t(B,C), \mathfrak{m}[2;r]_t(A,B)) = (B \,\mathfrak{m}_{r,t} C) \;\mathfrak{m}_{r,t}\; (A \,\mathfrak{m}_{r,t} B) \\ &= \Big((1-t)\big((1-t)B^r + tC^r\big) + t\big((1-t)A^r + tB^r\big)\Big)^{1/r} \\ &= \Big(t(1-t)A^r + \big((1-t)^2 + t^2\big)B^r + (1-t)tC^r\Big)^{1/r}. \end{aligned}$$

These coefficients in each procedure coincide with those of the arithmetic Lawson-Lim mean, so that we have it converges to

$$\left(\frac{1-t}{2-t}A^r + \frac{1-t+t^2}{(2-t)(1+t)}B^r + \frac{t}{1+t}C^r\right)^{1/r} = \mathfrak{m}[3;r]_t(A,B,C).$$

Thus we have $(**)$ inductively. In particular, the chaotically geometric mean is just

$$\mathfrak{m}[n;0]_t(A_1, \cdots, A_n) = \exp\left(\sum_{k=1}^n t[n]_k \log A_k\right).$$

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