「作用素論・作用素環論」研究集会報告集

Proceedings of the Workshop on Operator Theory and Operator Algebras

平成 19 年 11 月 7 日(水)—11 月 9 日(金)

千葉大学自然科学研究棟

世話人： 渚　　勝（会場責任者・千葉大学）

藤井正俊（大阪教育大学）

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作用素論・作用素環論研究集会プログラム

１１月７日（水）［座長　藤井　正俊（大阪教育大学）］

１４：００−１４：５０　藤井　淳一（大阪教育大学）

One-parameter families of multivariable operator means

１５：００−１５：５０　柳田　昌宏（東京理科大学）

Generalizations and extensions of Furuta inequality

１６：００−１６：５０　山崎　丈明（神奈川大学）

Recent topics on Aluthge transformation

１１月８日（木）［座長　岡安　類（大阪教育大学）］

９：００−　９：５０　斎藤　吉助（新潟大学）

ノルム空間の三角不等式を巡って

１０：００−１０：５０　Mikael Pichot（東大数理）

Intermediate rank and group C* algebras

１１：００−１１：５０　中里　博（弘前大学）

行列の q−数域　−−−4 次元空間の凸体の２次元空間への射影−−

１２：００−１２：４０　林　倫弘（名古屋工大）

作用素の三角不等式の等号成立条件について

［座長　佐野　隆志（山形大学）］

１４：００−１４：５０　M. B. Ruskai（Tufts Univ.）

Complementary and Degradable Channels in Quantum Information Theory

１５：００−１５：４０　EunYoung Lee（Kyungpook National Univ., Daegu, Korea）

Concave functions and symmetric norms

１５：５０−１６：３０　Jean-Christophe Bourin (Univ. Franche-Comte)

Matrix subadditivity and monotony inequalities

１６：４０−１７：４０　森吉　仁志（慶応大学）

Twisted Index Theorem and type III factors

１８：００−　懇親会

厚生会館２階職員食堂

１１月９日（金）　［座長　伊藤　隆（群馬大学）］

９：３０－１０：２０　梶原　毅（岡山大学）

Algebraic correspondence から作られる C*-環

１０：３０－１１：２０　大野　博道（九大数理）

Quasi-orthogonal subalgebras of matrix algebras

１１：３０－１２：２０　松本　健吾（横浜市大）

Orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras

研究集会開催にあたって多くの方々の科研費から多大の援助をいただきました。この場を借りて厚く御礼申し上げます。

基盤研究(A)　「作用素環と数理物理学の総合的研究」
研究代表者　東京大学　河東泰之　(課題番号 19204015)

基盤研究(B)　「複素力学系と作用素環の研究」
研究代表者　九州大学　綿谷安男　(課題番号 19340040)

基盤研究(B)　「作用素論と作用素平均の研究」
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基盤研究(B)　「ランダム行列に基づく自由確率論と作用素環の研究」
研究代表者　東北大学　日合文雄　(課題番号 17340043)

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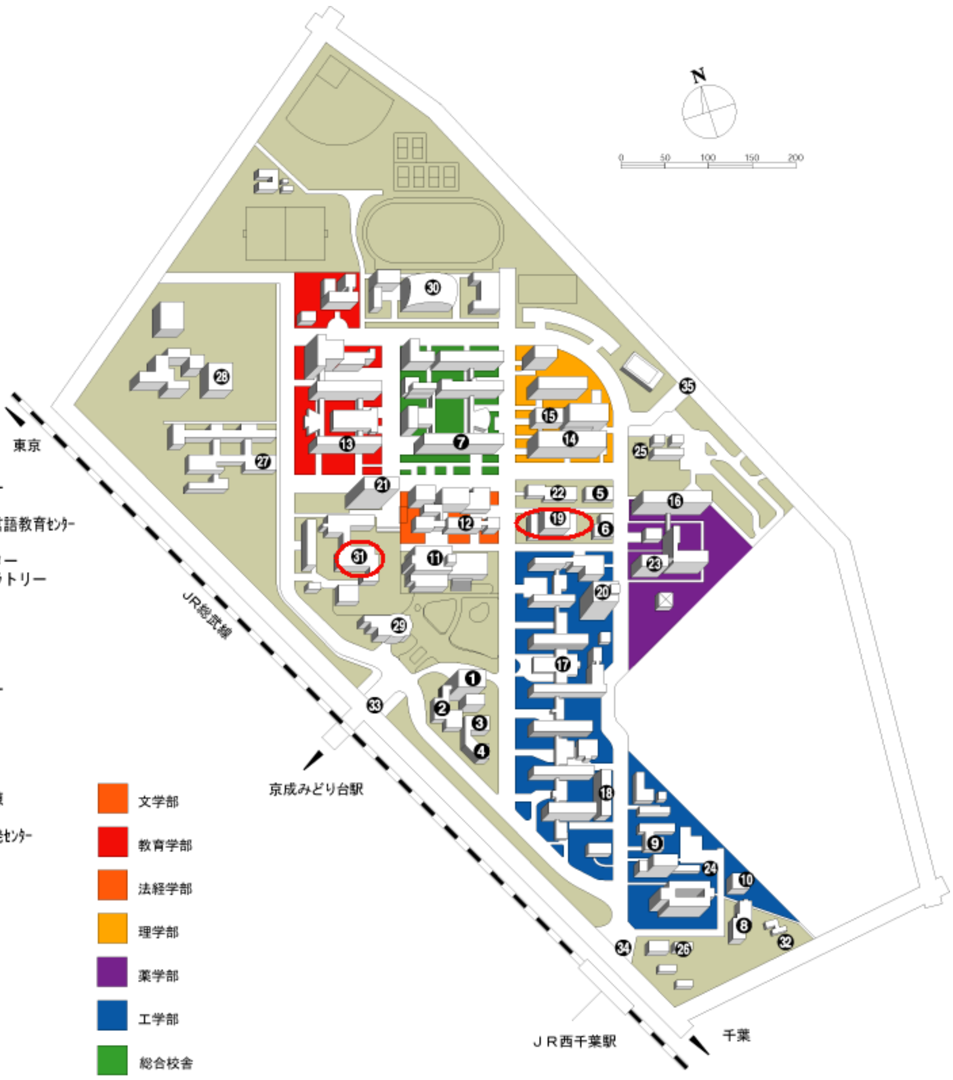

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㉝ 正門
㉞ 南門
㉟ 北門

Workshop on Operator Theory and Operator Algebras in Japan

Seminar Room at the 3rd Floor,
Graduate School of Science and Technology,
Chiba University.

November 7, 2007 (Wednesday)

14:00 -- 14:50 Jun-ichi FUJII (Osaka Kyoiku Univ.)
One-parameter families of multivariable operator means

15:00 -- 15:50 Masahiro YANAGIDA (Tokyo Univ. of Science)
Generalizations and extensions of Furuta inequality

16:00 -- 16:50 Takeaki YAMAZAKI (Kanagawa Univ.)
Recent topics on Aluthge transformation

November 8, 2007 (Thursday)

9:00 -- 9:50 Kichi-suke SAITO (Niigata Univ.)
On the triangle inequality in normed spaces

10:00 -- 10:50 Mikael Pichot (Univ. of Tokyo)
Intermediate rank and group C* algebras

11:00 -- 11:50 Hiroshi NAKAZATO (Hirosaki Univ.)
q-numerical range of a matrix

12:00 -- 12:40 Tomohiro HAYASHI (Nagoya Inst. of Tech.)
Triangular inequality for operators

14:00 -- 14:50 M. B. Ruskai (Tufts Univ.)
Complementary and Degradable Channels in Quantum Information Theory

15:00 -- 15:40 EunYoung Lee （Kyungpook National Univ., Daegu, Korea)
Concave functions and symmetric norms

15:50 -- 16:30 Jean-Christophe Bourin. (Univ. Franche-Comt'e)
Matrix subadditivity and monotony inequalities

16:40 -- 17:40 Hitoshi MORIYOSHI (Keio Univ.)
Twisted Index Theorem and type III factors

18:00 -- Reception

November 9, 2007 (Friday)

9:30 -- 10:20 Tsuyoshi KAJIWARA (Okayama Univ.)
C*-algebras associated with algebraic correspondences

10:30 -- 11:20 Hiromichi OHNO (Kyushu Univ.)
Quasi-orthogonal subalgebras of matrix algebras

11:30 -- 12:20 Kengo MATSUMOTO (Yokohama City Univ.)
Orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras

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- Administration Bureau
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- Center for Frontier Science
- Center for General Education
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- Center Gate
- Chemical Analysis Center
- Department of Student Affairs
- Division of Environmental Safety (Safety and Health Organization)
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- Faculty of Engineering
- Faculty of Letters / Faculty of Law and Economics
- Faculty of Pharmaceutical Sciences
- Faculty of Science
- General Studies Complex
- Graduate School of Humanities and Social Sciences
- Graduate School of Science and Technology, Building 1
- Graduate School of Science and Technology, Building 2
- Guest House
- Gymnasium
- Innovation Plaza, Faculty of Engineering
- Institute of Media and Information Technology
- Junior High School
- Keyaki Kaikan (University Hall)
- Kindergarten
- North Gate
- Office of Research Administration
- Radioisotope Research Center
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- Venture Business Laboratory

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Faculty of Law and Economics
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General Studies Complex

参加者リスト

M. B. Ruskai			E.-Y. Lee		
J. C. Bourin			M. Pichot		東京大学
池澤　聖海		千葉大学	伊藤　隆		群馬大学
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One-parameter families of multivariable operator means

Jun Ichi FUJII, Osaka Kyoiku University

1 Means and paths

In this talk, we use operator means, in particular, the Kubo-Ando mean [10] plays a central role: A binary operation $\mathbf{m}$ on positive operators on a Hilbert space is called the *Kubo-Ando (operator) mean* if $\mathbf{m}$ satisfies the following axioms:

$$\begin{aligned}
\textbf{monotonicity}: &\quad A \le C,\ B \le D \implies A\,\mathbf{m}\,B \le C\,\mathbf{m}\,D. \\
\textbf{semicontinuity}: &\quad A_n \downarrow A,\ B_n \downarrow B \Longrightarrow A_n\,\mathbf{m}\,B_n \downarrow A\,\mathbf{m}\,B. \\
\textbf{transformer inequality}: &\quad T^*(A\,\mathbf{m}\,B)T \le T^*AT\,\mathbf{m}\,T^*BT. \\
\textbf{normalization}: &\quad A\,\mathbf{m}\,A = A.
\end{aligned}$$

By semicontinuity, we may assume positive operators are invertible. The *representing function* $f_{\mathbf{m}}(x) = 1\,\mathbf{m}\,x$ for a Kubo-Ando mean $\mathbf{m}$ is operator monotone (concave) on $(0,\infty)$ and $\mathbf{m}$ is represented by

$$A\,\mathbf{m}\,B = A^{\frac{1}{2}} f_{\mathbf{m}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

A *path* $A\,\mathbf{m}_t B$ means parametrized operator means which is usually differentiable for t with $A\,\mathbf{m}_0 B = A$ and $A\,\mathbf{m}_1 B = 0$. A path is called *symmetric* if

$$A\,\mathbf{m}_t B = B\,\mathbf{m}_{1-t} A$$

holds for all $t \in [0,1]$. Typical example is (*quasi-arithmetic*) *power means* for $r \in [-1,1]$:

$$A \#_{r,t} B = A^{\frac{1}{2}} \left((1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\right)^{\frac{1}{r}} A^{\frac{1}{2}},$$

which include important means:

$$\begin{aligned}
\text{arithmetic mean:} &\quad A\nabla_t B = A\#_{1,t}B = (1-t)A + tB \\
\text{geometic mean:} &\quad A\#_t B = A\#_{0,t}B \equiv \lim_{\varepsilon\to 0} A\#_{\varepsilon,t}B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}} \\
\text{harmonic mean:} &\quad A\ !_t B = A\#_{-1,t}B = ((1-t)A^{-1} + tB^{-1})^{-1}.
\end{aligned}$$

Moreover the above paths are *interpolational* in the sense that

$$(A\#_{r,p}B)\#_{r,t}(A\#_{r,q}B) = A\#_{r,(1-t)p+tq}B$$

for all $p, q, t \in [0,1]$.

2 CPR geometry and Thompson metric

Here the *CPR geometry* represents the one on the Finsler manifold $\mathcal{A}^+$, the positive invertible elements in a unital C*-algebra $\mathcal{A}$, discussed by Corach-Porta-Recht [4, 5]. Corach himself reformulated it in [6]: The base manifold is $\mathcal{A}^+$ with the tangent vector bundle $\mathcal{A}^h$ (the tangent space at $A \in \mathcal{A}^+$ is $\mathcal{A}^+ - A$). For the invertible elements $\mathcal{G}$ in $\mathcal{A}$, the principal fibre bundle $\{\mathcal{G}, \mathcal{A}^+, \mathcal{U}_A, \pi_A\}$ for fixed $A \in \mathcal{A}$ is defined by

projection	$\pi_A \colon \mathcal{G} \to \mathcal{A}^+$, $G \mapsto GAG^*$
structure group	$\mathcal{U}_A = \{V \in \mathcal{G} \mid VAV^* = A\} = A^{1/2}\mathcal{U}A^{-1/2}$ with the action $L_V A = VAV^*$, which shows $\mathcal{A}^+$ is homogeneous.
fiber	$\pi_A^{-1}(B) = B^{1/2}A^{-1/2}\mathcal{U}_A = B^{1/2}\mathcal{U}A^{-1/2}$
tangent map	$\tau_G : \mathcal{A} \to \mathcal{A}^h, X \mapsto XG^* + GX^*$ (The tangent space for G is identified with $\mathcal{A}$ itself.)

Since $\mathcal{A}^+$ is a homogeneous space, we may usually assume the fixed element A is the identity I. Then the principal fibre bundle has a natural connection, which induces the covariant derivative D_t of a tangent field $X(t)$ along the curve $\gamma(t)$ in $\mathcal{A}^+$

$$D_t X = \dot{X} - \frac{1}{2}(\dot{\gamma}\gamma^{-1}X + X\gamma^{-1}\dot{\gamma}).$$

Then the geodesic equation

$$O = D_t\dot{\gamma} = \ddot{\gamma} - \dot{\gamma}\gamma^{-1}\dot{\gamma}$$

implies that the geodesic from A to B is the path of geometric Kubo-Ando means:

$$A\#_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}.$$

Moreover the above manifold $\mathcal{A}^+$ is the Finsler space with a Finsler metric

$$L(X; A) = \|X\|_A = \|A^{-1/2}XA^{-1/2}\| :$$

Since $\|X\|_A$ is an equvalent norm to $\|X\|$, it is a Finsler metric if

Finsler condition: $\|P_t X\|_{\gamma(t)} = \|X\|_{\gamma(0)}$

holds for all curves γ and parallel transports P_t along γ [15]. In this case,

$$P_t X = \Gamma(t)\Gamma(0)^{-1}X(\Gamma(0)^*)^{-1}\Gamma(t)^*$$

for a parallel lift Γ for γ. For the case $A = I$ for simplicity, a lift Γ satisfies

$$\gamma = \pi_I(\Gamma) = \Gamma\Gamma^*,$$

so that $U_t = \gamma(t)^{-1/2}\Gamma(t)$ defines a unitary for each t. Therefore we show the Finsler condition by

$$\|P_t X\|_{\gamma(t)} = \|U_t U_0^* \gamma(0)^{-1/2} X \gamma(0)^{-1/2} U_0 U_t^*\| = \|\gamma(0)^{-1/2} X \gamma(0)^{-1/2}\| = \|X\|_{\gamma(0)}.$$

Then the geodesic is the shortest path with respect to this metric: The length $\ell(\gamma)$ of path $\gamma(t)$ is defined by

$$\ell(\gamma) \equiv \int_0^1 L(\gamma'(t);\gamma(t))dt = \int_0^1 \|\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}\|dt.$$

If $\gamma(t)$ is a path from A to B, then

$$\begin{aligned} d(A,B) &\equiv \inf_\gamma \ell(\gamma) = \ell(A\#_t B) = \|\log(A^{-1/2}BA^{-1/2})\| \\ &= \log\left(\max\{\|A^{-1/2}BA^{-1/2}\|,\ \|B^{-1/2}AB^{-1/2}\|\}\right) \\ &= \log\left(\max\{r(A^{-1}B),\ r(B^{-1}A)\}\right). \end{aligned}$$

Also the homogeneity of $\mathcal{A}^+$ implies

$$d(A,B) = d(X^*AX, X^*BX) = d(I, A^{-1/2}BA^{-1/2})$$

for invertible X. The metric d makes $\mathcal{A}^+$ a complete metric space and it is called the *Thompson (part) one* [17, 14].
Remark. Batia-Holbrook [3] shows $A\#_t B$ is also the shortest for the metric $\|A^{-\frac{1}{2}}XA^{-\frac{1}{2}}\|_2$ for 2-norm. The essential part is the inequality

$$(*) \qquad \ell(\gamma) \equiv \int_0^1 \|\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}\|_2 dt \geqq \int_0^1 \|H'(t)\|_2 dt \geqq \|\log B - \log A\|_2$$

where $H(t) = \log\gamma(t)$. Though they showed the case for 2-norm, we can show the cases $(*)$ for all unitarily invariant norms by the logarithmic-geometric mean inequality in the Hiai-Kosaki means [9]:

$$\left|\!\left|\!\left| \int_0^1 H^t X K^{1-t} dt \right|\!\right|\!\right| \geqq |||H^{1/2}XK^{1/2}|||.$$

Also, Corach et.al. [5, 2] showed the convexity for the metric: For geodesics γ and δ, the followings are equivalent:

(i) $F(t) = d(\gamma(t),\delta(t)) = \log\|\gamma(t)^{-1/2}\delta(t)\gamma(t)^{-1/2}\|$: convex:

(ii) $d(\gamma(t),\delta(t)) \leqq (1-t)d(\gamma(0),\delta(0)) + td(\gamma(1),\delta(1))$.

The above equivalence is guaranteed by the interpolationality for the path $A\#_t B$. This convexity suggests that the curvature of $\mathcal{A}^+$ is negative. In Riemannian geometry, the above convexity implies exactly the negativity of the curvature. But, in Finsler geometry, the notion of it has not been completely established yet.

3 Lawson-Lim's operator mean

Recently, Lawson-Lim [12, 13, 11] defines multivariable operator means parametrized by $t \in [0,1]$ which is an extension of Ando-Li-Mathius' geometric operator mean [1]: For a symmetric path $\mathbf{m}_t$ in Kubo-Ando means, it is defined inductively:

$$(n=2): \quad \mathbf{m}[2,t](A_1,A_2) = A_1\, \mathbf{m}_t\, A_2$$

$$(n+1): \quad \mathbf{m}[n+1,t](A_1,\cdots,A_{n+1}) = \lim_{r\to\infty} A_{\mathbf{m}}(r)_k \ \text{if the limit exists}$$

$$\text{where} \quad \begin{cases} A_{\mathbf{m}}(r)_k = \mathbf{m}[n,t]((A_{\mathbf{m}}(r-1)_j)_{j\neq k}) \\ (A_{\mathbf{m}}(1)_k = A_k). \end{cases}$$

Then they showed that $\#[n,t](A_1,\cdots,A_n)$ always exists making use of the Thompson metric and that it coincides with Ando-Li-Mathius' one for $t=1/2$. In [8], we pointed out that the arithmetic mean plays an essential part. In fact, it is expressed by the weight $\{t[n]_k\}$:

$$\nabla[n,t](A_1,\cdots,A_n) = \sum_{k=1}^{n} t[n]_k A_k.$$

Also the harmonic mean is

$$![n,t](A_1,\cdots,A_n) = \left(\sum_{k=1}^{n} t[n]_k A_k^{-1}\right)^{-1}.$$

If A_k are commuting, then the geometric mean is

$$\#[n,t](A_1,\cdots,A_n) = \prod_{k=1}^{n} A_k^{t[n]_k}.$$

Moreover we extend the convexity

$$d(A_1\#_t B_1, A_2\#_t B_2) \leqq d(A_1,B_1)\nabla_t d(A_2,B_2)$$

of the Thompson metric:

$$\begin{aligned} d(\#[n,t](A_1,\cdots,A_n), \#[n,t](B_1,\cdots,B_n) &\leqq \nabla[n,t](d(A_1,B_1),\cdots,d(A_n,B_n)) \\ &= \sum_{k=1}^{n} t[n]_k d(A_k,B_k), \end{aligned}$$

which shows the existence of the Lawson-Lim geometric mean.

Then we obtain the formulae for $t[n]_k$ in [8]:

Lemma.

$$t[n]_n = \frac{t}{1+(n-2)t}$$

$$t[n]_1 = \frac{1-t}{1+(n-2)(1-t)} = \frac{1-t}{(n-1)-(n-2)t)}$$

Theorem.

$$\text{(i)} \quad t[n]_{n-m} = \frac{m(m+1)+2m(n-2m-2)t+\left(n^2-(4m+1)n+4m(m+1)\right)t^2}{(n-1)\,(m+(n-2m)t)\,(m+1+(n-2(m+1))t)}$$

$$\text{(ii)} \quad \sum_{j>n-m-1} t[n]_j = t[n]_n + \cdots + t[n]_{n-m} = \frac{(m+1)\,(m+(n-2m-1)t)}{(n-1)\,(m+1+(n-2m-2)t)}.$$

Here we give another short proof of the above to show the probability distribution distribution function

$$F_n(k) = \sum_{j<k+1} t[n]_j = 1 - \frac{(n-k)\,(n-k-1+(2k-n+1)t)}{(n-1)\,(n-k+(2k-n)t)}.$$

Proof. Suppose the formula for $F_N(k)$ is valid for all k. Putting $v = F_N(k-1)$ and $w = F_N(k)$, we have

$$a_{n+1} = va_n + (1-v)b_n \quad \text{and} \quad b_{n+1} = wa_n + (1-w)b_n.$$

Thereby

$$a_{n+1} - b_{n+1} = (v-w)a_n + (w-v)b_n = (v-w)(a_n - b_n) = \cdots = (v-w)^n,$$

and hence $b_n = a_n - (v-w)^{n-1}$. Then we have $a_{n+1} - a_n = -(1-v)(v-w)^{n-1}$ and

$$a_{n+1} = a_1 - (1-v)\sum_{k=0}^{n-1}(v-w)^k \longrightarrow 1 - \frac{1-v}{1-v+w},$$

which coincides with $F_{N+1}(k)$. THerefore, the formulae $F_n(k)$ are valid by induction. Thus (ii) in Theorem is obtained by $1 - F_n(k)$ and (i) by $t[n]_k = F_n(k) - F_n(k-1)$. $\square$

Now we give the table for the density function $t[n]_k$:

$1-t$	t					
$\frac{1-t}{2-t}$	$\frac{1-t+t^2}{(2-t)(1+t)}$	$\frac{t}{1+t}$				
$\frac{1-t}{3-2t}$	$\frac{3-4t+2t^2}{3(3-2t)}$	$\frac{1+2t^2}{3(1+2t)}$	$\frac{t}{1+2t}$			
$\frac{1-t}{4-3t}$	$\frac{6-9t+4t^2}{2(4-3t)(3-t)}$	$\frac{3-2t+2t^2}{2(3-t)(2+t)}$	$\frac{1+t+4t^2}{2(2+t)(1+3t)}$	$\frac{t}{1+3t}$		
$\frac{1-t}{5-4t}$	$\frac{10-16t+7t^2}{5(5-4t)(2-t)}$	$\frac{2-2t+t^2}{5(2-t)}$	$\frac{1+t^2}{5(1+t)}$	$\frac{1+2t+7t^2}{5(1+t)(1+4t)}$	$\frac{t}{1+4t}$	
$\frac{1-t}{6-5t}$	$\frac{15-25t+11t^2}{3(5-3t)(6-5t)}$	$\frac{10-12t+5t^2}{3(4-t)(5-3t)}$	$\frac{2-t+t^2}{(4-t)(3+t)}$	$\frac{3+2t+5t^2}{3(3+t)(2+3t)}$	$\frac{1+3t+11t^2}{3(2+3t)(1+5t)}$	$\frac{t}{1+5t}$

The table for $t[n]_k$

4 Appendix 1: binomial mean $\mathbf{m}[n]_t$ for $\mathbf{m}_t$

From the viewpoint of probability distribution, a simple one-parameter extension of symmetric path can be defined inductively:

$$\begin{aligned}\mathbf{m}[2]_t(A_1, A_2) &= A_1\,\mathbf{m}_t A_2\\ \mathbf{m}[3]_t(A_1, A_2, A_3) &= (\,\mathbf{m}[2]_t(A_1, A_2))\,\mathbf{m}_t(\,\mathbf{m}[2]_t(A_2, A_3))\\ \mathbf{m}[n+1]_t(A_1, \cdots, A_{n+1}) &= (\,\mathbf{m}[n]_t(A_1, \cdots, A_n))\,\mathbf{m}_t(\,\mathbf{m}[n]_t(A_2, \cdots, A_{n+1})).\end{aligned}$$

This path is *symmetric* in the sense of

$$\mathbf{m}[n]_t(A_1, \cdots, A_n) = \mathbf{m}[n]_{1-t}(A_n, \cdots, A_1)$$

The binomial arithmetic mean is

$$\nabla[n]_t(A_1, \cdots, A_n) = \sum_{k=1}^{n} {}_{n-1}C_{k-1}(1-t)^{n-k}t^{k-1}A_k,$$

and the barycenter is the usual arithmetic mean:

$$\int_0^1 \nabla[n]_t(A_1, \cdots, A_n) = \sum_{k=1}^{n} {}_{n-1}C_{k-1}B(n-k+1, k)A_k = \frac{1}{n}\sum_{k=1}^{n} A_k$$

where $B(p, q)$ is the beta function. As in [16], a multivariable extension of *logarithmic mean*

$$L[2](a, b) = \frac{b-a}{\log b - \log a}$$

is a fascinating one. Considering

$$L[2](A, B) = \int_0^1 A\#_t B\ dt$$

holds in Kubo-Ando means, we might define

$$L[n](A_1, \cdots, A_n) = \int_0^1 \#[n]_t(A_1, \cdots, A_n)dt.$$

5 Appendix 2: chaotic power mean $\mathfrak{m}[n;r]_t$

In [7], we extend the Kubo-Ando means: A sequence $\{A_n\}$ of positive (invertible) operators is called *chaotically decreasing* and denoted by $A_n \downdownarrows$ if $A_n \gg A_{n+1}$ for all n. If a chaotically decreasing sequence $\{A_n\}$ is lower bounded; $\log A_n \geq c$ for some scalar c, then it converges to some positive (invertible) operator A, which is denoted by $A_n \downdownarrows A$. Now, following the Kubo-Ando theory, we define a *chaotic mean* $\mathfrak{m}$ as a binary operation on positive operators satisfying:

monotonicity: $A \leq C$ and $B \leq D$ imply $A\,\mathfrak{m}B \ll C\,\mathfrak{m}D$.

semicontinuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \,\mathfrak{m} B_n \downarrow\downarrow A \,\mathfrak{m} B$.

normalization: $A \,\mathfrak{m} A = A$.

Clearly, all Kubo-Ando means are chaotic ones.

Then the path for the (quasi-arithmetic) chaotic power means is

$$A \,\mathfrak{m}_{r,t} B = A^{\frac{1}{2}} \left((1-t)A^r + tB^r\right)^{\frac{1}{r}} A^{\frac{1}{2}},$$

and it is also interpolational

$$(A \,\mathfrak{m}_{r,p} B) \,\mathfrak{m}_{r,t} (A \,\mathfrak{m}_{r,q} B) = A \,\mathfrak{m}_{r,(1-t)p+tq} B.$$

In particular, the chaotically geometric mean is

$$A \,\mathfrak{m}_{0,t} B = \exp((1-t)\log A + t \log B).$$

Now consider the metric $d_r(A,B) = \|A^r - B^r\|/r$. Then,

$$d_r(A \,\mathfrak{m}_{r,t} B, C \,\mathfrak{m}_{r,t} D) \leqq (1-t) d_r(A,C) + t d_r(B,D),$$

that is, $D_r(t) = d_r(A \,\mathfrak{m}_{r,t} B, C \,\mathfrak{m}_{r,t} D)$ is convex. So we can define a multivariable chaotic operator mean parametrized by $t \in [0,1]$ like the Lawson-Lim construction. Then we have

$$(**) \qquad \mathfrak{m}[n;r]_t(A_1, \cdots, A_n) = \left(\sum_{k=1}^{n} t[n]_k A_k^r\right)^{1/r}.$$

In fact, to get $\mathfrak{m}[3;r]_t(A,B,C)$, we have

$$\begin{aligned} A_{\mathfrak{m}}(2)_k &= \mathfrak{m}[2;r]_t(\mathfrak{m}[2;r]_t(B,C), \mathfrak{m}[2;r]_t(A,B)) = (B \,\mathfrak{m}_{r,t} C) \,\mathfrak{m}_{r,t}\, (A \,\mathfrak{m}_{r,t} B) \\ &= \Big((1-t)\big((1-t)B^r + tC^r\big) + t\big((1-t)A^r + tB^r\big)\Big)^{1/r} \\ &= \Big(t(1-t)A^r + \big((1-t)^2 + t^2\big)B^r + (1-t)tC^r\Big)^{1/r}. \end{aligned}$$

These coefficients in each procedure coincide with those of the arithmetic Lawson-Lim mean, so that we have it converges to

$$\left(\frac{1-t}{2-t}A^r + \frac{1-t+t^2}{(2-t)(1+t)}B^r + \frac{t}{1+t}C^r\right)^{1/r} = \mathfrak{m}[3;r]_t(A,B,C).$$

Thus we have $(**)$ inductively. In particular, the chaotically geometric mean is just

$$\mathfrak{m}[n;0]_t(A_1, \cdots, A_n) = \exp\left(\sum_{k=1}^{n} t[n]_k \log A_k\right).$$

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Generalizations and extensions of Furuta inequality

東京理科大・理　柳田 昌宏 (Masahiro Yanagida)

Tokyo University of Science

1 Furuta inequality

In what follows, an operator means a bounded linear operator on a Hilbert space H. An operator T is positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$.

The following result was published in 1987 and has been applied in a lot of papers.

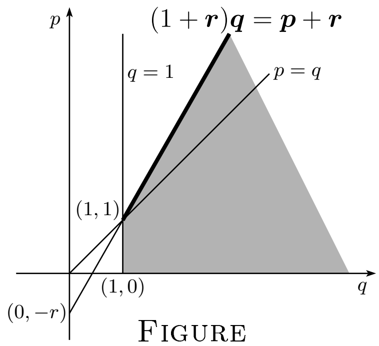
Theorem F (Furuta inequality [2]).
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



FIGURE

We remark that Löwner-Heinz theorem "$A \geq B \geq 0 \Longrightarrow A^\alpha \geq B^\alpha$ *for any* $\alpha \in [0,1]$" is the case $r = 0$ of Theorem F. Other proofs are given in [1][4] and also an elementary one-page proof in [3]. It is shown in [5] that the domain of p, q and r in Theorem F is the best possible for the inequalities (i) and (ii) to hold under the assumption $A \geq B$.

2 M. Uchiyama's results

A real-valued continuous function f defined on an interval $I \subseteq \mathbb{R}$ is operator monotone if $A \geq B$ implies $f(A) \geq f(B)$ for any self-adjoint operators A and B such that $\sigma(A), \sigma(B) \subseteq I$. Let $\mathbb{P}_+[a,b)$ and $\mathbb{P}_+^{-1}[a,b)$ be the families of functions defined as follows:

- $\mathbb{P}_+[a,b)$ is the set of all non-negative operator monotone functions defined on $[a,b)$.
- $\mathbb{P}_+^{-1}[a,b)$ is the set of all increasing functions h defined on $[a,b)$ such that $h([a,b)) = [0,\infty)$ and its inverse h^{-1} is operator monotone on $[0,\infty)$.

Uchiyama [6] introduced a new concept of majorization and showed a quite interesting result on operator monotone functions.

Definition ([6]). Let h be a non-decreasing function on I and k an increasing function on J. Then

$$h \preceq k \Longleftrightarrow J \subseteq I \text{ and the composite } h \circ k^{-1} \text{ is operator monotone on } k(J).$$

Theorem A (Product theorem [6]). *Suppose $-\infty < a < b \leq \infty$. Then*

$$\mathbb{P}_+[a,b) \cdot \mathbb{P}_+^{-1}[a,b) \subseteq \mathbb{P}_+^{-1}[a,b), \quad \mathbb{P}_+^{-1}[a,b) \cdot \mathbb{P}_+^{-1}[a,b) \subseteq \mathbb{P}_+^{-1}[a,b).$$

Further, let $h_i \in \mathbb{P}_+^{-1}[a,b)$ for $1 \leq i \leq m$, and let g_j be a finite product of functions in $\mathbb{P}_+[a,b)$ for $1 \leq j \leq n$. Then for $\psi_i, \phi_j \in \mathbb{P}_+[0,\infty)$

$$\prod_{i=1}^m h_i(t) \prod_{j=1}^n g_j(t) \in \mathbb{P}_+^{-1}[a,b), \quad \prod_{i=1}^m \psi_i(h_i(t)) \prod_{j=1}^n \phi_j(g_j(t)) \preceq \prod_{i=1}^m h_i(t) \prod_{j=1}^n g_j(t).$$

He also obtained generalizations of Theorem F as applications of Theorem A.

Proposition B ([6]). *Let $h \in \mathbb{P}_+^{-1}[0,\infty)$, and let $\tilde{h}$ be a non-negative non-decreasing function on $[0,\infty)$ such that $\tilde{h} \preceq h$. Let g be a finite product of functions in $\mathbb{P}_+[0,\infty)$. Then for the function φ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$*

$$A \geq B \geq 0 \Longrightarrow \begin{cases} \varphi(g(B)^{\frac{1}{2}} h(A) g(B)^{\frac{1}{2}}) \geq g(B)^{\frac{1}{2}} \tilde{h}(A) g(B)^{\frac{1}{2}}, \\ \varphi(g(A)^{\frac{1}{2}} h(B) g(A)^{\frac{1}{2}}) \leq g(A)^{\frac{1}{2}} \tilde{h}(B) g(A)^{\frac{1}{2}}. \end{cases}$$

Theorem C ([6]). *Let $h \in \mathbb{P}_+^{-1}[0,\infty)$, and let $\tilde{h}$ be a non-negative non-decreasing function on $[0,\infty)$ such that $\tilde{h} \preceq h$. Let g_n be a finite product of functions in $\mathbb{P}_+[0,\infty)$ for each n, and let the sequence $\{g_n\}$ converge pointwise to g. Suppose $g \neq 0$ and $g(0+) = g(0)$. Then for the function φ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$*

$$A \geq B \geq 0 \Longrightarrow \begin{cases} \varphi(g(B)^{\frac{1}{2}} h(A) g(B)^{\frac{1}{2}}) \geq g(B)^{\frac{1}{2}} \tilde{h}(A) g(B)^{\frac{1}{2}}, \\ \varphi(g(A)^{\frac{1}{2}} h(B) g(A)^{\frac{1}{2}}) \leq g(A)^{\frac{1}{2}} \tilde{h}(B) g(A)^{\frac{1}{2}}. \end{cases}$$

3 Our results

We obtain extensions of Proposition B and Theorem C. In fact, Theorem 2 yields Theorem C by putting $\hat{h}(t) = t$ and $g(t) = 1$.

Proposition 1. *Let f_i be non-negative non-decreasing functions on $[0,\infty)$ and $g_j(t) = \prod_{i=1}^{j} f_i(t)$. Let h, $\hat{h}$ and $\tilde{h}$ be non-negative non-decreasing functions on $[0,\infty)$ such that $f_n(t) \preceq \hat{h}(t)g_{n-1}(t)$, $\tilde{h} \preceq h$ and $h(0)g_{n-1}(0) = 0$. Then for the functions ψ_j and φ_j defined by*

$$\psi_j(h(t)g_j(t)) = \hat{h}(t)g_j(t) \quad \text{and} \quad \varphi_j(h(t)g_j(t)) = \tilde{h}(t)g_j(t),$$

if $A, B \geq 0$ satisfy

$$\psi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}}) \geq \hat{h}(B)g_{n-1}(B),$$

then

$$\varphi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) \geq f_n(B)^{\frac{1}{2}}\varphi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}})f_n(B)^{\frac{1}{2}}$$

holds. Furthermore,

$$\psi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) \geq \hat{h}(B)g_n(B)$$

holds if $\hat{h} \preceq h$.

Theorem 2. *Let $\hat{h} \in \mathbb{P}_+^{-1}[0,\infty)$, and let h and $\tilde{h}$ be non-negative non-decreasing functions on $[0,\infty)$ such that $\tilde{h} \preceq h$ and $\hat{h} \preceq h$. Let g be a finite product of functions in $\mathbb{P}_+[0,\infty) \cup \mathbb{P}_+^{-1}[0,\infty)$ and γ_n a finite product of functions in $\mathbb{P}_+[0,\infty)$ for each n, and let the sequence $\{g(t)\gamma_n(t)\}$ converge pointwise to $\bar{g}(t)$. Suppose $\bar{g} \neq 0$ and $\bar{g}(0+) = \bar{g}(0)$. Then for the functions ψ, $\bar{\psi}$, φ and $\bar{\varphi}$ defined by*

$$\psi(h(t)g(t)) = \hat{h}(t)g(t), \quad \bar{\psi}(h(t)\bar{g}(t)) = \hat{h}(t)\bar{g}(t),$$

$$\varphi(h(t)g(t)) = \tilde{h}(t)g(t), \quad \bar{\varphi}(h(t)\bar{g}(t)) = \tilde{h}(t)\bar{g}(t),$$

if $A, B \geq 0$ satisfy

$$\psi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \geq \hat{h}(B)g(B),$$

then

$$g(B)^{\frac{1}{2}}\bar{\varphi}(\bar{g}(B)^{\frac{1}{2}}h(A)\bar{g}(B)^{\frac{1}{2}})g(B)^{\frac{1}{2}} \geq \bar{g}(B)^{\frac{1}{2}}\varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})\bar{g}(B)^{\frac{1}{2}}$$

and

$$\bar{\psi}(\bar{g}(B)^{\frac{1}{2}}h(A)\bar{g}(B)^{\frac{1}{2}}) \geq \hat{h}(B)\bar{g}(B)$$

holds.

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Recent topics on Aluthge transform

山崎　丈明 (神奈川大)
Takeaki Yamazaki (Kanagawa University)

Abstract

Aluthge transform is a good tool for studying some operator classes. Especially, it is used in the reserach on semi-hyponormal and p-hyponormal operators by many authors. Recently, many authors are interested in Aluthge sequence (i.e., operator sequence of iterated Aluthge transform). In this talk, we introduce recent topics on Aluthge transform.

1. Introduction

The research on some operator classes which include the class of normal operators on a complex Hilbert space $\mathcal{H}$ is developed by many authors. Especially, the classes of normal, quasinormal, subnormal, hyponormal and paranormal operators are very famous. It is well known that every normal operator has the spectral decomposition, and then the structure of normal operators is well-known. The structure of quasinormal operators is also known as a direct sum of normal and operator valued weighted shift in [5]. It is also well known that every subnormal operator has a nontrivial invariant subspace in [6]. On the other hand, there are a lot of problem about hyponormal operators, for example, whether any hyponormal operator has nontrivial invariant subspace or not.

In 1990, A. Aluthge [1] defined an operator transform in the research on hyponormal operators (we call it Aluthge transform). It is a good tool for the study on hyponormal operators and used in many paper. Especially, we call the operator sequence of iterated Aluthge transform of an operator Aluthge sequence, and recently some papers on Aluthge sequence have been published in [2, 3, 4, 7, 8, 13, 18, 20]. In this talk, we shall introduce recent topics on Aluthge transform.

Here, we shall introduce the definitions needed in the talk. Let $\mathcal{H}$ and $B(\mathcal{H})$ be a complex Hilbert space and algebra of all bounded linear operators on $\mathcal{H}$, respectively. We shall introduce the definition of Aluthge transform as follows:

Definition 1 (Aluthge transform [1]). *Let $T = U|T|$ be the polar decomposition of an operator $T \in B(\mathcal{H})$. Then the Aluthge transform $\Delta(T)$ of T is defined as follows:*

$$\Delta(T) = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}.$$

Moreover, for each nonnegative integer n, the n-th Aluthge transform $\Delta^n(T)$ of T is defined as follows:

$$\Delta^n(T) = \Delta(\Delta^{n-1}(T)), \qquad \Delta^0(T) = T.$$

Here we call the operator sequence $\{\Delta^n(T)\}_{n=0}^{\infty}$ Aluthge sequence.

Remark.

(i) The symbol of Aluthge transform has been used in $\widetilde{T}$, but in this talk we write $\Delta(T)$ as Aluthge transform of T.

(ii) Aluthge transform does not depend on the partial isometry part of the polar decomposition of an operator.

Example 1. (i) *Let T be a unilateral weighted shift on $l^2(\mathbb{N})$ such that*

$$T(f_1, f_2, f_3, ...) = (0, \alpha_1 f_1, \alpha_2 f_2, ...).$$

Then

$$\Delta(T)(f_1, f_2, f_3, ...) = (0, \sqrt{\alpha_1\alpha_2} f_1, \sqrt{\alpha_2\alpha_3} f_2, ...).$$

(ii) *Let T be a bilateral weighted shift on $l^2(\mathbb{Z})$ such that*

$$T(..., f_{-2}, f_{-1}, \boxed{f_0}, f_1, f_2, ...) = (..., \alpha_{-3}f_{-3}, \alpha_{-2}f_{-2}, \boxed{\alpha_{-1}f_{-1}}, \alpha_0 f_0, \alpha_1 f_1, ...).$$

Then

$$\begin{aligned}&\Delta(T)(..., f_{-2}, f_{-1}, \boxed{f_0}, f_1, f_2, ...)\\ &\quad = (..., \sqrt{\alpha_{-3}\alpha_{-2}} f_{-3}, \sqrt{\alpha_{-2}\alpha_{-1}} f_{-2}, \boxed{\sqrt{\alpha_{-1}\alpha_0} f_{-1}}, \sqrt{\alpha_0\alpha_1} f_0, \sqrt{\alpha_1\alpha_2} f_1, ...).\end{aligned}$$

Next, we shall introduce some operator classes which will be used in the talk.

Definition 2. *Let $T \in B(\mathcal{H})$.*

(i) *T is normal $\Longleftrightarrow T^*T = TT^*$,*
(ii) *T is quasinormal $\Longleftrightarrow T^*TT = TT^*T$,*
(iii) *T is subnormal $\Longleftrightarrow T$ has a normal extension,*
(iv) *for $p > 0$, T is p-hyponormal $\Longleftrightarrow (T^*T)^p \geq (TT^*)^p$.*

Especially, we call 1-hyponormal hyponormal, simply, and also we call $\frac{1}{2}$-hyponormal semi-hyponormal. The following inclusion relations are well known and they are proper.

{Normal} $\subset$ {quasinormal} $\subset$ {subnormal} $\subset$ {hyponormal} $\subset$ {semi-hyponormal}.

We remark that every quasinormal operators is fixed point for Aluthge transform as follows:

Proposition 2 ([12]). *Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition. Then the following conditions are equivalent:*

(i) T *is quasinormal,*
(ii) $|T|U = U|T|$,
(iii) $\Delta(T) = T$.

Lastly, we shall introduce definitions related to the operator norm. Let $\sigma(T)$ be the spectrum of T, and let $W(T) = \{(Tx, x) : \|x\| = 1\}$ be the numerical range of T. The following relation between the spectrum and numerical range is well known.

$$\mathrm{co}\sigma(T) \subseteq \overline{W(T)}, \tag{1.1}$$

where co$X$ means the convex hull of a subset $X \in \mathbb{C}$.

Definition 3. *Let $T \in B(\mathcal{H})$.*

(i) $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$ *(operator norm),*
(ii) $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$ *(numerical radius),*
(iii) $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ *(spectral radius).*

The following relations are very famous:

$$r(T) \le w(T) \le \|T\| \text{ and } \frac{1}{2}\|T\| \le w(T) \le \|T\|. \tag{1.2}$$

2. The First Aluthge transform

In this section, we shall introduce some properties of Aluthge transform without ones of Aluthge sequence. By the definition of Aluthge transform, we can obtain the following proposition, easily.

Proposition 3. *Let $T \in B(\mathcal{H})$. Then the following assertions hold:*

(i) $\sigma(T) = \sigma(\Delta(T))$,
(ii) $\|\Delta(T)\| \le \|T^2\|^{\frac{1}{2}} \le \|T\|$.

Aluthge transform has been defined in the paper discussed on hyponormal operators by A. Aluthge [1] as follows:

Theorem 4 ([1]). *For $p \in (0, 1]$, let $T \in B(\mathcal{H})$ be a p-hyponormal operator. Then the following assertions hold:*

(i) $\Delta(T)$ *is* $(p + \frac{1}{2})$*-hyponormal if* $0 < p \le \frac{1}{2}$,
(ii) $\Delta(T)$ *is hyponormal if* $\frac{1}{2} < p \le 1$.

Especially, if T is semi-hyponormal, then $\Delta(T)$ is hyponormal. Hence by (i) of Proposition 3, Aluthge transform of an operator T has good properties than T with the same spectrum. So we can study on some spectral properties of semi-hyponormal operators by using ones of hyponormal operators via Aluthge transform. Hence we can expect that Aluthge transform is a useful tool for studying operator theory. For example, Aluthge transform may contribute to nontrivial invariant subspace problem as follows:

Proposition 5 ([12]). *If $T \in B(\mathcal{H})$ and $\Delta(T)$ has a nontrivial invariant subspace, then so does T.*

Next, we shall introduce properties of numerical range of Aluthge transform. Firstly, the following result has been obtained:

Theorem 6 ([12, 21, 19]). *Let $T \in B(\mathcal{H})$. Then $\overline{W(\Delta(T))} \subseteq \overline{W(T)}$ holds. Especially, $w(\Delta(T)) \leq w(T)$ holds.*

Theorem 6 has been shown in [12] in the case of 2-by-2 matrices. Then n-by-n matrices case was shown in [21], and then [19] shows it in the general case. Now Theorem 6 is extended to C-numerical range version in the case that C is rank one or self-adjoint matrix in [9]. Another extension of Theorem 6 is shown in [15, 16] which shows some relations for ρ-radius between a matrix T and $\Delta(T)$. Recently, as an application of Aluthge transform, it has been obtained a more exact estimation of numerical range as follows:

Theorem 7. [22] *For any $T \in B(\mathcal{H})$,*

$$\frac{1}{2}\|T\| \leq w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}w(\Delta(T)).$$

Relation between operator norm and numerical range is known as $w(T) \leq \|T\|$ (1.2), but Kittaneh [14] has refined it to $w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}} \leq \|T\|$. Theorem 7 is a more exact estimation than Kittaneh's result by the following inequalities:

$$w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}w(\Delta(T)) \leq \frac{1}{2}\|T\| + \frac{1}{2}\|\Delta(T)\| \leq \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}} \leq \|T\|.$$

Lastly, as a function $\Delta(\cdot)$ on $B(\mathcal{H})$, the following results are shown. Let $R(\Delta) \subseteq B(\mathcal{H})$ be the range of Aluthge transform.

Theorem 8 ([4]). *Let T be an operator with closed range. Then the Aluthge transform $\Delta(\cdot)$ is continuous at T.*

Theorem 9 ([10, 11]). *Let $\mathcal{H} = \mathbb{C}^p$ for $p > 2$. Then $R(\Delta)$ is neither closed nor dense in $B(\mathcal{H})$.*

Theorem 10 ([10]). *Let $\mathcal{H}$ be a complex separable infinite dimensional Hilbert space. Then $R(\Delta)$ is neither closed nor dense in norm topology but strongly dense in $B(\mathcal{H})$.*

3. Aluthge sequence

In this section, we shall introduce properties of Aluthge sequence. On the properties of Aluthge sequence, the following formula has been shown in [20], firstly.

Theorem 11 ([20]). *For each $T \in B(\mathcal{H})$, $\lim_{n\to\infty} \|\Delta^n(T)\| = r(T)$ holds.*

The simplified proof is given in [18]. Moreover, in the matrix case, more generalization is shown as follows: For a n-by-n matrix T, arrange the eigenvalues $\lambda(T)$ of T in modulus non-increasing order, counting multiplicities:

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots \geq |\lambda_n(T)| \geq 0,$$

and the singular values $s(T)$ of T, that is, the eigenvalues of $|T|$, non-increasing order, counting multiplicities:

$$s_1(T) \geq s_2(T) \geq \cdots \geq s_n(T) \geq 0.$$

Then the following formula holds:

Theorem 12 ([2]). *For each n-by-n matrix T, $\lim_{n\to\infty} S_i(\Delta^n(T)) = |\lambda_i(T)|$ for $i = 1, 2, ..., n$.*

As a parallel result to the above ones, Ando gives a characterization of the convex hull of spectrum radius as follows:

Theorem 13 ([2]). *For each $T \in B(\mathcal{H})$, $\bigcap_n \overline{W(\Delta^n(T))} = co\sigma(T)$.*

Moreover. the following characterization is obtained:

Theorem 14 ([2]). *$co\sigma(T) = \overline{W(T)}$ is equivalent to $\overline{W(T)} = \overline{W(\Delta(T))}$.*

By (1.1), we can regards that Aluthge sequence converges to normal-like operator, and by Proposition 2, every Aluthge sequence may converge to a quasinormal operator.

Now we discuss on convergency of Aluthge sequence. Firstly, by Example 1 in introduction we have that Aluthge sequence of weighted shit converges in the strong operator topology if its weight sequence $\{\alpha_n\}$ converges. Next, the following result has been shown.

Theorem 15 ([3]). *For each 2-by-2 matrix T, there exists a normal matrix N such that $\lim_{n\to\infty} \Delta^n(T) = N$ and $\sigma(T) = \sigma(N)$.*

For the general operator, every Aluthge sequence converges? But there is a counterexample as follows:

Theorem 16 ([7]). *There exists an operator T such that the Aluthge sequence does not converge in the weak operator topology.*

Moreover, there exists a hyponormal operator whose Aluthge sequence converge in strong operator topology not norm topology as follows:

Theorem 17 ([7]). *Let T be a hyponormal bilateral weighted shift on $l^2(\mathbb{Z})$ with a weight sequence $\{\alpha_n\}$. Let $a = \sup\{\alpha_n\}$ and $b = \inf\{\alpha_n\}$. Then the Aluthge sequence converges to a quasinormal operator in the norm topology if and only if $a = b$.*

Example 18. [7] *Let T be a bilateral shift with weight sequence $\{\alpha_n\}$, where α_n is given by*

$$\alpha_n := \begin{cases} \frac{1}{2} & (n < 0), \\ 1 & (n \geq 0). \end{cases}$$

Then the Aluthge sequence does not converges to a quasinormal operator in the norm topology but converges in the strong operator topology.

Hence we have the more refinement problem: Every Aluthge sequence of a hyponormal operator converges to a quasinormal operator in the strong operator topology?

By the way, in the finite dimensional Hilbert space case, the convergency problem still remains. Recently the following partial solution has been shown:

Theorem 19 ([8]). *If the nonzero eigenvalues of a $n \times n$ matrix T have distinct moduli, then the Aluthge sequence converges to a normal matrix with the same eigenvalues (counting multiplicity) as T.*

Relating to the problem, the following result are very interesting:

Theorem 20 ([17]). *If the Aluthge sequence stabilizes for an $n \times n$ matrix, then it does so in at $n - 1$ steps.*

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Takeaki Yamazaki, Department of Mathematics, Kanagawa University, Yokohama 221-8686, Japan
- : `yamazt26@kanagawa-u.ac.jp`

On the triangle inequality in normed spaces

Kichi-Suke Saito

Department of Mathematics

Faculty of Science

Niigata University

saito@math.sc.niigata-u.ac.jp

1. Introduction

The triangle inequality is undoubtedly one of the most fundamental inequalities in mathematics. Let X be a normed (Banach) space. For any vectors $x, y \in X$,

$$||x + y|| \leq ||x|| + ||y|| \text{ (Triangle inequality).}$$

Several authors have been treating its generalizations and reverse inequalities (cf. Hudzik–Landes[7], S. Saitoh[16], Dragomir[2] and etc). Recently, Kato-Saito-Tamura [9] found the sharp triangle inequality and its reverse inequality with n elements in a normed space to study the geometrical structure of Banach spaces. After that, we have several papers about the triangle inequalities (cf. J. Pečarić–R. Rajić[15], Dragomir[3, 4] and Hsu–Shaw–Wong [6]). Very recently, Mitani-Saito-Kato-Tamura [13] proved the refinement of sharp triangle inequality and the reverse inequality.

Our aim in this talk is to present the recent results of sharp triangle inequalities in [8, 13].

2. Sharp triangle inequalities and the reverse

At first, we consider two non-zero vectors x, y of a normed space X. Then we have

Theorem 1 For two non-zero vectors $x, y \in X$ such that $\|x\| \geq \|y\|$,

$$\|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|y\|$$

$$\leq \|x\| + \|y\| \tag{1}$$

$$\leq \|x+y\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)\|x\|. \tag{2}$$

The first inequality with two elements (1) was given earlier in Hudzik and Landes [7]; the inequalites (1) and (2) are also found in a recent paper of Maligranda [10].

We next consider three non-zero vectors x, y, z of a normed space X. Then we have

Theorem 2. *For all nonzero elements x, y, z in a Banach space X with $\|x\| \geq \|y\| \geq \|z\|$,*

$$\|x+y+z\| + \left(3 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)\|z\|$$

$$+\left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)(\|y\| - \|z\|)$$

$$\leq \|x\| + \|y\| + \|z\|$$

$$\leq \|x+y+z\| + \left(3 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)\|x\|$$

$$-\left(2 - \left\|\frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)(\|x\| - \|y\|).$$

In general, we have the following triangle inequalities for n nonzero vectors $x_1, \cdots, x_n \in X$.

Theorem 3 ([13]). Let $n \geq 3$. For any non-zero vectors $x_1, \cdots, x_n$ of a normed space X,

$$\left\|\sum_{j=1}^{n} x_j\right\| + \sum_{k=2}^{n}\left(k - \left\|\sum_{j=1}^{k} \frac{x_j^*}{\|x_j^*\|}\right\|\right)(\|x_k^*\| - \|x_{k+1}^*\|)$$

$$\leq \sum_{j=1}^{n} \|x_j\|$$

$$\leq \left\| \sum_{j=1}^{n} x_j \right\| - \sum_{k=2}^{n} \left(k - \left\| \sum_{j=n-(k-1)}^{n} \frac{x_j^*}{\|x_j^*\|} \right\| \right) (\|x_{n-k}^*\| - \|x_{n-(k-1)}^*\|),$$

where $x_1^*, x_2^*, \cdots, x_n^*$ are the rearrangement of $x_1, x_2, \cdots, x_n$ satisfying $\|x_1^*\| \geq \|x_2^*\| \geq \cdots \geq \|x_n^*\|$, and $x_0^* = x_{n+1}^* = 0$.

In Theorem 3, we may assume that $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_n\|$: that is, we have the following

Theorem 3a. Let $n \geq 3$. For all nonzero elements $x_1, \cdots, x_n$ in a normed space X such that $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_n\|$,

$$\left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \|x_n\|$$

$$+ \sum_{k=2}^{n-1} \left(k - \left\| \sum_{j=1}^{k} \frac{x_j}{\|x_j\|} \right\| \right) (\|x_k\| - \|x_{k+1}\|)$$

$$\leq \sum_{j=1}^{n} \|x_j\|$$

$$\leq \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \|x_1\|$$

$$- \sum_{k=2}^{n-1} \left(k - \left\| \sum_{j=n-(k-1)}^{n} \frac{x_j}{\|x_j\|} \right\| \right) (\|x_{n-k}\| - \|x_{n-(k-1)}\|).$$

Corollary 4([8]) For all nonzero elements $x_1, \cdots, x_n$ in a normed space X

$$\left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leq j \leq n} \|x_j\|$$

$$\leq \sum_{j=1}^{n} \|x_j\|$$

$$\leq \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \max_{1 \leq j \leq n} \|x_j\|.$$

3. Equality of Corollary 4

In [8], Kato-Saito-Tamura considerd equality attainedness for each of our inequalities in a strictly convex Banach space. The following lemma is quite powerful in our subsequent discussions.

Lemma 5. *Let X be a strictly convex Banach space. Let $x_1, x_2, \ldots, x_n$ be nonzero elements in X. Then the following are equivalent.*

(i) $\left\| \sum_{j=1}^{n} \alpha_j x_j \right\| = \sum_{j=1}^{n} \alpha_j \|x_j\|$ *with any positive numbers* $\alpha_1, \alpha_2, \ldots, \alpha_n$.

(ii) $\left\| \sum_{j=1}^{n} \alpha_j x_j \right\| = \sum_{j=1}^{n} \alpha_j \|x_j\|$ *with some positive numbers* $\alpha_1, \alpha_2, \ldots, \alpha_n$.

(iii) $\frac{x_1}{\|x_1\|} = \frac{x_2}{\|x_2\|} = \cdots = \frac{x_n}{\|x_n\|}$.

Theorem 6. *Let X be a strictly convex Banach space and $x_1, x_2, \ldots, x_n$ nonzero elements in X. Let $\|x_{j_0}\| = \min\{\|x_j\| : 1 \leq j \leq n\}$ and $\|x_{j_1}\| = \max\{\|x_j\| : 1 \leq j \leq n\}$. Let $J_0 = \{j : \|x_j\| = \|x_{j_0}\|,\ 1 \leq j \leq n\}$. Then*

$$\left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leq j \leq n} \|x_j\| = \sum_{j=1}^{n} \|x_j\| \tag{3}$$

if and only if either

(a) $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$

or

(b) $\frac{x_j}{\|x_j\|} = \frac{x_{j_1}}{\|x_{j_1}\|}$ *for all* $j \in J_0^c$ *and* $\sum_{j=1}^{n} \frac{x_j}{\|x_j\|} = \| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \| \frac{x_{j_1}}{\|x_{j_1}\|}$.

Theorem 7. *Let X be a strictly convex Banach space and $x_1, x_2, \ldots, x_n$ nonzero elements in X. Let $\|x_{j_0}\| = \min\{\|x_j\| : 1 \leq j \leq n\}$ and $\|x_{j_1}\| = \max\{\|x_j\| : 1 \leq j \leq n\}$. Let $J_1 = \{j : \|x_j\| = \|x_{j_1}\|, 1 \leq j \leq n\}$. Then*

$$\sum_{j=1}^{n} \|x_j\| = \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \max_{1 \leq j \leq n} \|x_j\| \tag{4}$$

if and only if either

(a) $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$

or

(b) $\frac{x_j}{\|x_j\|} = \frac{x_{j_0}}{\|x_{j_0}\|}$ *for all* $j \in J_1^c$ *and* $\sum_{j=1}^n x_j = \|\sum_{j=1}^n x_j\| \frac{x_{j_0}}{\|x_{j_0}\|}$.

Theorem 8. *Let X be a strictly convex Banach space and $x_1, x_2, \ldots, x_n$ nonzero elements in X. Then the equality*

$$\left\|\sum_{j=1}^n x_j\right\| + \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right) \min_{1\le j\le n} \|x_j\| \tag{5}$$

$$= \sum_{j=1}^n \|x_j\| = \left\|\sum_{j=1}^n x_j\right\| + \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right) \max_{1\le j\le n} \|x_j\| \tag{6}$$

holds if and only if

(a) $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$

or

(b) $\dfrac{x_1}{\|x_1\|} = \dfrac{x_2}{\|x_2\|} = \cdots = \dfrac{x_n}{\|x_n\|}$.

4. Applications

For non-zero vectors $x, y \in X$, we define the angular distance $\alpha[x, y]$ between x and y by

$$\alpha[x, y] = \|\frac{x}{\|x\|} - \frac{y}{\|y\|}\|.$$

Then the well-known Dunkl-William inequality [5] states that for any two non-zero elements x, y,

$$\alpha[x, y] \le \frac{4\|x - y\|}{\|x\| + \|y\|}.$$

The refinment established by Maligranda [10] is

$$\alpha[x, y] \le \frac{\|x - y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}}.$$

More generally, J. Pečarić and R. Rajić [15] showed that, for n nonzero elements $x_1, \cdots, x_n$,

$$\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\| \le \min_{1\le i\le n}\{\frac{1}{\|x_i\|}(\|\sum_{j=1}^n x_j\| + \sum_{j=1}^n |\|x_j\| - \|x_i\||)\}.$$

We next apply Corollary 4 to a geometric property of Banach spaces. Recall that a Banach space X is called *uniformly non-ℓ_1^n* provided there exists ε $(0 < \varepsilon < 1)$ such that for any $x_1, \cdots, x_n$ in the *unit sphere* of X there exists $\theta = (\theta_j)$ of n signs ± 1 for which

$$\left\| \sum_{j=1}^{n} \theta_j x_j \right\| \leq n(1-\varepsilon). \tag{7}$$

When $n = 2$, X is called *uniformly non-square.* By virtue of Corollary 4 we immediately have the following fact.

Proposition 9. *For a Banach space X the following are equivalent.*

(i) *X is uniformly non-ℓ_1^n.*

(ii) *There exists ε $(0 < \varepsilon < 1)$ such that for any $x_1, \cdots, x_n$ in the unit ball of X there exists $\theta = (\theta_j)$ of n signs ± 1 for which* (8) *holds true.*

Indeed, assume that there exists ε $(0 < \varepsilon < 1)$ such that for any $x_1, \cdots, x_n$ in the *unit sphere* of X there exist $\theta = (\theta_j)$ of n signs ± 1 for which (8) is valid. Take $x_1, \cdots, x_n$ from the *unit ball* of X. If $\|x_{j_0}\| := \min\{\|x_1, \| \ldots, \|x_n\|\} \leq 1/2$, we have

$$\left\| \sum_{j=1}^{n} \sigma_j x_j \right\| \;\leq\; \sum_{j \neq j_0} \|x_j\| + \|x_{j_0}\| \leq (n-1) + \frac{1}{2} \leq n(1 - \frac{1}{2n}).$$

Let $\|x_{j_0}\| \geq \frac{1}{2}$. According to our assumption there exists n signs (θ_j) for which (8) is valid for $x_1/\|x_1\|, \cdots, x_n/\|x_n\|$. Therefore by the first inequality of Corollary 4

$$\begin{aligned} \left\| \sum_{j=1}^{n} \theta_j x_j \right\| &\leq \sum_{j=1}^{n} \|x_j\| - \left(n - \left\| \sum_{j=1}^{n} \theta_j \frac{x_j}{\|x_j\|} \right\| \right) \|x_{j_0}\| \\ &\leq n - \frac{n\varepsilon}{2} = n\left(1 - \frac{\varepsilon}{2}\right). \end{aligned}$$

Consequently by letting $\varepsilon_0 = \min\{\frac{\varepsilon}{2}, \frac{1}{2n}\}$ we have the conclusion.

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INTERMEDIATE RANK AND GROUP C^*-ALGEBRAS

(ABSTRACT FOR THE TALK AT THE CHIBA CONFERENCE)

MIKAEL PICHOT

ABSTRACT. In a recent paper with Sylvain Barré [1] we introduced concepts of intermediate rank for countable groups (essentially non positively curved) that "interpolate" between consecutive values of the classical integer-valued rank.

The goal of the talk is to present the general framework of rank interpolation, to give concrete examples, and to explain what can be derived from this at the C^*-algebraic level.

The text below is an extended abstract for this talk and is extracted from [1], to which I refer for more details and references.

1. Interpolating the rank

We seek for a sufficiently 'smooth' interpolation of successive values of the rank (as we shall see there are actually several possible dimensions of interpolation). The rank of a space captures the presence of 'maximally flat portions' in that space.

The following concepts, which differ by the scale at which interpolated rank is detected in the group, are considered in the paper:

- growth rank (at the asymptotic scale),
- local rank (at the infinitesimal scale),
- mesoscopic rank (in between).

Various classes of groups are proved to have intermediate rank behaviors. We are especially interested in interpolation between rank 1 and rank 2.

For instance, we construct groups *of rank* $\frac{7}{4}$, which find their origin in the following graph.

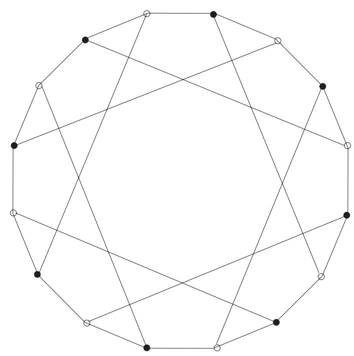

FIGURE 1. Rank $\frac{7}{4}$

Date: November 8, 2007.

(Note that this graph is *not* the incidence graph of the Fano plane and in fact is not a spherical building.)

Our setting is essentially that of non positively curved (i.e. CAT(0)) spaces. We also construct groups of intermediate rank (called *groups of friezes*) that are archetypal as far as mesoscopic rank is concerned. (Some groups of rank $\frac{7}{4}$ also exhibit mesoscopic rank phenomena.)

Precise definitions and statements of the results will be given during the talk.

The following graphic shows what happens for 'mesoscopic flats' under some local rank assumptions. Let X be a CAT(0) space of dimension 2 (without boundary) and A be a point of X. Then by definition the mesoscopic profile of X at A is the function $\varphi_A : \mathbf{R}_+ \to \mathbf{N}$ which associated to an $r \in \mathbf{R}_+$ the number of flat disks in X of center A which are not included in a flat of X.

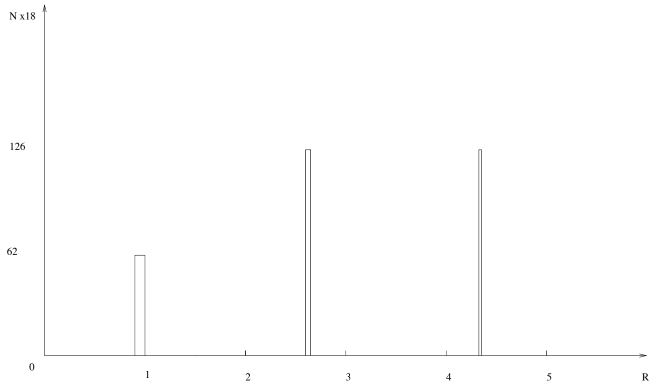

FIGURE 2. Example of mesoscopic profile when the local rank is $\leq \frac{3}{2}$

Mesoscopic rank, by definition, requires that the support of φ_A contains a neighborhood of infinity.

2. Property RD

Recall that a length ℓ on a countable group Γ is a non negative function ℓ on Γ such that $\ell(e) = 0$, $\ell(x) = \ell(x^{-1})$ and $\ell(xy) \leq \ell(x) + \ell(y)$ for $x, y \in \Gamma$.

Definition 1. Let Γ be a countable group endowed with a length ℓ. One says that Γ has Property RD with respect to ℓ if there is a polynomial P such that for any $r \in \mathbf{R}_+$ and $f, g \in \mathbf{C}\Gamma$ with $\mathrm{supp}(f) \subset B_r$ one has

$$\|f * g\|_2 \leq P(r)\|f\|_2\|g\|_2$$

where $B_r = \{x \in \Gamma,\ \ell(x) \leq r\}$ is the ball of radius r in Γ.

Here are two fundamental examples of groups with property RD:

(1) free groups on finitely many generators have property RD (with respect to the usual word length), as was proved by U. Haagerup in [3],

(2) groups acting freely isometrically on Bruhat-Tits buildings of type $\tilde{A}_2$ (also called triangle buildings) have property RD with respect to the length induced from the 1-skeleton, as was proved by J. Ramagge, G. Robertson and T. Steger in [5].

In [1] we prove property RD for groups interpolating between (1) and (2), including:

- groups of rank $\frac{7}{4}$,
- groups of friezes.

(In fact RD is shown to hold for all *triangle groups*, which by definition acts on 2-dimensional CAT(0) simplicial complex with **equilateral faces**.)

3. Application to the Baum-Connes conjecture

This gives new examples where V. Lafforgue's Banach KK-theory approach to establish the Baum-Connes conjecture applies.

Thus by combining the above and Lafforgue's Theorem in [4] we get the following result.

Theorem 2. Let Γ be a countable group admitting a proper, isometric, and cocompact action on a triangle polyhedron. Then Γ satisfies the Baum-Connes conjecture, i.e. the Baum-Connes assembly map

$$\mu_r : K_*^{\mathrm{top}}(\Gamma) \to K_*(C_r^*(\Gamma))$$

is an isomorphism.

Theorem 2 includes groups of rank $\frac{7}{4}$ and groups of friezes, for instance.

4. Stable rank, real rank

Let A be a unital C^*-algebra. The stable rank $\mathrm{sr}(A)$ of A is an invariant of A taking values in $\{1, 2, \ldots\} \cup \{\infty\}$. In the commutative case $\mathrm{sr}(A)$ behaves as a dimension. Thus for a compact space X and $A = C(X)$ the C^*-algebra of complex-valued function on X one has

$$\mathrm{sr}(A) = \lfloor \dim X/2 \rfloor + 1.$$

In particular

$$\mathrm{sr}(C_r^*(\mathbf{Z}^2)) = 2$$

where $C_r^*(\mathbf{Z}^2) \simeq C(\boldsymbol{T}^2)$ is the C^*-algebra of the abelian free group $\mathbf{Z}^2$.

Definition 3. A unital C^*-algebra A has stable rank 1 if and only if the group $\mathrm{GL}(A)$ of invertible elements of A is norm dense in A.

In [2] K. Dykema, U. Haagerup and M. Rørdam proved that if Γ_1 and Γ_2 are two countable groups with $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$ then

$$\mathrm{sr}(C_r^*(\Gamma_1 * \Gamma_2)) = 1.$$

In particular for the free groups F_n on $n \geq 2$ generators one has

$$\mathrm{sr}(C_r^*(F_n)) = 1.$$

K. Dykema and P. de la Harpe then generalized this result and proved that if Γ is a torsion free non elementary hyperbolic group, or a cocompact lattice in a real, noncompact, simple, connected Lie group of real rank one with trivial center, one has

$$\mathrm{sr}(C_r^*(\Gamma)) = 1.$$

By results of Sudo, C^*-algebras of real Lie groups of higher rank have stable rank 2.

Our main results here concerns the groups of rank $\frac{7}{4}$ of Section 1.

Theorem 4. Let X be a complex of rank $\frac{7}{4}$ and let $\Gamma = \pi_1(X)$ be the fundamental group of X. Then the reduced C^*-algebra $C_r^*(\Gamma)$ of Γ has stable rank 1.

This provides the first examples of dicrete group of 'higher rank' whose reduced C^*-algebra has stable rank 1.

Corollary 5. Let X be a complex of rank $\frac{7}{4}$ and let $\Gamma = \pi_1(X)$ be the fundamental group of X. Then the reduced C^*-algebra $C_r^*(\Gamma)$ of Γ has real rank 1.

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Mikaël Pichot, Department of Mathematical Sciences University of Tokyo 3-8-1 Komaba, Tokyo, 153-8914 JAPAN

E-mail address: pichot@ms.u-tokyo.ac.jp

q-numerical range of a matrix –the projection of a convex body in a 4-dimensional space onto a plane–

Hiroshi Nakazato, Hirosaki University

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1. numerical range of a matrix and the equation of its boundary

Definition Suppose that T is a bounded linear on a complex Hilbert space H. The range

$$W(T) = \{\langle Tx, x\rangle : x \in H, ||x|| = 1\}, \tag{1.1}$$

is called the *numerical range* of T.

It is known that the spectrum $\sigma(T)$ of T is contained in $W(T)$. The range $W(T)$ is a bounded convex subset of $\mathbf{C}$ (Hausdorff, 1919 [H];[T],[AL]). If H is finite-dimensional, then the set $W(T)$ is compact and the boundary of $W(T)$ lies on an algebraic curve. We assume that H is finite-dimensional. Set

$$M_R = \max \sigma(\Re(T)), \quad m_R = \min \sigma(\Re(T)),$$
$$M_I = \max \sigma(\Im(T)), \quad m_I = \min \sigma(\Im(T)).$$

In 1902, Bendixson showed the inclusion

$$\sigma(T) \subset \{x + iy : (x, y) \in \mathbf{R}^2, m_R \le x \le M_R, m_I \le y \le M_I\}.$$

This relation was refined as

$$W(T) = \cap_{0\le\theta\le\pi}\{z \in \mathbf{C} : \min \sigma(\Re(\exp(-i\theta)T)) \le \Re(z \exp(-i\theta))$$

$$\leq \max \sigma(\Re(\exp(-i\theta)T))\}.$$

A concrete procedure to determine the equation of $\partial W(T)$ is provided in the following. We use an algebraic or a convex analytic method. Suppose that Δ is a compact convex set in $\mathbf{C} \cong \mathbf{R}^2$ and 0 is an interior point of Δ. Then the set

$$\Delta^{\perp} = \{x + iy : (x, y) \in \mathbf{R}^2, xu + yv + 1 \geq 0 \,\text{for every}\, u + iv \in \Delta\}$$

is also a compact convex set. If p, q are positive real numbers satisfying $1/p + 1/q = 1$, then the sets

$$B_p = \{x+iy : (x, y) \in \mathbf{R}^2, |x|^p+|y|^p \leq 1\}, B_q = \{x+iy : (x, y) \in \mathbf{R}^2, |x|^q+|y|^q \leq 1\}$$

satisfy

$$B_p^{\perp} = B_q, \quad B_q^{\perp} = B_p.$$

If $W(T)$ contains 0 as an interior point, then

$$\partial(W(T)^{\perp}) \subset \{x + iy : \det(I_n + x\Re(T) + y\Im(T)) = 0\}.$$

Even if 0 is not an interior point of $W(T)$, the boundary of $W(T)$ is the dual curve of the algebraic curve:

$$\{x + iy : \det(I_n + x\Re(T) + y\Im(T)) = 0\},$$

and hence the degree of the equation of $\partial W(T)$ is at most $n(n-1)$.

2. the q-numerical range of a matrix

Definition Suppose that T is a bounded linear operator on a complex Hilbert space H, and q is a real number $0 \leq q \leq 1$. The q-numerical range $W_q(T)$ is defined as

$$W_q(T) = \{\langle Tx, y\rangle : x, y \in H, ||x|| = ||y|| = 1, \langle x, y\rangle = q\}. \qquad (2.1)$$

We assume that H is finite-dimensional. In 1984, N. K. Tsing proved the following formula and proved the convexity of $W_q(T)$:

$$W_q(T) = \{q\, z + \sqrt{1 - |q|^2} w : z \in W(T), w \in \mathbf{C}, |w| \leq \phi(z)\},$$

where

$$\phi(z) = \max\{\langle T^*Tx, x\rangle : x \in H, ||x|| = 1, \langle Tx, x\rangle = z\}$$

is convex on $W(T)$ as a result of Binding in 1985 (cf.[B],[LP]). (Tsing used a slightly different method.) $W_1(T) = W(T)$.

Tsing's formula provides a basis to compute the boundary of the equation. We set

$$\Gamma(T) = \{(x, y, u, v) \in \mathbf{R}^2, x+iy \in W(T), x^2+y^2+u^2+v^2 \leq \phi(x+iy)\}. \quad (2.2)$$

Then $\Gamma(T)$ is a compact convex set in $\mathbf{R}^4$. For $0 < q < 1$, we consider an orthogonal projection $\Pi_q : \mathbf{R}^4 \to \mathbf{R}^2 \cong \mathbf{C}$ defined by

$$\Pi_q(x, y, u, v) = (qx + \sqrt{1-q^2}u, qy + \sqrt{1-q^2}v).$$

Then the following equation holds:

$$W_q(T) = \Pi_q(\Gamma(T)) \quad (2.3)$$

(cf. [CN1]). The boundary of $\Gamma(T)$ lies on an algebraic hypersurface in the 4-dimensional space $\mathbf{R}^4$. The set $W_q(T)$ is also the image of this boundary by Π_q.

The order of the equation of the boundary of $\Gamma(T)$ is at most $2n(n-1)^2$. The equation $F(x, y, u, v) = 0$ is obtained by

$$F(x, y, u, v) = G(x, y, x^2 + y^2 + u^2 + v^2),$$

where $G(x, y, z) = 0$ is the equation of the dual algebraic surface of the algebraic surface defined by

$$\det(I_n + X\,\Re(T) + Y\,\Im(T) + Z\,T^*T) = 0.$$

If T, T are respective $n \times n$, $m \times m$ matrices. Then the following two conditions are mutually equivalent

(i) $\Gamma(T) \subset \Gamma(S)$,

(ii) $W_q(T) \subset W_q(S)$ for $0 \leq q \leq 1$.

In this talk, we provide a concrete method to compute the equation of the boundary of $W_q(T)$ for an irreducible 3×3 matrix , for example

$$T = \begin{pmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{pmatrix}.$$

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作用素の三角不等式の等号成立条件について

名古屋工業大学　林　倫弘（Tomohiro Hayashi）
Nagoya Institute of Technology

In this talk I would like to explain my paper which is a joint work with Tsuyoshi Ando.

First I would like to x some notations. We denote by H an in nite dimensional complex Hilbert space. The set of all bounded linear operators on H is denoted by $B(H)$. For $X \in B(H)$, we de ne its "absolute value" by $|X| = (X \ X)^{1 \ 2}$.

We would like to consider a triangle inequality for this absolute value. It is well-known that the inequality

$$|A + B| \leq |A| + |B|$$

is **wrong**. But we have

Theorem 0.1. *(Thompson, '76 PJM) For two $n \times n$matrices $A, B \in M_n(\mathbb{C})$, we can nd two unitaries $U, V \in M_n(\mathbb{C})$ s.t.*

$$|A + B| \leq U|A|U \ \ + VBV \ .$$

Moreover

Theorem 0.2. *(Thompson, '79 PJM) For some unitaries $U, V \in M_n(\mathbb{C})$, the equality*

$$|A + B| = U|A|U \ \ + V|B|V$$

holds if and only if there exists a unitary $W \in M_n(\mathbb{C})$ s.t. $A = W|A|$ and $B = W|B|$. (That is, A and B have a common phase part W.)

Therefore in both cases A and B satisfy

$$|A + B| = |A| + |B|.$$

The aim of this talk is to generalize the second theorem for in nite dimensional setting. Our main result is:

Theorem 0.3. *For two bounded linear operators $A, B \in B(H)$, the triangle equality $|A+B| = |A| + |B|$ holds if and only if there exists a partial isometry W such that $A = W|A|$ and $B = W|B|$.*

Remark 0.1. (1) Why do we consider $|A+B| = |A| + |B|$ instead of $|A+B| = U|A|U^* + V|B|V^*$?

Take a projection $P \in B(H)$ and subprojection $Q \le P$ s.t. four projections

$$P, Q, P - Q, 1 - P$$

are infinite rank. Then it is easy to find unitaries U, V s.t.

$$|P + (-Q)| = U|P|U^* + V|(-Q)|V^*.$$

However P and $-Q$ cannot have a common phase part. Hence Thompson's theorem does not hold for this triangle equality.

(2) The proof of our theorem is simple if we have some faithful finite trace or H is finite dimensional as follows. (The following argument does work for the equality $|A_1 + \cdots A_n| = |A_1| + \cdots |A_n|$.)

Take polar decompositions

$$A = U|A|, \; B = V|B|, \; A + B = W|A+B|.$$

We may assume that U, V, W are unitaries. By using the triangle equality

$$|A+B| = |A| + |B|,$$

we have

$$\begin{aligned} W^*U|A| + W^*V|B| &= W^*(A+B) \\ &= |A+B| = |A| + |B|. \end{aligned}$$

Then

$$\begin{aligned}|A|+|B| =& \frac{1}{4}\{(1+W\ U)|A|(1+W\ U)\\ &-(1-W\ U)|A|(1-W\ U)\\ &+(1+W\ V)|B|(1+W\ V)\\ &-(1-W\ V)|B|(1-W\ V)\ \}\\ \leq& \frac{1}{4}\{(1+W\ U)|A|(1+W\ U)\\ &+(1+W\ V)|B|(1+W\ V)\ \}\\ =& \frac{1}{4}\{(W\ U|A|+W\ V|B|)\\ &+(W\ U|A|+W\ V|B|)\\ &+(|A|+|B|)\\ &+(W\ U|A|U\ W+W\ V|B|V\ W)\}\\ =& \frac{1}{4}\{3(|A|+|B|)\\ &+(W\ U|A|U\ W+W\ V|B|V\ W)\}.\end{aligned}$$

That is,

$$|A|+|B| \leq W\ U|A|U\ W+W\ V|B|V\ W.$$

Since we have a faithful trace, this inequality implies

$$|A|+|B| = W\ U|A|U\ W+W\ V|B|V\ W$$

and hence

$$\begin{aligned}(1-W\ U)|A|(1-W\ U)\ &= 0,\\ (1-W\ V)|B|(1-W\ V)\ &= 0,\end{aligned}$$

in other words

$$W\ A = W\ U|A| = |A|$$

and

$$W\ B = W\ V|B| = |B|.$$

So we are done.

This argument heavily depends on **niteness**. We need another method for general cases.

<u>Proof of Main Result</u>

Take polar decompositions

$$A = U|A|, \ B = V|B|.$$

By the triangle equality we have

$$\begin{aligned}(|A| + |B|)^2 &= |A + B|^2 \\ &= (U|A| + V|B|) \ (U|A| + V|B|)\end{aligned}$$

and hence

$$|A|(U \ V - 1)|B| + |B|(V \ U - 1)|A| = 0.$$

That is

$$i|A|(U \ V - 1)|B|$$

is self-adjoint. Since $|A|, |B| \leq |A| + |B|$, we can nd two contractions K, L satisfying

$$\begin{aligned}|A|^{1 \ 2} &= K(|A| + |B|)^{1 \ 2} = K|A + B|^{1 \ 2}, \\ |B|^{1 \ 2} &= L(|A| + |B|)^{1 \ 2} = L|A + B|^{1 \ 2}\end{aligned}$$

and the support of $K \ K + L \ L$ is dominated by that of $|A| + |B|$.

Then we have

$$|A| + |B| = (|A| + |B|)^{1 \ 2}(K \ K + L \ L)(|A| + |B|)^{1 \ 2}.$$

Thus $K \ K + L \ L$ is equal to the support projection of $|A| + |B|$ and hence

$$K \ KL \ L = L \ LK \ K.$$

Direct computations show

$$\begin{aligned}&i|A|(U \ V - 1)|B| \\ =&|A + B|^{1 \ 2} \\ &\times \{iK \ K|A + B|^{1 \ 2}(U \ V - 1)|A + B|^{1 \ 2}L \ L\} \\ &\times |A + B|^{1 \ 2}.\end{aligned}$$

Since the left-hand side is self-adjoint, we conclude that

$$iK \ K|A + B|^{1 \ 2}(U \ V - 1)|A + B|^{1 \ 2}L \ L$$

is a self-adjoint operator. In particular

$$(iK \ K|A + B|^{1 \ 2}(U \ V - 1)|A + B|^{1 \ 2}L \ L) \subset \mathbb{R}$$

We de ne a positive operator D by

$$D = [|A+B|^{1\ 2}(L\ L)(K\ K)|A+B|^{1\ 2}]^{1\ 2}.$$

Then we have

$$\begin{aligned}
&\mathbb{R} \supset \\
&\quad (iK\ K|A+B|^{1\ 2}(U\ V-1)|A+B|^{1\ 2}L\ L) \setminus \{0\} \\
&\quad = i\ \ (D(U\ V-1)D) \setminus \{0\} \\
&\quad \subset i\mathbb{W}(D(U\ V-1)D) \text{ (the numerical range)} \\
&\quad \subset i\{z \in \mathbb{C};\ |z+||D||^2| \leq ||D||^2\} \\
&\quad = \text{the circle with center } -i||D|| \text{ and radius } ||D||.
\end{aligned}$$

Therefore we conclude

$$iK\ K|A+B|^{1\ 2}(U\ V-1)|A+B|^{1\ 2}L\ L = 0$$

and hence

$$i|A|(U\ V-1)|B| = 0.$$

Let

$$W(|A|\xi + |B|\eta) = A\xi + B\eta.$$

$(\xi, \eta \in H)$

Since

$$\begin{aligned}
&||A\xi + B\eta||^2 - |||A|\xi + |B|\eta||^2 \\
&\quad = 2Re\langle |A|(U\ V-1)|B|\eta, \xi\rangle \\
&\quad = 0,
\end{aligned}$$

W is a partial isometry. Obviously W satis es

$$A = W|A|, \quad B = W|B|.$$

<u>Final Remark</u>

Since we deal with only two operators, we can get some commutativity. This is the crucial point in our argument. I have no idea to attack this problem for 3 (or n) operators.

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- : hayashi.tomohiro@nitech.ac.jp

Complementary and Degradable Channels in Quantum Information Theory

Mary Beth Ruskai (Tufts University)

Abstract: In qauntum information theory, noise is represented by a completely positive trace preseving map, typically referred to as a "channel". Using Stinespring's reprentation theory and Arveson's commutant lifting theorem, the complement Φ^C of a channel Φ can be defined and shown to represent the environment's view. A channel N is called degradable if there is another channel X whose action following that of the channel yields the complement, i.e., there is a channel X such that $X \circ N = N^C$.

Both degradable and complementary channels have implications for the following important question.
When can the (asympotitic) capacity be reduced to a a "one-short" formula, as in classical information theory ? It should be noted that quantum Shannon theory is much richer with many different types of capacity, some of which can not always be reduced to a simple formula. This talk will try to give a flavor for this subject and an indication of the many challenging mathematical questions remaining.

Title: Concave functions and symmetric norms

Authors: *Jean-Christophe Bourin* and *Eun-Young Lee*

Jean-Christophe Bourin's Address: Universit*é* De Franche-Comt*é* Besancon
Jean-Christophe Bourin's e-mail address: bourinjc@club-internet.fr

Eun Young Lee's Address: Department of Mathematics Kyungpook National University
Taegu 702-701, Rep. of Korea
Eun Young Lee's e-mail address: eylee8910@hotmail.com

Keyword: Hermitian operators, symmetric norms, operator inequalities

Abstract:
Given $A_i \geq 0$ and $Z_i \geq 0$ for all $i = 1, \ldots, n$, we have this theorem:

$$|| \sum f(Z_i A_i Z_i)|| \leq || \sum Z_i f(A_i) Z_i||.$$

This theorem contains the next two well-known recent inequalities: Let A, B, Z be positive semidefinite matrices of same size and suppose Z is expansive, i.e., $Z \geq I$. Two remarkable inequalities are

$$\|f(A+B)\| \leq \|f(A)+f(B)\| \quad \text{and} \quad \|f(ZAZ)\| \leq \|Zf(A)Z\|$$

for all non-negative concave function f on $[0,\infty)$ and all symmetric norms $\|\cdot\|$ (in particular for all Schatten p-norms). In this paper we survey several related results and we show that these inequalities are two aspects of a unique theorem. For the operator norm, our result also holds for operators on an infinite dimensional Hilbert space.

Matrix Subadditivity and Monotony Inequalities

(J.-C. Bourin, Univ. Franche-Comté)

This talk will start with two recent subaddivity result extending in two different way Rotfeld's trace inequality for non-negative concave functions $f(t)$ on $[0, \infty)$ and positive semi-definite matrices A, B. The first result states that

$$f(A + B) \le Uf(A)U^* + Vf(B)V^*$$

for some unitary matrices U and V. The second result involves symmetric (unitarily invariant) norms:

$$\|f(A + B) \le \|f(A) + f(B)\|$$

The proof of this norm inequality use some monotony inequality considered in the second part of this talk. A pair of positive matrices (A, B) is monotone if we have $A = f(C)$, $B = g(C)$ for some non-decreasing functions f, g and some positive matrix C. A typical example is (A^p, A^q), $p, q > 0$. For such pairs (A, B) we have several inequalities, for instance with E an (ortho-)projection

$$|AEB| \le V|ABE|V^*$$

for some unitary V. By considering the range of E we can derive inequalities for compressions and unital positive linear maps Φ. Many open questions then naturally arise. For instance, is it true that, for some unitary V

$$|\Phi(A^p)\Phi(A^q)| \le V\Phi(A^{p+q})V^*$$

for all positive A and $p, q > 0$? Answer is positive when $p/3 \le q \le 3p$.

TWISTED INDEX THEOREM AND TYPE III FACTORS

HITOSHI MORIYOSHI

ABSTRACT

As is well known, the von Neumann algebras are classified into three types, namely, type I, II and III. Type III factors are further classified into type III_λ $(0 \leq \lambda \leq 1)$ according to the value λ determined from Connes's S-set. It is also known that there is a unique type III_λ hyperfinite factor R_λ for $0 < \lambda \leq 1$. Then it is interesting to study such hyperfinite factors from Geometric point of view. For instance, the type III_1 hyperfinite factor can be constructed from the Anosov foliation on the unit tangent bundle of a closed surface. For $0 < \lambda < 1$ there exist a foliated T^2-bundle $(M_\mu, \mathcal{F}_\mu)$ on a closed surface whose foliation W^*-algebra $W^*(M_\mu, \mathcal{F}_\mu)$ is isomorphic to R_λ with $\lambda = \mu^2$. In this talk we shall prove Twisted Index theorm on such type III_λ hyperfinite factors. Then we can recapture Connes's S-set via the evaluation between the twisted index and the transvese fundamental cyclic cocycles in the framework of Noncommutative Geometry.

Department of Mathematics, Keio University, 3-14-1 Hiyoshi, Yokohama 223-8522, JAPAN

E-mail address: moriyosi@math.keio.ac.jp

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ALGEBRAIC CORRESPONDENCE から作られる C*-環

梶原 毅 (TSUYOSHI KAJIWARA)
岡山大学環境学研究科 (OKAYAMA UNIVERSITY)

Abstract. This is a joint work with Yasuo Watatani.

Let $p(z,w)$ be a two variable polynomial. We consider the algebraic equation $p(z,w) = 0$ on the Riemannian sphere. We call the solution of the equation algebraic correspondence. We can construct Hilbert C*-bimodule from p, and Cuntz-Pimsner C*-algebra from it. Let J be a p -invariant closed subset of $\hat{\mathbb{C}}$. Restricting to J, we can construct Hilbert C*-bimodule and Cuntz-Pimser C*-algebra.

Under the assumption that p is expansive on J and free on J for a p-invariant closed subset J, we prove that the corresponding Cuntz-Pimsner C*-algebra is simple and purely infinite. Moreover we present some examples satisfying the above assumptions.

1. Introduction

本研究は綿谷安男氏との共同研究である．

$p(z,w)$ を 2 変数の多項式として，リーマン球面 $\hat{\mathbb{C}}$ 上の代数方程式 $p(z,w) = 0$ を考える．この方程式で決まる $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ の部分集合を algebraic correspondence という．p に適切な条件を付した上で，p から Hilbert C*-bimodule を構成し，これより Cuntz-Pimsner 環を構成することができる．p 不変な閉集合 J に対して p を J に制限して Hilbert C*bimodule, Cuntz-Pimsner 環を考えることもできる．従来調べてきたリーマン球面上の有理関数 $R(z) = P(z)/Q(z)$ のグラフは，$p(z,w) = Q(z)w - P(z)$ とすることによって，algebraic correspondence の特別の例と考えることができる．

p-不変集合 J に対して p が expansive on J かつ free on J とする．そのとき p を J に制限して構成した Cuntz-Pimsner 環が simple かつ purely infinite になることを示す．有理関数のときには J としてジュリア集合を取れば上の 2 つの条件は自動的に満たされているが，一般の algebraic correspondence の場合にはそうではなく，検証する必要がある．

最初に algebraic correspondence の基本的な性質とヒルベルト C*-bimodule の構成について説明する．次に，expansive と free の条件のもとで，ヒルベルト C*-bimodule から構成した Cuntz-Pimsner 環が simple かつ purely infinite になることを証明する．最後にいくつかの example について報告する．

2. Construction of C*-algebras

$p(z,w)$ を 2 変数の多項式とする．$p(z,w) = 0$ を複素変数の代数方程式と考える．さらにこれは，1 次元射影平面上の代数方程式に拡張できる．リーマン球面 $\hat{\mathbb{C}}$ は 1

次元射影平面と同一視できるので, $\hat{\mathbb{C}}$ 上の代数方程式とみなすことができ, 以下そのように考える.

$p(z,w)$ を z, w の多項式として既約分解する. すなわち,

$$p(z,w) = g_1(z,w)^{n_1} \cdots g_p(z,w)^{n_p}$$

であって, 各 $g_i(z,w)$ は z, w について既約であるとする.

仮定 $p(z,w) = 0$ は被約 (reduced) すなわち, 全ての i について, $n_i = 1$ とする. さらに, 各 $g_i(z,w)$ は z だけ, また w だけの関数ではないとする.

$p(z,w)$ を z の多項式と見たときの次数を m とする.

Lemma 2.1. *$w \in \hat{\mathbb{C}}$ を固定する. そのとき $p(z,w) = 0$ は $z \in \hat{\mathbb{C}}$ の方程式とみて, 重複度を込めてちょうど m 個の解をもつ. さらに, それらの解は, w に連続的に依存する.*

$e(z_0,w)$ で, $p(z,w) = 0$ を w を定数と思って z の方程式とみなしたときの解 $z = z_0$ の分岐指数を表す. $\mathcal{C}_p = \{(z,w) \in \hat{\mathbb{C}} \times \hat{\mathbb{C}} \,|\, p(z,w) = 0\}$ とおく.

Lemma 2.2. *$\mathcal{C}_p$ は $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ のコンパクト部分集合で ある.*

Definition 2.3. $\mathcal{C}_p$ を p によって決まる algebraic correspondence とよぶ.

$A = \mathrm{C}(\hat{\mathbb{C}})$, $X = \mathrm{C}(\mathcal{C}_p)$ とおく. A は可換 C* 環である. w を固定すると $p(z,w) = 0$ となる z が有限個であるので, f, $g \in X$, $a, \in A$ に対して, 次のように, $\mathcal{C}$ の関数 $a \cdot f \cdot b$ と $\hat{\mathbb{C}}$ の関数 $(f|g)_A(w)$ を定義することができる.

$$(a \cdot f \cdot b)(z,w) = a(z)f(z,w)b(w)$$

$$(f|g)_A(w) = \sum_{\{\, z \,|\, (z,w) \in \mathcal{C}_p \,\}} e(z,w)\overline{f(z,w)}g(z,w)$$

$(a \cdot f \cdot b)(z,w)$ は, X の元である. 一方, Lemma 2.1 により次がなりたつ.

Lemma 2.4. *任意の f, $g \in X$ に対して, 写像 $w \to (f|g)_A(w)$ は連続であり, $(f|g)_A$ は A の元になる.*

Proposition 2.5. *A の右作用と X 上の A-値内積によって X は Hilbert A-module になる.*

$\mathcal{L}(X_A)$ で, X 上の線形写像全体で, A 内積に関して adjoint を持つものを表す. A から $\mathcal{L}(X_A)$ への写像 ϕ を a, $f \in X$ に対して $\phi(a)f = a \cdot f$ で定義する.

Proposition 2.6. *(X,ϕ) は, Hilbert C*-bimodule (または C*-correspondence) である.*

この Hilbert C*-bimodule に対して, Pimsner construction によって C*-環を定義する.

Definition 2.7. (X,ϕ) から作られる Pimsner 環を $\mathcal{O}_p(\hat{\mathbb{C}})$ とかく.

Definition 2.8. $\hat{\mathbb{C}}$ の部分集合 J が p-不変であるとは, $z \in J$ に対して $p(z,w)=0$ なら $w \in J$ となり, また $w \in J$ に対して $p(w,z)=0$ でも $z \in J$ となることである.

p-不変な閉集合が J に対して, $\mathcal{C}_p(J) = \{\,(z,w) \in J \times J \mid p(z,w)=0\,\}$, $A_J = \mathrm{C}(J)$, $X_J = \mathrm{C}(\mathcal{C}_J)$ が同様に定義でき, (X_J, ϕ) は Hilbert C*-bimodule になる.

Definition 2.9. (X_J, ϕ) に対して作られる Pimsner 環を $\mathcal{O}_p(J)$ と書く.

ここで, 有理関数における分岐点にあたるものを定義する.

$$B(p) = \{\, z \in \hat{\mathbb{C}} \mid e(z,w) \geq 2 \quad \text{for some} \quad w \quad p(z,w)=0\,\}$$

Lemma 2.10. *$B(p)$ は, 有限集合である.*

$\mathcal{L}(X_A)$ の部分集合で, 有限階作用素のノルム極限で表されるもの全体を $\mathcal{K}(X_A)$ とする. $I_X = \phi^{-1}(\phi(A) \cap \mathcal{K}(X_A))$ とおく.

Proposition 2.11. *p が仮定を満たすとき,*
$I_X = \{\, f \in \mathrm{C}(\hat{\mathbb{C}}) \mid f|_{B(p)}\,\} = 0$ となる. X_J に対しても, 同様に記述される.

$X_A^{\otimes 2} = X \otimes_A X, \cdots, X_A^{\otimes n} = X^{\otimes n-1} \otimes_A X$ と書く.
J を不変集合として次を定義する.

$$\mathcal{P}_n = \{\,(z_1, z_2, \ldots, z_{n+1}) \in J^{n+1} \mid p(z_i, z_{i+1}) = 0,\ i = 1, \ldots, n\,\}$$
$$\mathcal{P}'_n = \{(z,w) \in J \times J \mid \exists z_2, \ldots, z_n \quad \text{such that} \quad (z, z_2, z_3, \ldots, z_n, w) \in \mathcal{P}_n\}$$

Lemma 2.12. *$\mathcal{P}_n$, $\mathcal{P}'_m$ は, $J \times \cdots \times J$, $J \times J$ のコンパクト部分集合である.*

φ を $f_1, \cdots, f_n \in X$ に対して,

$$\varphi(f_1 \otimes f_2 \otimes \cdots \otimes f_n)(z_1, z_2, \ldots, z_n) = f_1(z_1, z_2) f_2(z_2, z_3) \cdots f_n(z_n, z_{n+1})$$

と定める.

Proposition 2.13. *$\mathrm{C}(\mathcal{P}_n)$ は自然に Hilbert A-A-bimodule であり, φ は $X^{\otimes n}$ から $\mathrm{C}(\mathcal{P}_n)$ への Hilbert A-A-bimodule としての同型である.*

この Proposition は $X^{\otimes n}$ の関数としての記述を与え, 後の証明において重要である.

3. SIMPLICITY AND PURE INFINITENESS

n を自然数とし, p-不変集合 J の部分集合 U に対して, J の部分集合 $U^{(n)}$ を,

$$U^{(n)} = \{\, w \in J \mid (z_1, z_2, \ldots, z_n, w) \in \mathcal{P}_n \quad \text{for some} \quad z_1 \in U,\ z_2 \ldots z_n \in J\,\}$$

と定義する.
N を自然数とするとき, the set of N-generalized periodic points $\mathrm{GP}(N)$ を,

$$\begin{aligned}\mathrm{GP}(N) = \{ & w \in J \mid \exists m, n \quad 0 \leq m \neq n \leq N, \exists (z, z_2, z_3, \ldots, z_n, w) \in \mathcal{P}_n, \\ & \exists (z, u_2, u_3, \ldots, u_m, w) \in \mathcal{P}_m\,\}\end{aligned}$$

と定義する.

Definition 3.1. p が J-expansive on J であるとは, J の任意の空でない開集合 U に対して自然数 n が存在して $U^{(n)} = J$ となることである.

p が有理関数 $R(z)$ によってあたえられている場合, J を R のジュリア集合にとれば, p は J 拡大的である. (Beardon [1])

Definition 3.2. p が free on J であるとは, 任意の自然数 N に対して, $\mathrm{GP}(N)$ が有限集合になることである.

Lemma 3.3. p は J-拡大的とする. 任意の $a \in A_J^+$, $a \neq 0$, 任意の $\varepsilon > 0$ に対して, $n \in \mathbb{N}$ と $f \in X^{\otimes n}$ で $(f|f)_A = 1$ で,

$$\|a\| - \varepsilon \leq f^* a f \leq \|a\|$$

となるようなものが取れる.

この Lemma を用いて次の Lemma を示すことができる.

Lemma 3.4. *(Kajiwara-Watatani [8])* 任意の $a \in A_J^+$, $a \neq 0$ と任意の ε で $0 < \varepsilon < \|a\|$ をみたすものに対して, $n \in \mathbb{N}$ と $u \in X \otimes n$ で

$$\|u\|_2 \leq (\|a\| - \varepsilon)^{-1/2}, \qquad u^* a u = I$$

となるものが取れる.

これは, p が expansive on J であることの帰結である.

$x \in X$ に対して, 対応する Toepliz 環の元を T_x, Cuntz-Pimsner 環の元を S_x とかく.

Lemma 3.5. i, j は 0 以上の整数で $i \neq j$ とする. $x \in X^{\otimes i}$, $y \in X^{\otimes j}$ とする. もし, $a \in \mathrm{C}(J)$ が $(z_1, z_2, \ldots, z_i, w) \in \mathcal{P}_i$, $(u_1, u_2, \ldots, u_j, w) \in \mathcal{P}_j$ と なるような全ての z_1, u_1 に対して $a(z_1)a(u_1) = 0$ となるならば, $aT_xT_y^*a^* = 0$ となる.

この Lemma は Toepliz 環に対するものであるが, quotient をとることにより Cuntz-Pimsner 環でも成立する.

Lemma 3.6. p が *free on* J であるとする. 自然数 r を固定する. J の任意の開集合 U に対して, U の空でない開部分集合 V で, 次をみたすものがとれる.

(1) J の互いに素な m^r 個の開集合 W_i で, 各 $w \in V$ に対して, $(z, w) \in \mathcal{P}'_n$ となるような $z \in W_i$ がただひとつ存在し, w から各 z_i への対応が, V から W_i への局所同相写像 Φ_i となるものがとれる.

(2) $b(w)$ を V 内にサポートを持つ関数とする. 連続関数 $a(z)$ を

$$a(z) = \begin{cases} b(\Phi_i^{-1}(z)) & z \in W_i \\ 0 & \text{otherwize} \end{cases}$$

と定義することができる. そのとき任意の $x \in X^{\otimes i}$, $y \in X^{\otimes j}$, $0 \leq i, j \leq r$, $i \neq j$ に対して, $aS_xS_y^*a^* = 0$ が成り立つ.

こちらは, p が free on J であることの帰結である.

これらをあわせて, 次の Proposition が成り立つ.

Proposition 3.7. *p は expansive on J free on J とする. 任意の $r \in \mathbb{N}$, 任意の $T \in \mathcal{L}(X^{\otimes r})$, 任意の $\varepsilon > 0$ に対して, $a \in A_J^+$, $\|a\| = 1$ で,*

$$\|\phi(a)T\|^2 \geq \|T\|^2 - \varepsilon,$$
$$aS_xS_y^*a = 0 \quad \forall x \in X^{\otimes i}, \forall y \in X^{\otimes j}, 0 \leq i, j \leq r, i \neq j$$

となるものが取れる.

この Proposition の前半の部分は, $X^{\otimes n}$ と $\mathrm{C}(\mathcal{P}_n)$ の同型により, 関数を使った議論によって示すことができる.

$\mathcal{L}(X^{\otimes r})$ の部分環 $A \otimes I^r + K(X) \otimes I^{r-1} + \cdots + \mathcal{K}(X^{\otimes r})$ から $\mathcal{O}_p(J)^{\mathbf{T}}$ への *-準同型が well-defined であり, しかも等距離となる ([5]). 従って, $b_0 \in \mathcal{O}_p(J)^{\mathbf{T}}$ が代数的な元であるとき, 任意の $\varepsilon > 0$ に対して $\|Pb_0P\| \geq \|b_0\| - \varepsilon$ となるような $P \in A_J^+$, $\|P\| = 1$ で 2 番めの条件も同時にみたすものが取れる.

以上の準備のもとで, あとは Kajiwara-Watatani ([8]) の証明と全く同様に, $\mathcal{O}_p(J)$ が simple かつ purely inifinite であることが証明できる.

Theorem 3.8. *J は p 不変閉集合, p は expansive on J かつ free on J となるとき, $\mathcal{O}_p(J)$ は simple かつ purely infinite である.*

4. EXAMPLES

$p(z,w) = w^m - z^n = 0$ に対して, $\mathbf{T} = \{z \in \mathbb{C} \,|\, |z| = 1\}$ は両側 p-不変である.

Lemma 4.1. *$p(z,w) = w^m - z^n = 0$ を考える. $m \neq n$ のとき, $J = \hat{\mathbb{C}}$ または $J = \mathbf{T}$ のとき p は free on J である. $J = \mathbf{T}$ とするとき, n が m の約数の場合をのぞいて, p は J-拡大的である.*

$p(z,w) = w^m - z^n = 0$ を $J = \mathbf{T}$ に制限する. m, n が互いに素のときには, $\mathcal{C}_p$ は連結となり, Katsura([7]) の例と同じである. m, n が互いに素でないときには, 複数の連結成分が現れ, Katsura([7]) の例とは別のものになる.

Lemma 4.2. *$R_i(z) = P_i(z)/Q_i(z)$ $i = 1, \ldots, n$ を有理関数とする. $p(z,w) = (Q_1(z)w - P_1(z)) \cdots (Q_n(z)w - R_n(z)) = 0$ とする. もし各 R_i の次数が全て 2 以上で互いに素であれば, p は全ての i に対して R_i 不変な完全閉集合 J に対して free on J である.*

次は, ふたつの有理関数に対して共通のジュリア集合が存在する場合である.

Example 4.3. m, n を互いに素な自然数とし, $p(z,w) = (w - z^m)(w - z^n) = 0$ とする. $J = \mathbf{T}$ とする. そのとき, p は free on J かつ J-拡大的となり, $\mathcal{O}_p(\mathbf{T})$ は simple かつ purely infinte である.

この場合, 分岐点のない correspondence の積から新たに分岐点 $(1,1)$ が出現する.

二つの有理関数から correspondece を構成する場合, Julia 集合は必ずしも共通でなくても free かつ expansive になる場合がある.

Example 4.4. $R_1(z) = \frac{(z2+1)2}{4z(z2-1)}$ (Latte の有理関数) であり, $R_2(z) = P_2(z)/Q_2(z)$ は有理関数とする. $p(z,w) = ((4z(z2-1))w - (z2+1)2)(Q_2(z)w - P_2(z))$ とする. $J = \hat{\mathbb{C}}$ として, p は expansive on J である. もし R_2 の次数が奇数なら, p は free on $\hat{\mathbb{C}}$ ともなるので, $\mathcal{O}_p(\hat{\mathbb{C}})$ は simple かつ purely infinite である.

一般に free 条件を判定することは大変だが, 冪関数の積を $\mathbf{T}$ に制限した場合については判定可能である. 次は１つの十分条件である.

Example 4.5. $i_1, \cdots, i_n, j_1, \cdots, j_n$ は自然数とし, i_k と j_k はともに 1 ではなく, また 1 でないものは全て互いに素であるとする. そのとき, $J = \mathbf{T}$ として,

$$p(z,w) = (z^{i_1} - w^{j_1})(z^{i_2} - w^{j_2}) \cdots (z^{i_n} - w^{j_n}) = 0$$

は free on J である.

free でないものは容易に出現する.

Example 4.6. m を 2 以上の自然数として, $p(z,w) = (w - z^m)(w^m - z)$ は, free on $\mathbf{T}$ でない.

この他, 具体例の K-群についても報告する.

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Quasi-orthogonal subalgebras of matrix algebras

Hiromichi Ohno
Graduate School of Mathematics,
Kyushu University

The motivation of this work comes from the algebraic or matrix formalism of finite quantum systems. An n-level system is described by the algebra $M_n = M_n(\mathbb{C})$ of $n \times n$ complex matrices. The matrix algebra of a composite system consisting of an n-level and an m-level system is $M_n \otimes M_m \simeq M_{nm}$. A subalgebra of M_k corresponds to a subsystem of a k-level quantum system.

In this lecture, subalgebras contain the identity and closed under the adjoint operation of matrices, that is, they are unital $*$-subalgebras. The algebra M_k can be endowed by the inner product $\langle A, B\rangle = \mathrm{Tr}(A^*B)$ and it becomes a Hilbert space. Two subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are called quasi-orthogonal if $\mathcal{A}_1 \ominus \mathbb{C}I \perp \mathcal{A}_2 \ominus \mathbb{C}I$.

The aim of this lecture is to show the maximal number of (pairwise) quasi-orthogonal subalgebras which are isomorphic to M_d in M_{d^n} with some special d.

1 Preliminaries

$\mathcal{A}$ is a finite dimensional C^*-algebra with usual trace Tr and is considered as a Hilbert space under the inner product

$$\langle A, B\rangle = \mathrm{Tr}(A^*B)$$

for any $A, B \in \mathcal{A}$.

Definition 1.1 Two subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are called *quasi-orthogonal* (MQOA) if

$$\mathcal{A}_1 \ominus \mathbb{C}I \perp \mathcal{A}_2 \ominus \mathbb{C}I.$$

The equivalent conditions of this definition are following:
(i) For any $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$,

$$\mathrm{Tr}(A_1A_2) = \frac{\mathrm{Tr}(A_1)\mathrm{Tr}(A_2)}{\mathrm{Tr}(I)}.$$

(ii) For any $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$ with $\mathrm{Tr}(A_1) = \mathrm{Tr}(A_2) = 0$,

$$\mathrm{Tr}(A_1A_2) = 0.$$

If e, f, g are vectors of a Hilbert space, then the linear operator $|e\rangle\langle f|$ acts as $|e\rangle\langle f|g := \langle f, g\rangle e$.

Theorem 1.2 *Let E_i be an orthonormal basis in M_n and let $W = \sum_i E_i \otimes W_i \in M_n \otimes M_m$ be a unitary. The subalgebra $W(I \otimes M_m)W^*$ is quasi-orthogonal to $I \otimes M_m$ if and only if*

$$\frac{m}{n} \sum_k |W_k\rangle\langle W_k|$$

is the identity mapping on M_m. This condition cannot hold if $m < n$ and in the case $n = m$ the condition means that $\{W_k : 1 \le k \le n^2\}$ is an orthonormal basis in M_m.

Proof. Assume that $A, B \in M_m$ and $\mathrm{Tr} B = 0$. Then the condition

$$W(I \otimes A^*)W^* \perp (I \otimes B)$$

is equivalently written as

$$\mathrm{Tr}(W(I \otimes A)W^*(I \otimes B)) = \sum_{k,l} \mathrm{Tr}(E_k E_l^*)\mathrm{Tr}(W_k A W_l^* B) = \sum_k \mathrm{Tr}(W_k A W_k^* B) = 0.$$

Putting $B - \mathrm{Tr}(B) I_m/m$ in place of B, we get

$$\sum_k \mathrm{Tr}(W_k A W_k^* B) = \frac{\mathrm{Tr} B}{m} \sum_k \mathrm{Tr}(W_k A W_k^*)\,.$$

for every $B \in M_m$. Let $\mathcal{E}_2 : M_n \otimes M_m \to M_m$ be the linear mapping defined as

$$\mathcal{E}_2(K \otimes L) = \frac{\mathrm{Tr} K}{n} L.$$

Since $\mathcal{E}_2$ is unit-preserving and W is a unitary,

$$I_m = \mathcal{E}_2(W^*W) = \mathcal{E}_2\Big(\sum_{k,l} E_k^* E_l \otimes W_k^* W_l\Big) = \frac{1}{n}\sum_{k,l} \mathrm{Tr}(E_k^* E_l) W_k^* W_l = \frac{1}{n}\sum_k W_k^* W_k,$$

and we arrive at the relation

$$\sum_k \mathrm{Tr} W_k A W_k^* B = \frac{n}{m} \mathrm{Tr} A \mathrm{Tr} B. \tag{1}$$

We can transform this into another equivalent condition in terms of the left multiplication, right multiplication and $|W_k\rangle\langle W_k|$ operators.

For $A, B \in M_m$, the operator R_A is the right multiplication by A and the operator L_B is the left multiplication by B: $R_A, L_B : M_m \to M_m$, $R_A X = XA$, $L_B X = BX$. If λ_i's are the igenvalues of A and μ_j's are the igenvalues of B, then $\lambda_i \mu_j$'s are the eigenvalues of $R_A L_B$. Therefore

$$\mathrm{Tr} R_A L_B = \Big(\sum_i \lambda_i\Big)\Big(\sum_j \mu_j\Big) = \mathrm{Tr} A \mathrm{Tr} B.$$

We have

$$\begin{aligned}\sum_k \mathrm{Tr}|W_k\rangle\langle W_k|R_A L_B &= \sum_k \langle W_k, R_A L_B W_k\rangle = \sum_k \mathrm{Tr} W_k^* B W_k A \\ &= \frac{n}{m}\mathrm{Tr}A\,\mathrm{Tr}B = \frac{n}{m}\mathrm{Tr}R_A L_B\end{aligned}$$

for every $A, B \in M_m$. Since the operators $R_A L_B$ linearly span the space of all linear operators on M_m, we have

$$\frac{m}{n}\sum_k |W_k\rangle\langle W_k| = I_{m^2}.$$

This is our statement. □

2 Quasi-orthogonal subalgebras in the case $d = 2$

In this section, we consider the quasi-orthogonal subalgebras in $M_4 = M_2 \otimes M_2$ which are isomorphic to M_2. A natural orthogonal basis of M_2 consists of the Pauli matrices:

$$\sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is easy to construct four pairwise quasi-orthogonal subalgebras by using the Pauli matrices. For example:

$$\begin{aligned}&\mathrm{span}\{I, \sigma_1 \otimes \sigma_0, \sigma_2 \otimes \sigma_0, \sigma_3 \otimes \sigma_0\}, \quad \mathrm{span}\{I, \sigma_0 \otimes \sigma_1, \sigma_1 \otimes \sigma_2, \sigma_1 \otimes \sigma_3\} \\ &\mathrm{span}\{I, \sigma_2 \otimes \sigma_1, \sigma_0 \otimes \sigma_2, \sigma_2 \otimes \sigma_3\}, \quad \mathrm{span}\{I, \sigma_3 \otimes \sigma_1, \sigma_3 \otimes \sigma_2, \sigma_0 \otimes \sigma_3\}.\end{aligned}$$

Next, we prove that the maximal number of such quasi-orthogonal subalgebras is 4.

Theorem 2.1 *Let $I \otimes M_2$ and $\mathcal{A}$ be quasi-orthogonal subalgebras of M_4 which are isomorphic to M_2. Then the intersection $M_2 \otimes I \cap \mathcal{A}$ is an at least two dimensional subspace of M_4.*

Proof. The 4×4 matrices

$$C = \begin{bmatrix} a & 0 & 0 & b \\ 0 & c & d & 0 \\ 0 & d & c & 0 \\ b & 0 & 0 & a \end{bmatrix}$$

form a commutative algebra $\mathcal{C}$. Since

$$\sum_{i=0}^{3} c_i\,\sigma_i \otimes \sigma_i = \begin{bmatrix} c_0 + c_3 & 0 & 0 & c_1 - c_2 \\ 0 & c_0 - c_3 & c_1 + c_2 & 0 \\ 0 & c_1 + c_2 & c_0 - c_3 & 0 \\ c_1 - c_2 & 0 & 0 & c_0 + c_3 \end{bmatrix},$$

$\mathcal{C}$ is the linear span of the matrices $\sigma_i \otimes \sigma_i$, $0 \le i \le 3$. (These are the matrices which are diagonal in the so-called Bell basis.)

The algebra $\mathcal{C}$ plays a special role. Any unitary in M_4 can be written in the form

$$(L_1 \otimes L_2)N(L_3 \otimes L_4), \tag{2}$$

where L_1, L_2, L_3, L_4 are 2×2 unitaries and the unitary N is in $\mathcal{C}$. This is called Cartan decomposition, see equation (11) in [7] or [3].

There is a unitary $W \in M_4$ such that

$$W(I \otimes M_2)W^* = \mathcal{A}.$$

W has a Cartan decomposition (2). Since the subalgebra $W(I \otimes M_2)W^*$ does not depend on L_3 and L_4, we may assume that $L_3 = L_4 = I$. Moreover, the quasi-orthogonality of $W(I \otimes M_2)W^*$ and $I \otimes M_2$ does not depend on L_1 and L_2. The quasi-orthogonality is determined by the factor $N \in \mathcal{C}$. Since the matrices $E_i = \sigma_i/\sqrt{2}$ form a basis in M_2, Theorem 1.2 is conveniently applied for the unitary $N = \sum_{i=0}^{3} c_i\, \sigma_i \otimes \sigma_i$, choose W_i as $c_i\sqrt{2}\sigma_i$. The theorem gives that

$$2\sum_{i=0}^{3} |c_i|^2 |\sigma_i\rangle\langle\sigma_i|$$

is the identity mapping on M_2 which implies $|c_i|^2 = 1/4$ $(0 \le i \le 3)$. In a trigonometric approach, let

$$\begin{aligned}
c_0 &= \cos\alpha\, \cos\beta \cos\gamma + i \sin\alpha\, \sin\beta\, \sin\gamma\,, \\
c_1 &= \cos\alpha\, \sin\beta \sin\gamma + i \sin\alpha\, \cos\beta\, \cos\gamma\,, \\
c_2 &= \sin\alpha\, \cos\beta \sin\gamma + i \cos\alpha\, \sin\beta\, \cos\gamma\,, \\
c_3 &= \sin\alpha\, \sin\beta \cos\gamma + i \cos\alpha\, \cos\beta\, \sin\gamma\,.
\end{aligned}$$

In order to get a proper unitary, two of the values of $\cos^2\alpha, \cos^2\beta$ and $\cos^2\gamma$ equal $1/2$ and the third one may be arbitrary. Let $\mathcal{N}$ be the set of all matrices such that the parameters α, β and γ satisfy the above condition, in other words two of the three values are of the form $\pi/4 + k\pi/2$. (k is an integer.) Let

$$\mathcal{N}_1 := \{N \in \mathcal{N} : \alpha \text{ is arbitrary, } \beta = \pi/4 + k_1\pi/2, \text{ and } \gamma = \pi/4 + k_2\pi/2\} \tag{3}$$

and define $\mathcal{N}_2$ and $\mathcal{N}_3$ similarly. ($\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$.). Since the subalgebra $N(I \otimes M_2)N^*$ does not depend on the integers k_1 and k_2, we simply take $k_1 = k_2 = 0$. This makes computations a bit more convenient. One computes that

$$N_i(I \otimes \sigma_i)N_i^* = \pm\sigma_i \otimes I$$

for $N_i \in \mathcal{N}_i$. It follows that

$$(L_1 \otimes L_2)N_i(I \otimes \sigma_i)N_i^*(L_1^* \otimes L_2^*) = \pm L_1\sigma_i L_1^* \otimes I$$

for every unitary $N_i \in \mathcal{N}_i$. Therefore $L_1\sigma_i L_1^* \otimes I \in \mathcal{A}(0)' \cap \mathcal{B}$. □

The theorem immediately gives that the maximal number of pairwise quasi-orthogonal subalgebras isomorphic to M_2 is at most 4.

3 Quasi-orthogonal subalgebras in M_{2^n}

Next we consider the pairwise quasi-orthogonal subalgebras $\mathcal{A}_i \simeq M_2$ in M_{2^n}. Let $m(n)$ be the maximal number of pariwise quasi-orthogonal subalgebras of M_{2^n} which are isomorphic to M_2. The question is their maximal number $m(n)$.

The traceless subspaces of M_2 and M_{2^n} are a 3-dimensional space and a (4^n-1)-dimensional space, respectively. Therefore,

$$m(n) \leq \frac{4^n - 1}{3} =: N_n.$$

Below, we construct $N_n - 1$ pairwise quasi-orthogonal subalgebras. We conjecture that this is the true value of $m(n)$.

The Hilbert space M_{2^n} has a natural orthogonal basis

$$\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n} =: (i_1, i_2, \dots, i_n),$$

where $i_j = 0,1,2,3$ and $1 \leq j \leq n$. We put

$$P_n = \{(i_1, i_2, \dots, i_n) \,:\, 0 \leq i_j \leq 3,\, 1 \leq j \leq n\} \setminus \{I\}.$$

A triplet $(A_1, A_2, A_3) \in P_n^3$ is called a weak Pauli triplet if $A_1 A_2 = \pm i A_3$. and $(A_1, A_2, A_3) \in P_n^3$ is a commuting triplet if $A_1 A_2 = \pm A_3$. The linear span of elements of a weak Pauli triplet and I is a subalgebra isomorphic to M_2.

Assume that $A = (A_1, A_2, A_3) \in P_n^3$ is a commuting triplet. Then we can construct three pairwise disjoint weak Pauli triplets: $\hat{A}^{(1)} := (\sigma_1 \otimes A_1, \sigma_2 \otimes A_2, \sigma_3 \otimes A_3)$ and $\hat{A}^{(2)} := (\sigma_2 \otimes A_1, \sigma_3 \otimes A_2, \sigma_1 \otimes A_3)$ and $\hat{A}^{(3)} := (\sigma_3 \otimes A_1, \sigma_1 \otimes A_2, \sigma_2 \otimes A_3)$ in P_{n+1}^3. Therefore, to construct pairwise quasi-orthogonal subalgebras isomorphic to M_2, it is useful to consider weak Pauli triplets and commuting triplets.

Example 3.1 *There are 5 pairwise disjoint commuting triplets in P_2^3. Indeed,*

$$\begin{aligned}&((0,1),(1,0),(1,1)), \quad ((0,2),(2,0),(2,2)), \quad ((0,3),(3,0),(3,3)),\\ &((1,2),(2,3),(3,1)), \quad ((1,3),(2,1),(3,2)).\end{aligned}$$

There are 21 pairwise disjoint commuting triplets in P_3^3. Indeed,

$$\begin{array}{lll}
((1,0,1),(2,0,3),(3,0,2)), & ((1,0,2),(2,0,1),(3,0,3)), & ((0,1,1),(0,2,3),(0,3,2)),\\
((0,1,3),(0,1,0),(0,0,3)), & ((0,2,2),(0,2,0),(0,0,2)), & ((0,3,1),(0,3,0),(0,0,1)),\\
((3,2,1),(3,0,0),(0,2,1)), & ((2,1,2),(2,0,0),(0,1,2)), & ((1,3,3),(1,0,0),(0,3,3)),\\
((3,3,1),(2,3,2),(1,0,3)), & ((3,1,1),(1,1,3),(2,0,2)), & ((2,2,2),(1,2,3),(3,0,1)),\\
((1,1,1),(2,2,1),(3,3,0)), & ((1,2,1),(2,3,1),(3,1,0)), & ((1,3,1),(2,1,1),(3,2,0)),\\
((1,1,2),(2,2,0),(3,3,2)), & ((1,2,2),(2,3,0),(3,1,2)), & ((1,3,2),(2,1,0),(3,2,2)),\\
((1,1,0),(2,2,3),(3,3,3)), & ((1,2,0),(2,3,3),(3,1,3)), & ((1,3,0),(2,1,3),(3,2,3)).
\end{array}$$

We show that P_n can be decomposed into commuting triplets.

Theorem 3.2 *For each $n \geq 2$, there is a family of commuting triplets*

$$\{A^{(i)} = (A_1^{(i)}, A_2^{(i)}, A_3^{(i)})\}_{i=1}^{N_n} \subset P_n^3$$

such that

$$\bigcup_{i=1}^{N_n} A^{(i)} = P_n.$$

Proof. In the case $n = 2$ and $n = 3$, it is already proven above. Assume it is proven in the case $n = k$, and we consider the case $n = k + 2$. Let $\{A^{(i)}\}_{i=1}^5$ and $\{B^{(j)}\}_{j=1}^{N_k}$ be the family of commuting triplets satisfying the theorem in the case of $n = 2$ and $n = k$, respectively. Then, for each $A^{(i)} = (A_1^{(i)}, A_2^{(i)}, A_3^{(i)})$ and $B^{(j)} = (B_1^{(j)}, B_2^{(j)}, B_3^{(j)})$, we can construct three commuting triplets in P_{k+2}^3, that is, $(A_1^{(i)} \otimes B_1^{(j)}, A_2^{(i)} \otimes B_2^{(j)}, A_3^{(i)} \otimes B_3^{(j)})$ and $(A_1^{(i)} \otimes B_2^{(j)}, A_2^{(i)} \otimes B_3^{(j)}, A_3^{(i)} \otimes B_1^{(j)})$ and $(A_1^{(i)} \otimes B_3^{(j)}, A_2^{(i)} \otimes B_1^{(j)}, A_3^{(i)} \otimes B_2^{(j)})$. Moreover, we have other commuting triplets, i.e., $(A_1^{(i)} \otimes I_k, A_2^{(i)} \otimes I_k, A_3^{(i)} \otimes I_k)$ and $(I_2 \otimes B_1^{(j)}, I_2 \otimes B_2^{(j)}, I_2 \otimes B_3^{(j)})$. Consequently, we have $5 + N_k + 3 \cdot 5 \cdot N_k = N_{k+2}$ commuting triplets. Since $\bigcup_{i=1}^5 A^{(i)} = P_2$ and $\bigcup_{i=1}^{N_k} B^{(i)} = P_k$, $\{A_1^{(i)}, A_2^{(i)}, A_3^{(i)}\}_{i=1}^5$ and $\{B_1^{(j)}, B_2^{(j)}, B_3^{(j)}\}_{j=1}^{N_k}$ are distinct. Hence, we obtain the union of the above N_{k+2} commuting triplets is P_{k+2}. □

The good point of this construction is that it is easy to use the induction.

Theorem 3.3 *There exist $N_n - 1$ quasi-orthogonal subalgebras in M_{2^n}.*

Proof. The case $n = 2$ is already proven in Theorem 3. Assume it is proven for $n = k$, and we consider the case $n = k + 1$.

From Theorem 3.2, let $\{A^{(i)} = (A_1^{(i)}, A_2^{(i)}, A_3^{(i)})\}_{i=1}^{N_k}$ be commuting triplets in P_k^3 such that $\bigcup_{i=1}^{N_k} A^{(i)} = P_k$. Then we have $3N_k$ pairwise disjoint weak Pauli triplets, that is, $(\sigma_1 \otimes A_1^{(i)}, \sigma_2 \otimes A_2^{(i)}, \sigma_3 \otimes A_3^{(i)})$ and $(\sigma_2 \otimes A_1^{(i)}, \sigma_3 \otimes A_2^{(i)}, \sigma_1 \otimes A_3^{(i)})$ and $(\sigma_3 \otimes A_1^{(i)}, \sigma_1 \otimes A_2^{(i)}, \sigma_2 \otimes A_3^{(i)})$. Furthermore, we obtain another weak Pauli triplet $(\sigma_1 \otimes I_k, \sigma_2 \otimes I_k, \sigma_3 \otimes I_k)$. These $3N_k + 1$ weak Pauli triplets are pairwise disjoint. Moreover, the complement space of above $3N_k + 1$ Pauli triplets is $I \otimes M_{2^k}$. Indeed, since $\bigcup_{i=1}^{N_k} A^{(i)} = P_k$, we have

$$\begin{aligned}
&\{(\sigma_1 \otimes A_1^{(i)}, \sigma_2 \otimes A_2^{(i)}, \sigma_3 \otimes A_3^{(i)}), (\sigma_2 \otimes A_1^{(i)}, \sigma_3 \otimes A_2^{(i)}, \sigma_1 \otimes A_3^{(i)}), \\
&(\sigma_3 \otimes A_1^{(i)}, \sigma_1 \otimes A_2^{(i)}, \sigma_2 \otimes A_3^{(i)}), (\sigma_1 \otimes I_k, \sigma_2 \otimes I_k, \sigma_3 \otimes I_k) \mid 1 \leq i \leq N_k\} \\
= \ &\{\sigma_i \otimes \sigma_{j_1} \otimes \cdots \otimes \sigma_{j_k} \mid i = 1, 2, 3, \ j_l = 0, 1, 2, 3, \ 1 \leq l \leq k\}.
\end{aligned}$$

Therefore, the complement space is $I \otimes M_{2^k}$ spanned by

$$\{\sigma_0 \otimes \sigma_{j_1} \otimes \cdots \otimes \sigma_{j_k} \mid j_l = 0, 1, 2, 3, \ 1 \leq l \leq k\}.$$

Now we use the assumption that there are $N_k - 1$ pairwise disjoint weak Pauli triplets $B^{(i)} = (B_1^{(i)}, B_2^{(i)}, B_3^{(i)})$ in M_{2^k} $(1 \leq i \leq N_k - 1)$. Then

$$(\sigma_0 \otimes B_1^{(i)}, \sigma_0 \otimes B_2^{(i)}, \sigma_0 \otimes B_3^{(i)})$$

give pairwise disjoint weak Pauli triplets in P^3_{k+1}. Summing up, we have $3N_k + 1 + N_k - 1 = 4N_k = N_{k+1} - 1$ pairwise disjoint weak Pauli triplets. $\square$

Similarly, we can prove the following. If there exist N_n pairwise quasi-orthogonal subalgebras in M_{2^n} for some n, then there exist N_k pairwise quasi-orthogonal subalgebras in M_{2^k} for all $k \geq n$.

4 Quasi-orthogonal subalgebras: d is prime

In this section, we consider the quasi-orthogonal subalgebras in $M_{p^2} = M_p \otimes M_p$ which are isomorphic to M_p, where p is a prime number with $p \geq 3$. In this case, we can construct $p^2 + 1$ pairwise quasi-orthogonal subalgebras.

Define the unitary operators W and S in M_p by

$$W = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^{p-1} \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

where $\lambda = e^{2\pi i/p}$. A natural orthogonal basis of M_p consists of $\{S^i W^j\}_{0 \leq i,j \leq p-1}$. Since $SW = \lambda^{-1} WS$, we have

$$S^{k_1} W^{l_1} S^{k_2} W^{l_2} = \lambda^{l_1 k_2} S^{k_1 + k_2} W^{l_1 + l_2}. \tag{4}$$

Therefore $S^{k_1} W^{l_1}$ and $S^{k_2} W^{l_2}$ commute if and only if $k_1 l_2 = k_2 l_1 \bmod p$. We consider this commutativity condition in the context of a vecter space over the finite field Z_p. Let $Z_p^4 = \{(k_1, l_1, k_2, l_2) \,|\, k_1, l_1, k_2, l_2 \in Z_p\}$ and define a nutural homomorphism π (up to scalar multiple) from Z_p^4 to $M_p \otimes M_p$ by

$$\pi(k_1, l_1, k_2, l_2) = S^{k_1} W^{l_1} \otimes S^{k_2} W^{l_2}.$$

We denote a *symplectic product* by

$$u \circ u' = k_1 l_1' - k_1' l_1 + k_2 l_2' - k_2' l_2 \qquad \bmod p,$$

where $u = (k_1, l_1, k_2, l_2)$ and $u' = (k_1', l_1', k_2', l_2')$. From (4),

$$\pi(u)\pi(u') = \lambda^{-u \circ u'} \pi(u')\pi(u). \tag{5}$$

Hence $\pi(u)$ and $\pi(u')$ commute if and only if their symplectic product equals zero.

Lemma 4.1 *If $\pi(u)$ and $\pi(u')$ are not commutative for $u = (k_1, l_1, k_2, l_2)$, $u' = (k_1', l_1', k_2', l_2') \in Z_p^4$, then the algebra $\mathcal{A}$ generated by $\pi(u)$ and $\pi(u')$ is isomorpic to M_p.*

Proof. From the assumption, $u \circ u' \neq 0$. We define a map ρ from $\{S, W^{u \circ u'}\}$ to $\mathcal{A}$ by

$$\rho(S) = \pi(u), \qquad \rho(W^{u \circ u'}) = \pi(u').$$

From (5) and $SW^{u \circ u'} = \lambda^{-u \circ u'} W^{u \circ u'} S$, the commutativity condition of $\pi(u)$, $\pi(u')$ and that of S, $W^{u \circ u'}$ are same. Therefore ρ can be extended to a isomorphism from M_p generated by S and $W^{u \circ u'}$ to $\mathcal{A}$. □

From this lemma, we need to find such u and u'. Let D be a non-zero interger in Z^p with the requirement that $D \neq k^2 \bmod p$ for all k in Z_p, i.e. D is not a quadratic residue of p. For any $a_0, a_1 \in Z_p$, we define subgroups of Z_p^4 by

$$C_{a_0,a_1} = \{b_0(1, a_1, 0, a_0) + b_1(0, a_0, -1, a_1 D) \mid b_0, b_1 \in Z_p\},$$

where scalar multipication and addition are defined by a natural way. Moreover define

$$C_\infty = \{b_0(0, 1, 0, 0) + b_1(0, 0, 0, 1) \mid b_0, b_1 \in Z_p\}.$$

Lemma 4.2 *The only vector common to any pair of above subgroups is* $(0, 0, 0, 0)$. *In particular, the subgroups partition* $Z_p^4 \backslash \{(0, 0, 0, 0)\}$.

Proof. Since there are p^2+1 subgroups and each subgroup has p^2 elements, it is enough to prove that the intersection of any two subgroups is $\{(0, 0, 0, 0)\}$. It is easy to see $C_{a_0,a_1} \cap C_\infty = \{(0, 0, 0, 0)\}$. Therefore we prove that $C_{a_0,a_1} \cap C_{a'_0,a'_1} = \{(0, 0, 0, 0)\}$ if $a_0 \neq a'_0$ or $a_1 \neq a'_1$.

Assume $b_0(1, a_1, 0, a_0) + b_1(0, a_0, -1, a_1 D) = b'_0(1, a'_1, 0, a'_0) + b'_1(0, a'_0, -1, a'_1 D)$, then from the first and third components we have $b_0 = b'_0$ and $b_1 = b'_1$. Similary, from the second and fourth components we are led to the equations

$$\begin{aligned} a_1 b_0 + a_0 b_1 &= a'_1 b_0 + a'_0 b_1 \\ a_0 b_0 + a_1 b_1 D &= a'_0 b_0 + a'_1 b_1 D. \end{aligned}$$

These equations can be rewritten as a matrix equation

$$\begin{bmatrix} b_1 & b_0 \\ b_0 & b_1 D \end{bmatrix} \begin{bmatrix} a_0 - a'_0 \\ a_1 - a'_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If $b_0 = b_1 = 0$, then the element is $(0, 0, 0, 0)$. Therefore we assume $b_0 \neq 0$ or $b_1 \neq 0$. Then the above matrix is invertible, indeed

$$\begin{bmatrix} b_1 & b_0 \\ b_0 & b_1 D \end{bmatrix}^{-1} = (b_1^2 D - b_0^2)^{-1} \begin{bmatrix} b_1 D & -b_0 \\ -b_0 & b_1 \end{bmatrix}.$$

Here we use that $b_1^2 D \neq b_0^2 \bmod p$ from the assumption of D. This implies $a_0 = a'_0$ and $a_1 = a'_1$ which is a contradiction. □

Since $(1, a_1, 0, a_0) \circ (0, a_0, -1, a_1 D) = 2a_0$, if $a_0 \neq 0$ then

$$\mathrm{span}\{\pi(C_{a_0,a_1})\} \simeq M_p$$

by Lemma 4.1. But if $a_0 = 0$, the algebra is commutative and hence $\mathrm{span}\{\pi(C_{a_0,a_1})\} \simeq \mathbb{C}^{p^2}$. Therefore we need reconstruct subgropus in the case $a_0 = 0$.

For any $a \in Z_p$, define subgroups by

$$\begin{aligned} D_a &= \{b_0(1,1,-a,aD) + b_1(1,2,-a,2aD) \mid b_0, b_1 \in Z_p\}, \\ D_\infty &= \{b_0(0,0,1,0) + b_1(0,0,0,1) \mid b_0, b_1 \in Z_p\}. \end{aligned}$$

Lemma 4.3 *The only vector common to any pair of above subgroups is* $(0,0,0,0)$*. Moreover we have*

$$\bigcup_{a \in Z_p} D_a \cup D_\infty = \bigcup_{a_1 \in Z_p} C_{0,a_1} \cup C_\infty.$$

Proof. It is easy to see the first assertion. Therefore to show the second assertion, it is enough to prove $D_a, D_\infty \subset \bigcup_{a_1 \in Z_p} C_{0,a_1} \cup C_\infty$.

First consider the element $(0,0,b_0,b_1)$ in D_∞. If $b_0 = 0$, then $(0,0,0,b_1) \in C_\infty$. If $b_0 \neq 0$, then

$$(0,0,b_0,b_1) = -b_0(0,0,-1,-b_0^{-1}b_1 D^{-1} D) \in C_{0,-b_0^{-1}b_1 D^{-1}}.$$

Hence $D_\infty \subset \bigcup_{a_1 \in Z_p} C_{0,a_1} \cup C_\infty$. Next we consider the element $b_0(1,1,-a,aD) + b_1(1,2,-a,2aD)$ in D_a. If $b_0 + b_1 = 0$, then $b_0(1,1,-a,aD) + b_1(1,2,-a,2aD) = (0, b_0 + 2b_1, 0, ab_0 D + 2ab_1 D) \in C_\infty$. If $b_0 + b_1 \neq 0$, then

$$\begin{aligned} & b_0(1,1,-a,aD) + b_1(1,2,-a,2aD) \\ = \ & (b_0+b_1)\big(1, (b_0+b_1)^{-1}(b_0+2b_1), 0, 0\big) \\ & + a(b_0+b_1)\big(0,0,-1,(b_0+b_1)^{-1}(b_0+2b_1)D\big) \\ \in \ & C_{0,(b_0+b_1)^{-1}(b_0+2b_1)}. \end{aligned}$$

Therefore we obtain $D_a \subset \bigcup_{a_1 \in Z_p} C_{0,a_1} \cup C_\infty$. □

Since $(1,1,-a,aD) \circ (1,2,-a,2aD) = 1 - a^2 D \neq 0$ by the assumption of D and $(0,0,1,0) \circ (0,0,0,1) = 1$, we obtain

$$\begin{aligned} \mathrm{span}\{\pi(D_a)\} &\simeq M_p \\ \mathrm{span}\{\pi(D_\infty)\} &\simeq M_p \end{aligned}$$

by Lemma 4.1. Consecently we have the next theorem.

Theorem 4.4 *There are* $p^2 + 1$ *pairwise quasi-orthogonal subalgebras of* M_{p^2} *which are isomorphic to* M_p*.*

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ORBIT EQUIVALENCE OF TOPOLOGICAL MARKOV SHIFTS AND CUNTZ-KRIEGER ALGEBRAS

Kengo Matsumoto

Department of Mathematical Sciences
Yokohama City University
Seto 22-2, Kanazawa-ku, Yokohama 236-0027, JAPAN

1. Introduction

This talk is based on my recent preprint [Ma4].

Study of orbit equivalence of ergodic finite measure preserving transformations was initiated by H. Dye [D], [D2], who proved that two such tranformations are orbit equivalent. W. Krieger [Kr] has proved that two ergodic non-singular transformations are orbit equivalent if and only if the associated von Neumann crossed products are isomorphic. In topological setting, Giordano-Putnam-Skau [GPS],[GPS2] (cf.[HPS]) have proved that two minimal homeomorphisms on Cantor sets are strong orbit equivalent if and only if the associated C^*-crossed products are isomorphic. In more general setting, J. Tomiyama [To2] (cf. [BT], [To3]) has proved that two topological free homeomorphisms (X, ϕ) and (Y, ψ) on compact Hausdorff spaces are continuously orbit equivalent if and only if there exists an isomorphism between the associated C^*-crossed products keeping their commutative C^*-subalgebras $C(X)$ and $C(Y)$. He also proved that it is equivalent to the condition that there exists a homeomorphism $h : X \to Y$ such that h preserves their topological full groups.

In this talk we will study relationship between orbit structure of one-sided topological Markov shifts and algebraic structure of the associated Cuntz-Krieger algebras. Let (X_A, σ_A) be the right one-sided topological Markov shift defined by an $N \times N$ square matrix A with entries in $\{0, 1\}$, where σ_A denotes the shift transformation on X_A. The one-sided topological Markov shifts are no longer homeomorphism in general and the Cuntz-Krieger algebras can not be written as a crossed product by $\mathbb{Z}$ in natural way. Hence Giordano-Putnam-Skau and Tomiyama's method can not apply to study one-sided topological Markov shifts and Cuntz-Krieger algebras. However, in this paper, similar type theorems to theirs will be proved in our setting by using a representation of $\mathcal{O}_A$ on a Hilbert space having its complete orthonormal basis consisting of all points of the shift space X_A. Let $\mathfrak{D}_A$ be the C^*-subalgebra consisting of all diagonal elements of the canonical AF-algebra $\mathcal{F}_A$ inside of $\mathcal{O}_A$. It is naturally isomorphic to the commutative C^*-algebra $C(X_A)$ of all complex valued continuous functions on X_A. Let $[\sigma_A]_c$ be the topological full group of (X_A, σ_A) whose elements consist of homeomorphisms τ on X_A such that $\tau(x)$ is contained in the orbit $orb_{\sigma_A}(x)$ of x under σ_A for all $x \in X_A$, and its orbit cocycles are continuous. We say that (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent if there exists a homeomorphism $h : X_A \longrightarrow X_B$ such that $h(orb_{\sigma_A}(x)) = orb_{\sigma_B}(h(x))$ for $x \in X_A$ and their orbit cocycles are continuous.

Theorem 1.1. *Let A, B be irreducible square matrices with entries in $\{0,1\}$ satisfying condition (I) in [CK]. Then the following are equivalent:*

1. *There exists an isomorphism $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\Psi(\mathfrak{D}_A) = \mathfrak{D}_B$.*
2. *(X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*
3. *There exists a homeomorphism $h : X_A \to X_B$ such that $h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$.*

To prove the above theorem, we study the normalizer $N(\mathcal{O}_A, \mathfrak{D}_A)$ of $\mathfrak{D}_A$ in $\mathcal{O}_A$, that is the group of all unitaries $u \in \mathfrak{D}_A$ such that $u\mathfrak{D}_A u^* = \mathfrak{D}_A$. We denote by $\mathcal{U}(\mathfrak{D}_A)$ the group of all unitaries in $\mathfrak{D}_A$.

Theorem 1.2. *Let A be a square matrix with entries in $\{0,1\}$ satisfying condition (I) in [CK]. Then there exists a short exact sequnce:*

$$1 \longrightarrow \mathcal{U}(\mathfrak{D}_A) \xrightarrow{\mathrm{id}} N(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\tau} [\sigma_A]_c \longrightarrow 1$$

that splits.

Let $\mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$ be the group of all automorphisms α of $\mathcal{O}_A$ such that $\alpha(\mathfrak{D}_A) = \mathfrak{D}_A$. Denote by $\mathrm{Inn}(\mathcal{O}_A, \mathfrak{D}_A)$ the subgroup of $\mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$ of inner automorphisms on $\mathcal{O}_A$. We set $\mathrm{Out}(\mathcal{O}_A, \mathfrak{D}_A)$ the quotient group $\mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)/\mathrm{Inn}(\mathcal{O}_A, \mathfrak{D}_A)$.

Theorem 1.3. *Let A be an irreducible square matrix with entries in $\{0,1\}$ satisfying condition (I) in [CK]. Then there exist short exact sequeneces:*

1. $1 \longrightarrow B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Inn}(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\phi} [\sigma_A]_c \longrightarrow 1,$
2. $1 \longrightarrow Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\phi} N([\sigma_A]_c) \longrightarrow 1,$
3. $1 \longrightarrow H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Out}(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\phi} N([\sigma_A]_c)/[\sigma_A]_c \longrightarrow 1.$

They all split. Hence $\mathrm{Out}(\mathcal{O}_A, \mathfrak{D}_A)$ is a semi-direct product

$$\mathrm{Out}(\mathcal{O}_A, \mathfrak{D}_A) = N([\sigma_A]_c)/[\sigma_A]_c \cdot H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)).$$

where $N([\sigma_A]_c)$ denotes the normalizer subgroup of $[\sigma_A]_c$ in the group $\mathrm{Homeo}(X_A)$ of all homeomorphisms on X_A, and $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$, $B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ and $H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ are the group of unitary one-cocycles for σ_A, the subgroup of $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ of coboundaries and the cohomology group $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))/B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ respectively.

Similar type theorems hold for the pair of the canonical AF-algebra $\mathcal{F}_A$ inside of $\mathcal{O}_A$ and its diagonal algebra $\mathfrak{D}_A$, that are studied in Section 7.

In [Ma5], the results of this talk is generalized to more general subshifts and the C^*-algebras associated with the subshifts considered in [Ma] (cf.[CM]) and [Ma3]. Throughout the paper, we denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the set of nonnegative integers and the set of positive integers respectively.

2. Preliminaries

Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries in $\{0,1\}$, where $1 < N \in \mathbb{N}$. Throughout the paper, we always assume that A satisfies condition (I) in the sense of Cuntz-Krieger [CK]. We denote by X_A the shift space

$$X_A = \{(x_n)_{n\in\mathbb{N}} \in \{1,\dots,N\}^{\mathbb{N}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}$$

over $\{1,\dots,N\}$ of the right one-sided topological Markov shift for A. It is a compact Hausdorff space in natural product topology. The shift transformation σ_A on X_A is defined by $\sigma_A((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$. It is a continuous surjective map on X_A. The topological dynamical system (X_A, σ_A) is called the (right one-sided) topological Markov shift for A. The condition (I) for A is equivalent to the condition that X_A is homeomorphic to a Cantor discontinuum. A word $\mu = \mu_1 \cdots \mu_k$ for $\mu_i \in \{1,\dots,N\}$ is said to be admissible for X_A if μ appears in somewhere in some element x in X_A. The length of μ is k and denoted by $|\mu|$. We denote by $B_k(X_A)$ the set of all admissible words of length $k \in \mathbb{N}$. For $k = 0$ we denote by $B_0(X_A)$ the empty word $\emptyset$. We set $B_*(X_A) = \cup_{k=0}^\infty B_k(X_A)$ the set of admissible words of X_A.

The Cuntz-Krieger algebra $\mathcal{O}_A$ for the matrix A has been defined by the universal C^*-algebra generated by N partial isometries $S_1,\dots,S_N$ subject to the relations:

$$\sum_{j=1}^N S_j S_j^* = 1, \qquad S_i^* S_i = \sum_{j=1}^N A(i,j) S_j S_j^*, \quad i = 1,\dots,N \qquad ([CK]).$$

If A satisfies condition (I), the algebra $\mathcal{O}_A$ is the unique C^*-algebra subject to the above relations. For a word $\mu = \mu_1 \cdots \mu_k \in B_k(X_A)$, we denote by $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$. It is well-known that the fixed point algebra of $\mathcal{O}_A$ under the gauge action ρ is the AF-algebra $\mathcal{F}_A$ generated by elements $S_\mu S_\nu^*, \mu, \nu \in B_*(X_A)$ with $|\mu| = |\nu|$ ([CK]). Let $\mathcal{F}_A^n$ be the C^*-subalgebra of $\mathcal{F}_A$ generated by elements $S_\mu S_\nu^*, \mu, \nu \in B_n(X_A)$. Hence $\mathcal{F}_A^{\mathrm{alg}} = \cup_{n=1}^\infty \mathcal{F}_A^n$ is a dense $*$-subalgebra of $\mathcal{F}_A$. Let $\mathfrak{D}_A$ be the C^*-subalgebra of $\mathcal{F}_A$ consisting of all diagonal elements of $\mathcal{F}_A$. It is generated by elements $S_\mu S_\mu^*, \mu \in B_*(X_A)$ and isomorphic to the commutative C^*-algebra $C(X_A)$ of all complex valued continuous functions on X_A through the correspondence $S_\mu S_\mu^* \in \mathfrak{D}_A \longleftrightarrow \chi_\mu \in C(X_A)$ where χ_μ denotes the characteristic function on X_A for the cylinder set $U_\mu = \{(x_n)_{n\in\mathbb{N}} \in X_A \mid x_1 = \mu_1, \dots, x_k = \mu_k\}$ for $\mu = \mu_1 \cdots \mu_k \in B_k(X_A)$. We identify $C(X_A)$ with the subalgebra $\mathfrak{D}_A$ of $\mathcal{O}_A$. Then the following lemma is well-known and basic in our further discussions.

Lemma 2.1([CK;Remark 2.18], cf.[Ma2;Proposition 3.3]). *The algebra $\mathfrak{D}_A$ is maximal abelian in $\mathcal{O}_A$.*

In [To2], [To3], Tomiyama has used structure of pure state extensions of point evaluations of the underlying space to study orbit structure of topological dynamical systems of homeomorphisms on compact Hausdorff spaces (cf. [To], [To4]). However for the Cuntz-Krieger algebras, it has not been clarified structure of pure state extensions of point evaluations of the underlying shift space. Instead of point evaluations, we will use a representation of the Cuntz-Krieger algebra $\mathcal{O}_A$ on a Hilbert space having the shift space X_A as a complete orthonormal basis, as in the following way. Let $\mathfrak{H}_A$ be the Hilbert space with its complete orthonormal system $e_x, x \in X_A$. The Hilbert space is not separable. Consider the partial isometries $T_i, i = 1, \dots, N$ defined by

$$T_i e_x = \begin{cases} e_{ix} & \text{if } ix \in X_A, \\ 0 & \text{otherwise,} \end{cases}$$

where ix denotes $ix = (i, x_1, x_2, \dots)$ for $x = (x_n)_{n\in\mathbb{N}} \in X_A$. The relations $\sum_{j=1}^N T_j T_j^* = 1, T_i^* T_i = \sum_{j=1}^N A(i,j) T_j T_j^*$ for $i = 1, \dots, N$ hold. Since A satisfies condition (I), the operator T_i is nonzero for each $i = 1, \dots, N$ so that the correspondence $S_i \to T_i$ yields a faithful representation of $\mathcal{O}_A$ on $\mathfrak{H}_A$. We regard the algebra $\mathcal{O}_A$ as the C^*-algebra generated by $T_i, i = 1, \dots, N$ on $\mathfrak{H}_A$ by this representation, and write T_i as S_i (cf. [Ma2;Lemma 4.1]).

3. Topological Full groups of Markov shifts

For $x = (x_n)_{n\in\mathbb{N}} \in X_A$, the orbit $orb_{\sigma_A}(x)$ of x under σ_A is defined by

$$orb_{\sigma_A}(x) = \cup_{k=0}^{\infty} \cup_{l=0}^{\infty} \sigma_A^{-k}(\sigma_A^l(x)) \subset X_A.$$

Hence $y = (y_n)_{n\in\mathbb{N}} \in X_A$ belongs to $orb_{\sigma_A}(x)$ if and only if there exists an admissible word $\mu_1 \cdots \mu_k \in B_k(X_A)$ such that $y = (\mu_1, \dots, \mu_k, x_{l+1}, x_{l+2}, \dots)$ for some $k, l \in \mathbb{Z}_+$. Denote by $\mathrm{Homeo}(X_A)$ the group of all homeomorphisms on X_A.
De nition. Let $[\sigma_A]$ be the set of all homeomorphism $\tau \in \mathrm{Homeo}(X_A)$ such that $\tau(x) \in orb_{\sigma_A}(x)$ for all $x \in X_A$. We call $[\sigma_A]$ the full group of (X_A, σ_A). Let $[\sigma_A]_c$ be the set of all τ in $[\sigma_A]$ such that there exist continuous functions $k, l : X_A \to \mathbb{Z}_+$ such that

$$\sigma_A^{k(x)}(\tau(x)) = \sigma_A^{l(x)}(x) \quad \text{for all } x \in X_A. \tag{3.1}$$

We call $[\sigma_A]_c$ the topological full group for (X_A, σ_A). The functions k, l above are called orbit cocycles for τ, and sometimes written as k_τ, l_τ respectively. We remark that the orbit cocyles are not necessarily uniquely determined by τ.
Example. Put $F = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$. Define $\tau \in \mathrm{Homeo}(X_F)$ by setting

$$\tau(x_1, x_2, \dots) = \begin{cases} (2,1,1,x_4,x_5,\dots) & \text{if } (x_1,x_2,x_3) = (1,1,1), \\ (1,1,1,x_4,x_5,\dots) & \text{if } (x_1,x_2,x_3) = (2,1,1), \\ (x_1,x_2,x_3,x_4,x_5,\dots) & \text{otherwise.} \end{cases}$$

Since $\sigma_F(\tau(x)) = \sigma_F(x)$ for all $x \in X_F$, by putting $k(x) = l(x) = 1$ for all $x \in X_F$, one sees that τ belongs to $[\sigma_F]_c$.

Let A be an arbitrary fixed $N \times N$ matrix with entries in $\{0, 1\}$ satisfying condition (I). Then one easily knows that $[\sigma_A]$ is a subgroup of $\mathrm{Homeo}(X_A)$ and $[\sigma_A]_c$ is a subgroup of $[\sigma_A]$.

Although σ_A itself does not belong to $[\sigma_A]$, the group $[\sigma_A]_c$ is not trivial in any case. Put $[\sigma_A]_c(x) = \{\tau(x) \in X_A \mid \tau \in [\sigma_A]_c\}$ for $x \in X_A$. Then we have

Lemma 3.1. *$[\sigma_A]_c(x) = orb_{\sigma_A}(x)$ for $x \in X_A$.*

4. Full groups and normalizers

Denote by $\mathcal{U}(\mathcal{O}_A), \mathcal{U}(\mathfrak{D}_A)$ the groups of unitaries of $\mathcal{O}_A$ and $\mathfrak{D}_A$ respectively. We will identify the algebra $C(X_A)$ with the subalgebra $\mathfrak{D}_A$ of $\mathcal{O}_A$. For $v \in \mathcal{U}(\mathcal{O}_A)$, we put $Ad(v)(a) = vav^*,\ a \in \mathcal{O}_A$. Then we have

Proposition 4.1. *For $\tau \in [\sigma_A]_c$, there exists a unitary $u_\tau \in N(\mathcal{O}_A, \mathfrak{D}_A)$ such that*

$$Ad(u_\tau)(f) = f \circ \tau^{-1} \qquad \text{for } f \in \mathfrak{D}_A,$$

and the correspondence $\tau \in [\sigma_A]_c \to u_\tau \in N(\mathcal{O}_A, \mathfrak{D}_A)$ is a homomorphism of group.

For $v \in N(\mathcal{O}_A, \mathfrak{D}_A)$, $Ad(v)$ induces an automorphism on both algebras $\mathcal{O}_A$ and $\mathfrak{D}_A$. Let τ_v denote the homeomorphism on X_A induced by $Ad(v) : \mathfrak{D}_A \to \mathfrak{D}_A$ satisfying $Ad(v)(f) = f \circ \tau_v{}^{-1}$ for $f \in \mathfrak{D}_A$. We will know that τ_v gives rise to an element of $[\sigma_A]_c$. We fix $v \in N(\mathcal{O}_A, \mathfrak{D}_A)$ for a while.

Lemma 4.2. *There exists a family $v_m, m \in \mathbb{Z}$ of partial isometries in $\mathcal{O}_A$ such that all but finitely many $v_m, m \in \mathbb{Z}$ are zero, and*

1. *$v = \sum_{m \in \mathbb{Z}} v_m$: finite sum.*
2. *$v_m \mathfrak{D}_A v_m^* \subset \mathfrak{D}_A$ and $v_m^* \mathfrak{D}_A v_m \subset \mathfrak{D}_A$ for $m \in \mathbb{Z}$.*
3. *$v_m^* v_m, v_m v_m^*$ are projections in $\mathfrak{D}_A$ for $m \in \mathbb{Z}$.*
4. *$v_m^* v_{m'} = v_m v_{m'}^* = 0$ for $m \neq m'$.*
5. *$v_0 \in \mathcal{F}_A$.*

Lemma 4.3. *For a fixed $n \in \mathbb{N}$, there exist partial isometries $v_\mu, v_{-\mu} \in \mathcal{F}_A$ for each $\mu \in B_n(X_A)$ satisfying the following conditions:*

1. *$v_n = \sum_{\mu \in B_n(X_A)} S_\mu v_\mu$ and $v_{-n} = \sum_{\mu \in B_n(X_A)} v_{-\mu} S_\mu^*$.*
2. *$v_\mu^* v_\mu$, $S_\mu v_\mu v_\mu^* S_\mu^*$, $S_\mu v_{-\mu}^* v_{-\mu} S_\mu^*$ and $v_{-\mu} v_{-\mu}^*$ are projections in $\mathfrak{D}_A$ such that*

$$v_n^* v_n = \sum_{\mu \in B_n(X_A)} v_\mu^* v_\mu, \qquad v_n v_n^* = \sum_{\mu \in B_n(X_A)} S_\mu v_\mu v_\mu^* S_\mu^*,$$

$$v_{-n}^* v_{-n} = \sum_{\mu \in B_n(X_A)} S_\mu v_{-\mu}^* v_{-\mu} S_\mu^*, \qquad v_{-n} v_{-n}^* = \sum_{\mu \in B_n(X_A)} v_{-\mu} v_{-\mu}^*.$$

3. *$v_\mu v_\nu^* = v_{-\mu}^* v_{-\nu} = 0$ for $\mu, \nu \in B_n(X_A)$ with $\mu \neq \nu$.*
4. *$v_\mu \mathfrak{D}_A v_\mu^*, v_\mu^* \mathfrak{D}_A v_\mu, v_{-\mu} \mathfrak{D}_A v_{-\mu}^*$ and $v_{-\mu}^* \mathfrak{D}_A v_{-\mu}$ are contained in $\mathfrak{D}_A$.*

Let $u \in \mathcal{O}_A$ be a partial isometry satisfying

$$u \mathfrak{D}_A u^* \subset \mathfrak{D}_A, \qquad u^* \mathfrak{D}_A u \subset \mathfrak{D}_A.$$

Put the projections $p_u = u^* u$, $q_u = u u^* \in \mathfrak{D}_A$ and clopen sets $X_u = \mathrm{supp}(p_u)$, $Y_u = \mathrm{supp}(q_u) \subset X_A$. Then $Ad(u) : \mathfrak{D}_A p_u \to \mathfrak{D}_A q_u$ yields an isomorphism and induces a homeomorphism $h_u : X_u \to Y_u$ such that

$$Ad(u)(g) = g \circ h_u{}^{-1} \in \mathfrak{D}_A q_u (= C(Y_u)) \qquad \text{for } g \in \mathfrak{D}_A p_u (= C(X_u)).$$

Lemma 4.4. *Keep the above situation. Assume that $u \in \mathcal{F}_A$. Then there exists $k \in \mathbb{N}$ such that for all $x = (x_n)_{n\in\mathbb{N}} \in X_u$*

$$y_n = x_n \qquad \text{for all } n > k$$

where $y = (y_n)_{n\in\mathbb{N}} = h_u(x)$.

Thus we have

Lemma 4.5. *For a partial isometry $u \in \mathcal{F}_A$ satisfying*

$$u\mathfrak{D}_A u^* \subset \mathfrak{D}_A, \qquad u^*\mathfrak{D}_A u \subset \mathfrak{D}_A,$$

there exists $k_u \in \mathbb{N}$ such that the homeomorphism $h_u : \mathrm{supp}(u^*u) \to \mathrm{supp}(uu^*)$ defined by $Ad(u)(g) = g \circ h_u^{-1}$ for $g \in \mathfrak{D}_A u^*u$ satisfies the condition

$$\sigma_A^{k_u}(h_u(x)) = \sigma_A^{k_u}(x) \qquad \text{for } x \in \mathrm{supp}(u^*u).$$

Therefore we have

Proposition 4.6. *For any $v \in N(\mathcal{O}_A, \mathfrak{D}_A)$, the homomorphism τ_v on X_A induced by the automorphism of $\mathfrak{D}_A$ defined by the restriction of $Ad(v)$ to $\mathfrak{D}_A$ gives rise to an element of the topological full group $[\sigma_A]_c$.*

The unitaries $\mathcal{U}(\mathfrak{D}_A)$ are naturally embedded into $N(\mathcal{O}_A, \mathfrak{D}_A)$. We denote the embedding by id. For $v \in N(\mathcal{O}_A, \mathfrak{D}_A)$, the induced homemorphism on X_A is denoted by τ_v, that gives rise to an element of $[\sigma_A]_c$ by the above proposition. We then have

Theorem 4.7. *The sequence*

$$1 \longrightarrow \mathcal{U}(\mathfrak{D}_A) \xrightarrow{\mathrm{id}} N(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\tau} [\sigma_A]_c \longrightarrow 1$$

is exact and splits.

5. Orbit equivalence

De nition. Let (X_A, σ_A) and (X_B, σ_B) be topological Markov shifts. If there exists a homeomorphism $h : X_A \to X_B$ such that $h(orb_{\sigma_A}(x)) = orb_{\sigma_B}(h(x))$ for $x \in X_A$, then (X_A, σ_A) and (X_B, σ_B) are said to be topologically orbit equivalent. In this case, for $x \in X_A$, $h(\sigma_A(x)) \in orb_{\sigma_B}(h(x))$ so that $h(\sigma_A(x)) \in \cup_{k=0}^{\infty} \cup_{l=0}^{\infty} \sigma_B^{-k}\sigma_B^{l}(h(x))$. Hence there exist $k_1, l_1 : X_A \to \mathbb{Z}_+$ such that $\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x))$. Similarly there exist $k_2, l_2 : X_B \to \mathbb{Z}_+$ such that $\sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y))$.

We say that (X_A, σ_A) and (X_B, σ_B) are *continuously orbit equivalent* if there exists a homeomorphism $h : X_A \to X_B$ and continuous functions $k_1, l_1 : X_A \longrightarrow \mathbb{Z}_+$ and $k_2, l_2 : X_B \longrightarrow \mathbb{Z}_+$ such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)), \qquad \sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y)) \tag{5.1}$$

for $x \in X_A$ and $y \in X_B$.

Example. Let $A_{[2]} = \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$, $F = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$. The subshift X_F is the set of all sequences $(x_n)_{n\in\mathbb{N}}$ of $1, 2$ such that the word $(2, 2)$ is forbidden. Define a homeomorphism $h : X_F \longrightarrow X_{A_{[2]}}$ by substituting the word 2 in $X_{A_{[2]}}$ for the word $(2, 1)$ in X_F from the leftmost. Then $h : X_F \longrightarrow X_{A_{[2]}}$ gives rise to a continuous orbit equivalence between (X_F, σ_F) and $(X_{A_{[2]}}, \sigma_{A_{[2]}})$. The following hold.

Proposition 5.1. *There exists a homeomorphism $h : X_A \longrightarrow X_B$ such that $h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$ if and only if then (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*

Proposition 5.2. *If there exists an isomorphism $\Psi : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ such that $\Psi(\mathfrak{D}_A) = \mathfrak{D}_B$, then there exists a homeomorphism $h : X_A \longrightarrow X_B$ such that $h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$.*

Proof. By Theorem 4.7, there exists an isomorphism $\widetilde{\Psi} : [\sigma_A]_c \longrightarrow [\sigma_B]_c$ of group such that the following diagrams are commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{U}(\mathfrak{D}_A) & \xrightarrow{\text{id}} & N(\mathcal{O}_A, \mathfrak{D}_A) & \xrightarrow{\tau} & [\sigma_A]_c & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \Psi|_{\mathcal{U}(\mathfrak{D}_A)}} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \widetilde{\Psi}} & & \\ 1 & \longrightarrow & \mathcal{U}(\mathfrak{D}_B) & \xrightarrow{\text{id}} & N(\mathcal{O}_B, \mathfrak{D}_B) & \xrightarrow{\tau} & [\sigma_B]_c & \longrightarrow & 1. \end{array}$$

Let $h : X_A \longrightarrow X_B$ be the homeomorphism satisfying $\Psi(f) = f \circ h^{-1}$ for $f \in \mathfrak{D}_A$. The identity $\Psi \circ Ad(v) \circ \Psi^{-1} = Ad(\Psi(v))$ implies that $h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$.

Then we have

Theorem 5.3. *Let A, B be irreducible square matrices with entries in $\{0, 1\}$ satisfying condition (I). The following three assertions are equivalent:*

(1) *There exists an isomorphism $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\Psi(\mathfrak{D}_A) = \mathfrak{D}_B$.*
(2) *(X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*
(3) *There exists a homeomorphism $h : X_A \to X_B$ such that $h \circ [\sigma_A]_c \circ h^{-1} = [\sigma_B]_c$.*

6. Normalizers of the full groups and automorphisms of $\mathcal{O}_A$

In this section, we will study the normalizer subgroup

$$N([\sigma_A]_c) = \{\varphi \in \text{Homeo}(X_A) \mid \varphi \circ \tau \circ \varphi^{-1} \in [\sigma_A]_c \text{ for all } \tau \in [\sigma_A]_c\}$$

of $[\sigma_A]_c$ in $\text{Homeo}(X_A)$, related to the automorphism group $\text{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$. We set

$$\begin{aligned} N[\sigma_A] =& \{h \in \text{Homeo}(X_A) \mid h(orb_{\sigma_A}(x)) = orb_{\sigma_A}(h(x)) \text{ for } x \in X_A\}, \\ N_c[\sigma_A] =& \{h \in \text{Homeo}(X_A) \mid \text{there exist continuous functions } k_1, l_1, k_2, l_2 : X_A \to \mathbb{Z}_+ \\ & \quad \text{such that } \sigma_A^{k_1(x)}(h(\sigma_A(x))) = \sigma_A^{l_1(x)}(h(x)), \\ & \qquad \sigma_A^{k_2(x)}(h^{-1}(\sigma_A(x))) = \sigma_A^{l_2(x)}(h^{-1}(x)) \text{ for } x \in X_A\} \end{aligned}$$

Lemma 6.1. $N_c[\sigma_A] = N([\sigma_A]_c)$.

Proposition 6.2. *For a homeomorphism $h \in N_c([\sigma_A])$ there exists an automorphism $\alpha_h \in \text{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$ such that $\alpha_h(f) = f \circ h^{-1}$ for $f \in \mathfrak{D}_A$, and the correspondence $h \in N_c([\sigma_A]) \to \alpha_h \in \text{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$ is a homomorphism.*

Conversely for any automorphism $\alpha \in \text{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$, we denote by ϕ_α the homeomorphism on X_A induced by the restriction of α to $\mathfrak{D}_A$ such that $\alpha(f) = f \circ \phi_\alpha^{-1}$ for $f \in \mathfrak{D}_A$. We then have

Proposition 6.3. *ϕ_α belongs to $N([\sigma_A]_c)$.*

We denote by $\varphi_A : \mathfrak{D}_A \to \mathfrak{D}_A$ the homomorphism defined by $\varphi_A(a) = \sum_{i=1}^N S_i a S_i^*$ for $a \in \mathfrak{D}_A$. In regarding $\mathfrak{D}_A$ with $C(X_A)$ as usual, one sees $\varphi_A(f) = f \circ \sigma_A$ for $f \in C(X_A)$. A unitary one-cocycle for φ_A is a $\mathcal{U}(\mathfrak{D}_A)$-valued function $U : \mathbb{Z}_+ \to \mathcal{U}(\mathfrak{D}_A)$ satisfying

$$U(k+l) = U(k)\varphi_A^k(U(l)), \qquad k, l \in \mathbb{Z}_+ \quad (cf.[Ma2]).$$

Let $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ be the set of all unitary one-cocycles for φ_A, that is an abelian group in natural way. For $U \in Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$, put

$$(U)(S_\mu) = U(k)S_\mu \qquad \text{for } \mu \in B_k(X_A).$$

Then (U) gives rise to an automorphism of $\mathcal{O}_A$ such that $(U)|_{\mathfrak{D}_A} = \mathrm{id}$. We note that the correspondence $U \in Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \to U(1) \in \mathcal{U}(\mathfrak{D}_A)$ yields an isomorphism of abelian group, and hence we may identify $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ with $\mathcal{U}(\mathfrak{D}_A)$. By [Ma2; Lemma 4.8],

$$: Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \to \mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A)$$

is an injective homomorphism of group.

Let $V : \mathbb{Z}_+ \to \mathcal{U}(\mathfrak{D}_A)$ be a $\mathcal{U}(\mathfrak{D}_A)$-valued function on $\mathbb{Z}_+$ satisfing

$$V(k) = v\varphi_A^k(v^*), \qquad k \in \mathbb{Z}_+$$

for some unitary $v \in \mathcal{U}(\mathfrak{D}_A)$. Then V is called a coboundary for φ_A. Note that a coboundary for φ_A is a unitary one-cocycle for φ_A. Let $B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ be the set of all coboundaries for φ_A. It is easy to see that $B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ is a subgroup of $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$. We remark that if $U \in Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ satisfies $U(1) = v\varphi_A(v^*)$ for some $v \in \mathcal{U}(\mathfrak{D}_A)$, then $U(k) = v\varphi_A^k(v^*)$ for $k \in \mathbb{N}$, and hence $U \in B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$. Define $H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$ by the quotient group $Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))/B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A))$.

Theorem 6.4. *There exist short exact sequences:*

(1) $1 \longrightarrow B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Inn}(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\phi} [\sigma_A]_c \longrightarrow 1,$

(2) $1 \longrightarrow Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Aut}(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\phi} N([\sigma_A]_c) \longrightarrow 1,$

(3) $1 \longrightarrow H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Out}(\mathcal{O}_A, \mathfrak{D}_A) \xrightarrow{\phi} N([\sigma_A]_c)/[\sigma_A]_c \longrightarrow 1.$

They all split.

7. Orbit equivalence and AF-algebras

In this section, we will show that the discussions in the previous sections can be applied to the pair $(\mathcal{F}_A, \mathfrak{D}_A)$ of the AF-algebra $\mathcal{F}_A$ and its diagonal algebra $\mathfrak{D}_A$, instead of the pair $(\mathcal{O}_A, \mathfrak{D}_A)$ that we have studied. For $x = (x_n)_{n\in\mathbb{N}} \in X_A$, the *uniform orbit* $orb_{\sigma_A}[x]$ of x under σ_A is defined by

$$orb_{\sigma_A}[x] = \cup_{k=0}^{\infty} \sigma_A^{-k}(\sigma_A^k(x)) \subset X_A.$$

Hence $y = (y_n)_{n\in\mathbb{N}} \in X_A$ belongs to $orb_{\sigma_A}[x]$ if and only if there exist $k \in \mathbb{Z}_+$ and an admissible word $\mu_1 \cdots \mu_k \in B_k(X_A)$ such that

$$y = (\mu_1, \dots, \mu_k, y_{k+1}, y_{k+2}, \dots).$$

Let $[[\sigma_A]]$ be the set of all homeomorphisms $\tau \in \mathrm{Homeo}(X_A)$ such that $\tau(x) \in orb_{\sigma_A}[x]$ for all $x \in X_A$. Let $[\sigma_A]_{AF}$ be the set of all τ in $[[\sigma_A]]$ such that there exists a continuous function $k : X_A \to \mathbb{Z}_+$ such that

$$\sigma_A^{k(x)}(\tau(x)) = \sigma_A^{k(x)}(x) \quad \text{for all } x \in X_A. \tag{7.1}$$

We call $[\sigma_A]_{AF}$ the AF-full group for (X_A, σ_A). As X_A is compact, for a homeomorphism $\tau \in \mathrm{Homeo}(X_A)$, τ belongs to $[\sigma_A]_{AF}$ if and only if there exists a constant number $k \in \mathbb{Z}_+$ such that $\sigma_A^k(\tau(x)) = \sigma_A^k(x)$ for all $x \in X_A$. We set for $x \in X_A$, $[\sigma_A]_{AF}(x) = \{\tau(x) \mid \tau \in [\sigma_A]_{AF}\}$. It is immediate to see that $[\sigma_A]_{AF}(x) = orb_{\sigma_A}[x]$. Let $N(\mathcal{F}_A, \mathfrak{D}_A)$ be the normalizer of $\mathfrak{D}_A$ in $\mathcal{F}_A$, that is defined as the group of all unitaries $u \in \mathcal{F}_A$ such that $u\mathfrak{D}_A u^* = \mathfrak{D}_A$. We note that the algebra $\mathfrak{D}_A$ is also maximal abelian in $\mathcal{F}_A$. By similar argument to the previous sections, there exists a short exact sequnce:

$$1 \longrightarrow \mathcal{U}(\mathfrak{D}_A) \xrightarrow{\mathrm{id}} N(\mathcal{F}_A, \mathfrak{D}_A) \xrightarrow{\tau} [\sigma_A]_{AF} \longrightarrow 1$$

that splits. We say that (X_A, σ_A) and (X_B, σ_B) are *uniformly orbit equivalent* if there exists a homeomorphism $h : X_A \to X_B$ such that $h(orb_{\sigma_A}[x]) = orb_{\sigma_B}[h(x)]$ for $x \in X_A$ and there exist constant numbers $k_1, k_2 \in \mathbb{Z}_+$ such that

$$\sigma_B^{k_1}(h(\sigma_A(x)) = \sigma_B^{k_1}(h(x)), \qquad \sigma_A^{k_2}(h^{-1}(\sigma_B(y)) = \sigma_A^{k_2}(h^{-1}(y))$$

for $x \in X_A$ and $y \in X_B$. Then we have

Theorem 7.1. *The following three assertions are equivalent:*

1. *There exists an isomorphism $\Psi : \mathcal{F}_A \to \mathcal{F}_B$ such that $\Psi(\mathfrak{D}_A) = \mathfrak{D}_B$.*
2. *(X_A, σ_A) and (X_B, σ_B) are uniformly orbit equivalent.*
3. *There exists a homeomorphism $h : X_A \to X_B$ such that $h \circ [\sigma_A]_{AF} \circ h^{-1} = [\sigma_B]_{AF}$.*

Let $\mathrm{Aut}(\mathcal{F}_A, \mathfrak{D}_A)$ be the group of all automorphisms α of $\mathcal{F}_A$ such that $\alpha(\mathfrak{D}_A) = \mathfrak{D}_A$. Denote by $\mathrm{Inn}(\mathcal{F}_A, \mathfrak{D}_A)$ the subgroup of $\mathrm{Aut}(\mathcal{F}_A, \mathfrak{D}_A)$ of inner automorphisms on $\mathcal{F}_A$. We set $\mathrm{Out}(\mathcal{F}_A, \mathfrak{D}_A)$ the quotient group $\mathrm{Aut}(\mathcal{F}_A, \mathfrak{D}_A)/\mathrm{Inn}(\mathcal{F}_A, \mathfrak{D}_A)$. By the same argument as Section 6, we have

Theorem 7.2. *There exist short exact sequeneces:*

1. $1 \longrightarrow B^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Inn}(\mathcal{F}_A, \mathfrak{D}_A) \xrightarrow{\phi} [\sigma_A]_{AF} \longrightarrow 1,$
2. $1 \longrightarrow Z^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Aut}(\mathcal{F}_A, \mathfrak{D}_A) \xrightarrow{\phi} N([\sigma_A]_{AF}) \longrightarrow 1$
3. $1 \longrightarrow H^1_{\sigma_A}(\mathcal{U}(\mathfrak{D}_A)) \xrightarrow{\lambda} \mathrm{Out}(\mathcal{F}_A, \mathfrak{D}_A) \xrightarrow{\phi} N([\sigma_A]_{AF})/[\sigma_A]_{AF} \longrightarrow 1.$

They all split, where $N([\sigma_A]_{AF})$ is the normalizer subgroup of $[\sigma_A]_{AF}$ in $[[\sigma_A]]$.

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